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Interior-point algorithm for semidefinite programming based on a logarithmic kernel function

by

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Abstract

In this work, we propose a primal-dual interior-point algorithm for semidefinite optimization based on a new kernel function with an efficient logarithmic barrier term. We show that the best result of iteration bounds can be achieved, namely $O(\sqrt{n}\log n \log \frac{n}{\varepsilon})$, for large update and $O(\sqrt{n}\log \frac{n}{\varepsilon})$ for small-update methods.

Key Words: Semidefinite optimization, kernel functions, primal-dual interior-point method, large and small-update methods, complexity bound.
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1 Introduction

Consider the standard semidefinite optimization (SDO) problem:

$$\min\{< C, X >: < A_i, X >= b_i \quad \text{for } i = 1, \ldots, m \ \text{ and } X \succeq 0\} \tag{P}$$

and its dual problem:

$$\max\{b^t y : S = C - \sum_{i=1}^{m} y_i A_i \ , \ S \succeq 0\}. \tag{D}$$

Where $b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$, the matrices C and $A_i,\ i = 1, \ldots, m$, are given and belong to the linear space of $n \times n$ symmetric matrices $\mathbb{S}_n$. The $< ., . >$ operation is the inner product on $\mathbb{S}_n$ of two matrices A and B which is the trace of their product, i.e., $\langle A, B\rangle = tr(AB) = \sum_{i,j} a_{ij} b_{ij}$ and the inequality constraint $X \succeq 0$ ($X \succ 0$) indicates that the matrix X belong to the cone of positive semidefinite matrices $\mathbb{S}_n^+$ (the cone of positive definite matrices $\mathbb{S}_n^{++}$).

Primal-dual interior-point method IPM is one of the most efficient numerical methods for solving large classes of optimization problems and highly efficient in both theory and practice. It is well known that the SDO has a variety of applications in engineering problems, such as optimal control, combinatorics, image processing, sensor networks, financial mathematics and statistics[13]. Many researchers have proposed interior-point algorithms for various optimization problems based on kernel functions. Most of the polynomial time interior-point algorithms are based on the logarithmic kernel function with $O(\sqrt{n}\log \frac{n}{\varepsilon})$ and $O(n \log \frac{n}{\varepsilon})$ iteration complexity for small- and large update methods, respectively[10]. Recently, Peng et al.[9] defined a class of self regular kernel functions, proposed primal-dual IPM for linear optimization LO and generalized to second order cone optimization SOCO and SDO. Roos et al.[2] defined eligible kernel functions which were defined by four conditions on the function and proposed a primal-dual IPM for LO and simplified the complexity analysis of Peng et al.'s in [9]. Bouafia et al.[3] proposed a primal-dual interior-point algorithm for LO based on a kernel function with a trigonometric barrier term and they obtain the best-known complexity result for small- and large update method.

Several interior-point methods IPMs for LO have been successfully extended to SDO. Wang et al.[11] proposed a primal-dual IPM for SDO based on a generalized version of the kernel function in[2] and obtained $O(q^2\sqrt{n}\log\frac{n}{\varepsilon})$, $q>1$, and $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ complexity results for small- and large update, respectively. EL Ghami et al.[5] extended the IPM for LO in[2] to SDO and obtained the similar iteration bounds as analog of LO. Lee et al.[7] defined a new class of kernel functions and obtained the best-known complexity results for small- and large update IPMs based on a kernel function for LO and SDO. EL Ghami [4] generalize the analysis presented in [3] for SDO and obtained the best-known complexity results for small- and large update method.

Motivated by their works, we proposed a primal-dual interior-point algorithm for SDO based on a new logarithmic kernel function and obtain the best-known complexity results of small- and large update methods.

This paper is organized as follows: In section 2, we introduce a fundamental concepts and give the classical Nesterov-Todd direction. In section 3, we present the kernel function based on Nesterov-Todd direction and describe the generic primal-dual algorithm. In section 4, a new kernel function and its growth properties for SDO are studied. In section 5, the complexity results of small- and large update algorithms for SDO are computed. Finally, a conclusion ends section 6.

We will make use of the following notations throughout the paper: $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of real, nonnegative real and positive real vectors with n components, respectively. $||.||$ denotes the Frobenius norm for matrices. For $Q\in\mathbb{S}^n_{++}$, $Q^{1/2}$ denotes the symmetric square root of Q. For any $V\in\mathbb{S}^n$, we denote by $\lambda(V)$ the vector of eigenvalues of V arranged in non-increasing order, that is, $\lambda_1(V)\geq\lambda_2(V)\geq\ldots\geq\lambda_n(V)$ and $\Lambda = diag(\lambda(V))$, i.e., the diagonal matrix from a vector $\lambda(V)$. I denotes an $n\times n$ identity matrix. For $f(x), g(x) : \mathbb{R}_{++}\to\mathbb{R}_{++}$, $f(x) = O(g(x))$ if $f(x)\leq c_1 g(x)$ for some positive constant c_1 and $f(x) = \Theta(g(x))$ if $c_2 g(x)\leq f(x)\leq c_3 g(x)$ for some positive constants c_2 and c_3.

2 Classical Nestrov-Todd search direction for SDO

In this section, we recall the notion of the central path with its properties and we drive the classical Nestrov-Todd search direction for SDO.

Throughout the paper, we assume that the matrix A_i, $i = 1,\ldots,m$, are linearly independent and the problems (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists $X\in F_P, S\in F_D$ with $X\succ 0, S\succ 0$, where F_P and F_D denote the feasible sets of the problem (P) and (D), respectively.

Finding an optimal solution of the problem (P) and (D) is equivalent to solving the following system:

$$\begin{cases} < A_i, X >= b_i, \quad i = 1, \ldots, m, \ X \succeq 0, \\ \sum\limits_{i=1}^{m} y_i A_i + S = C, \ S \succeq 0, \\ XS = 0. \end{cases} \tag{2.1}$$

The basic idea of primal-dual IPMs is to replace the complementarity condition of (2.1), $XS = 0$, by the parameterized equation $XS = \mu I$ with $X, S \succ 0$ and $\mu > 0$. So, we consider the following system:

$$\begin{cases} < A_i, X >= b_i, \quad i = 1, \ldots, m, \ X \succeq 0, \\ \sum\limits_{i=1}^{m} y_i A_i + S = C, \ S \succeq 0, \\ XS = \mu I. \end{cases} \tag{2.2}$$

Under the previous assumptions, the system (2.2) has a unique solution $(X(\mu), y(\mu), S(\mu))$ for each $\mu > 0$, we call it the $\mu-center$ of both problems (P) and (D). The set of $\mu-center$ defines a homotopy which is called the central path of (P) and (D) which is converges to the optimal solution of the problem (P) and (D) as μ goes to zero[13]. Now, to obtain the search direction, we apply Newton's method to the system (2.2), then we get the Newton system as follows:

$$\begin{cases} < A_i, \Delta X >= 0, \quad i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ X\Delta S + \Delta X S = \mu I - XS. \end{cases} \tag{2.3}$$

Since A_i are linearly independent and $X \succ 0, S \succ 0$, the system (2.3) has a unique search direction $(\Delta X, \Delta y, \Delta S)$. Note that ΔS is symmetric from the second equation of (2.3), but ΔX may be not symmetric. Various methods of symmetrizing the third equation of (2.3) are proposed so that the new system has a unique symmetric solution. In this paper, we use the NT symmetrizing scheme[8]. Let

$$P = X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2} = S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}$$

and $D = P^{1/2}$ where $P^{1/2}$ denotes the symmetric square root of P. The matrix D is used to scale both matrices X and S to the same matrix V defined by

$$V = \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} (D^{-1}XSD)^{1/2}. \tag{2.4}$$

Then, matrices D and V are symmetric positive definite. By using (2.4) the Newton system (2.3) can be rewritten as follows:

$$\begin{cases} < \overline{A}_i, D_X >= 0, \quad i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S = V^{-1} - V. \end{cases} \tag{2.5}$$

with

$$\overline{A}_i = \frac{1}{\sqrt{\mu}} D A_i D, \ i = \overline{1, m}, \ D_X = \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \ D_S = \frac{1}{\sqrt{\mu}} D \Delta S D. \tag{2.6}$$

The system (2.5) determines a uniquely symmetric NT direction with the matrices D_X and D_S be orthogonal, and it is evident that $Tr(D_X D_S) = Tr(D_S D_X) = 0$. The above Nesterov-Todd direction leads to the classical primal-dual IPM algorithms for SDO.

3 New search direction and the generic primal-dual IPM for SDO

In this section, we recall the definition of a matrix function and we derive the new kernel function based on NT direction and then we introduce our generic primal-dual IPM algorithm for SDO.

For $V = Q^T diag(\lambda_1(V), \lambda_2(V), \ldots, \lambda_n(V))Q$, the spectral decomposition of $V \in \mathbb{S}^n_{++}$, we generalize a function $\psi(t) : \mathbb{R}_{++} \to \mathbb{R}_+$ to the matrix function $\psi(V) : \mathbb{S}^n_{++} \to \mathbb{S}^n$ as follows:

$$\begin{aligned} \psi(V) &= Q^T diag(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \ldots, \psi(\lambda_n(V)))Q, \\ \psi'(V) &= Q^T diag(\psi'(\lambda_1(V)), \psi'(\lambda_2(V)), \ldots, \psi'(\lambda_n(V)))Q. \end{aligned} \tag{3.1}$$

Replacing the right hand side $V^{-1} - V$ of the third equation of (2.5) by $-\psi'(V)$. Then we have the linear system:

$$\begin{cases} < \overline{A}_i, D_X >= 0, \quad i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S = -\psi'(V). \end{cases} \tag{3.2}$$

where $\psi(t)$ is a given kernel function and $\psi(V), \psi'(V)$ are the associated matrix functions denote in (3.1), the system (3.2) has a unique symmetric solution. For any kernel function $\psi(t)$, we define $\Psi(V) : \mathbb{S}^n_{++} \to \mathbb{R}_+$ by

$$\Psi(V) = Tr(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)). \tag{3.3}$$

Then $\Psi(V)$ is strictly convex with respect to $V \succ 0$ and vanishes at its global minimal point $V = I$ and $\Psi(I) = 0$. Since D_X and D_S are orthogonal, for $\mu > 0$,

$$\Psi(V) = 0 \Leftrightarrow V = I \Leftrightarrow D_X = D_S = 0 \Leftrightarrow X = X(\mu), S = S(\mu).$$

Hence we can use $\Psi(V)$ as a proximity function to measure the distance between the current iteration and the corresponding $\mu-$center.

The primal-dual interior-point algorithm for SDO works as follows: Assume that $\tau \geq 1$ and there is a strictly feasible point (X, y, S) which is in a $\tau-$neighborhood of the given $\mu-$center[6]. We update μ to $\mu^+ = (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$, and then solve the system (3.2) and (2.6) to obtain the NT search direction. The positivity condition of a new iteration is ensured with the right choice of the step size α. This procedure is repeated until we find a new iteration (X^+, y^+, S^+) which is in a $\tau-$neighborhood of the μ^+-center and then we let $\mu = \mu^+$ and $(X, y, S) = (X^+, y^+, S^+)$. We repeat the process until $n\mu < \varepsilon$.

primal-dual Algorithm for SDO

Input
A threshold parameter $\tau \geq 1$; an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
a strictly feasible (X^0, S^0) and $\mu^0 = 1$ such that $\Psi(X^0, S^0, \mu^0) \leq \tau$;
begin
  $X = X^0; S = S^0; \mu = \mu^0;$
  while $n\mu \geq \varepsilon$ do
  begin
    $\mu = (1 - \theta)\mu;$
    while $\Psi(X, S, \mu) > \tau$ do
    begin
      solve the system (3.2) and (2.6) to obtain ΔX, Δy, ΔS;
      determine a step size α and take
      $X = X + \alpha\Delta X; y = y + \alpha\Delta y; S = S + \alpha\Delta S;$
    end
  end
end

4 Kernel function and its properties

In this section, we define a class of kernel functions and give its essential properties for complexity analysis.

Definition 1. *We call $\psi : \mathbb{R}_{++} \to \mathbb{R}_+$ a kernel function if ψ is twice differentiable and satisfies the following conditions:*

$$\psi'(1) = \psi(1) = 0, \ \psi''(t) > 0, \ t > 0, \quad \lim_{t \to 0^+} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty. \tag{4.1}$$

From the first tow conditions, it follows that $\psi(t)$ is strictly convex and minimal at $t = 1$, and $\psi(t)$ is expressed in term of its second derivative as follows:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi. \tag{4.2}$$

However, the third condition indicates the barrier property of $\psi(t)$.

Now, we consider our new kernel function $\psi(t)$ as follows:

$$\psi(t) = t^2 - 1 + \frac{t^{1-q} - 1}{q - 1} - \log t, \qquad q > 1, \quad t > 0. \tag{4.3}$$

It is easy to check that $\psi(t)$ is indeed a barrier kernel function and its three first derivatives are as follows:

$$\psi'(t) = 2t - t^{-q} - \frac{1}{t}, \psi''(t) = 2 + qt^{-q-1} + \frac{1}{t^2}, \psi^{(3)}(t) = -q(q+1)t^{-q-2} - \frac{2}{t^3}. \tag{4.4}$$

From (4.4), we have

$$\psi''(t) > 1, \quad t > 0. \tag{4.5}$$

Lemma 1. *Let $\psi(t)$ be defined as in (4.3), then*
(i) $t\psi''(t) + \psi'(t) > 0, \quad 0 < t < 1,$
(ii) $t\psi''(t) - \psi'(t) > 0, \quad t > 1,$
(iii) $\psi^{(3)}(t) < 0, \ \ t > 0.$

Proof. For (i), by using (4.4), it follows that $t\psi''(t) + \psi'(t) = 4t + (q-1)t^{-q} > 0$, for all $q > 1$ and $t > 0$.

For (ii), we have $t\psi''(t) - \psi'(t) = (q+1)t^{-q} + \frac{2}{t} > 0$, for all $q > 1$ and $t > 0$.

For (iii), it is clear from (4.4) that $\psi^{(3)}(t) < 0$, for $t > 0$. ∎

Remark 1. *(Lemma 2.4 in [2]) if $\psi(t)$ satisfy (ii) and (iii) in Lemma 1, then*

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \quad t > 0, \quad \beta > 1.$$

Lemma 2. *For $\psi(t)$, we have for $q > 1$,*
(i) $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}[\psi'(t)]^2, \ \ t > 0, \quad$ *(ii)* $\psi(t) \leq \frac{1}{2}(3+q)(t-1)^2, \ \ t \geq 1.$

Proof. For (i), using the first condition of (4.1) and (4.5), we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \geq \int_1^t \int_1^\xi d\zeta d\xi = \frac{1}{2}(t-1)^2,$$

the second inequality is obtained as follows:

$$\int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \quad \leq \quad \int_1^t \int_1^\xi \psi''(\xi)\psi''(\zeta) d\zeta d\xi = \int_1^t \psi''(\xi)\psi'(\xi) d\xi = \int_1^t \psi'(\xi) d\psi'(\xi) d\xi = \tfrac{1}{2}(\psi'(t))^2.$$

For (ii), using Taylor's theorem, the first condition of (4.1) and Lemma 1 (iii), we have

$$\begin{aligned} \psi(t) &= \psi(1) + \psi'(1)(t-1) + \tfrac{1}{2}\psi''(1)(t-1)^2 + \tfrac{1}{3!}\psi^{(3)}(c)(t-1)^3 \\ &= \tfrac{1}{2}\psi''(1)(t-1)^2 + \tfrac{1}{3!}\psi^{(3)}(c)(t-1)^3 < \tfrac{1}{2}\psi''(1)(t-1)^2 = \tfrac{1}{2}(3+q)(t-1)^2, \end{aligned}$$

for some c, $1 \leq c \leq t$. This completes the proof. ∎

Lemma 3. *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then we have*

$$1 + \sqrt{\frac{2s}{3+q}} \leq \varrho(s) \leq 1 + \sqrt{2s}, \quad q > 1, \ \ s \geq 0.$$

Proof. Let $s = \psi(t)$ for $t \geq 1$, i.e., $\varrho(s) = t$, $t \geq 1$. By the definition of $\psi(t)$, $s = t^2 - 1 + \frac{t^{1-q}-1}{q-1} - \log t$, $q > 1$, $t > 0$. Using Lemma 2 (i), we have $s = \psi(t) \geq \frac{1}{2}(t-1)^2$ this implies that $t = \varrho(s) \leq 1 + \sqrt{2s}$.

For the second inequality using Lemma 2 (ii), then $s = \psi(t) \leq \frac{1}{2}(3+q)(t-1)^2, \ \ t \geq 1$. It follows that $t = \varrho(s) \geq 1 + \sqrt{\frac{2s}{3+q}}$. This completes the proof. ∎

Lemma 4. *Let $\rho : [0, \infty) \to (0, 1]$ be the inverse function of $-\frac{1}{2}\psi(t)$ for $0 < t \leq 1$. Then we have*

$$\rho(z) \geq (\frac{1}{2z+1})^{\frac{1}{q-1}}, \qquad q > 1, \ z \geq 0.$$

Proof. Let $z = -\frac{1}{2}\psi(t)$ for $0 < t \leq 1$. By the definition of ρ, $\rho(z) = t$, for $z \geq 0$. So, we have $z = -\frac{1}{2}(2t - t^{-q} - \frac{1}{t}) \Longleftrightarrow \frac{t^{-q}+t^{-1}}{2} = z + t \Longleftrightarrow t^{-q+1} = 2t(z + t) - 1$, it follows that $t^{-q+1} \leq 2(z+1) - 1$. Hence, we obtain $t = \rho(z) \geq (\frac{1}{2z+1})^{\frac{1}{q-1}}$. This completes the proof. □

For the analysis of the algorithm, we also use the norm-based proximity measure $\delta(V)$ as follows:

$$\delta(V) = \frac{1}{2}||\psi'(V)|| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(\lambda_i(V)))^2} = \frac{1}{2}||D_X + D_S||, \quad V \in \mathbb{S}^n_{++}. \tag{4.6}$$

In the following lemma, we give a relationship between two proximity measures.

Lemma 5. *Let $\delta(V)$ and $\Psi(V)$ be defined as in* (4.6) *and* (3.3), *respectively. Then we have*

$$\delta(V) \geq \sqrt{\frac{1}{2}\Psi(V)}, \quad V \in \mathbb{S}^n_{++}.$$

Proof. Using (4.6) and the second inequality of Lemma 2 (i),

$$\delta^2(V) = \frac{1}{4}\sum_{i=1}^{n}(\psi'(\lambda_i(V)))^2 \geq \frac{1}{2}\sum_{i=1}^{n}\psi(\lambda_i(V)) = \frac{1}{2}\Psi(V).$$

Hence we have $\delta(V) \geq \sqrt{\frac{1}{2}\Psi(V)}$. This completes the proof. □

Remark 2. *Throughout the paper, we assume that $\tau \geq 1$. Using Lemma* 5 *and the assumption that $\Psi(V) \geq \tau$, we have $\delta(V) \geq \frac{1}{\sqrt{2}}$.*

In the following, using Remark 1, we estimate the effect of a $\mu-$update on the value of $\Psi(V)$.

Lemma 6. *(Lemma 4.16 in [12]) Let ϱ be defined as in Lemma* 3. *Then we have*

$$\Psi(\beta V) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right), \quad V \in \mathbb{S}^n_{++}, \quad \beta \geq 1.$$

Lemma 7. *Let $0 < \theta < 1$ and $V^+ = \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$, then for $q > 1$ we have*

(i) $\Psi(V^+) \leq \frac{3+q}{2(1-\theta)}\left(\sqrt{n\theta} + \sqrt{2\tau}\right)^2$, (ii) $\Psi(V^+) \leq \dfrac{n\theta + 2\tau + 2\sqrt{2\tau n}}{1-\theta}$.

Proof. For (i), since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi(V)}{n}\right) \geq 1$, we have $\frac{\varrho\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Lemma 6 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 2 (ii), Lemma 3 and $\Psi(V) \leq \tau$, we obtain

$$\begin{aligned}
\Psi(V^+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right)\right) \leq \frac{n(3+q)}{2}\left(\frac{\varrho\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2 \\
&\leq \frac{n(3+q)}{2}\left(\frac{1+\sqrt{\frac{2\tau}{n}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \leq \frac{n(3+q)}{2}\left(\frac{\theta+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right)^2 = \frac{3+q}{2(1-\theta)}\left(\sqrt{n\theta}+\sqrt{2\tau}\right)^2,
\end{aligned}$$

where the last inequality holds from $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta, \; 0 \leq \theta < 1$.

For (ii), we have $\psi(t) \leq t^2 - 1, \forall t \geq 1$. Using Lemma 6 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 3 and $\Psi(V) \leq \tau$, we have

$$\begin{aligned}
\Psi(V^+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right)\right) \leq n\left(\left[\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right)\right]^2 - 1\right) \\
&= \frac{n}{1-\theta}\left(\varrho\left(\frac{\Psi(V)}{n}\right)^2 - (1-\theta)\right) \leq \frac{n}{1-\theta}\left(\left[1+\sqrt{\frac{2\Psi(V)}{n}}\right]^2 - (1-\theta)\right) \\
&\leq \frac{n}{1-\theta}\left(\theta + 2\frac{\tau}{n} + 2\sqrt{\frac{2\tau}{n}}\right) = \frac{n\theta+2\tau+2\sqrt{2\tau n}}{1-\theta}.
\end{aligned}$$

□

Define for $q > 1$ and $0 < \theta < 1$,

$$\widetilde{\Psi}_0 = \frac{3+q}{2(1-\theta)}\left(\sqrt{n\theta}+\sqrt{2\tau}\right)^2, \qquad \overline{\Psi}_0 = \frac{n\theta+2\tau+2\sqrt{2\tau n}}{1-\theta}. \tag{4.7}$$

We will use $\widetilde{\Psi}_0$ and $\overline{\Psi}_0$ for the upper bounds of $\Psi(V)$ for small- and large- update methods, respectively.

Remark 3. *For small–update method with $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, and for large-update method with $\tau = O(n)$ and $\theta = \Theta(1)$.*

5 Complexity analysis

In this section, we compute a feasible step size α and the decrement of the proximity function during an inner iteration.

For fixed μ, if we take a step size α along the search direction $(\Delta X, \Delta y, \Delta S)$, we obtain a new iteration $X^+ = X + \alpha\Delta X, \quad y^+ = y + \alpha\Delta y, \quad S^+ = S + \alpha\Delta S, \quad \alpha > 0$.

Using (2.6), we can rewrite X^+ and S^+ as follows:

$$X^+ = \sqrt{\mu}D(V+\alpha D_X)D, \qquad S^+ = \sqrt{\mu}D^{-1}(V+\alpha D_S)D^{-1}. \tag{5.1}$$

From (2.5), we have $V^+ = \frac{1}{\sqrt{\mu}}(D^{-1}X^+S^+D)^{1/2}$. From (5.1), we have

$$(V^+)^2 = (V+\alpha D_X)(V+\alpha D_S).$$

Since $V+\alpha D_X \in \mathbb{S}^n_{++}$, $V+\alpha D_S \in \mathbb{S}^n_{++}$, $(V^+)^2$ is similar to the matrix

$$(V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}.$$

This implies that the eigenvalues of V^+ are the same as those of

$$\left((V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Then we have

$$\Psi(V^+) = \Psi\left(\left((V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right).$$

Lemma 8. *(Proposition 5.2.6 in [9]) Let $V_1, V_2 \in \mathbb{S}^n_{++}$. Then we have*

$$\Psi\left(\left[V_1^{\frac{1}{2}}V_2V_1^{\frac{1}{2}}\right]^{\frac{1}{2}}\right) \leq \frac{1}{2}(\Psi(V_1)+\Psi(V_2)).$$

By Lemma 8, we obtain

$$\Psi(V^+) \leq \frac{1}{2}(\Psi(V+\alpha D_X)+\Psi(V+\alpha D_S)). \tag{5.2}$$

Define for $\alpha > 0$, $f(\alpha) = \Psi(V^+) - \Psi(V)$, $f_1(\alpha) = \frac{1}{2}(\Psi(V+\alpha D_X)+\Psi(V+\alpha D_S)) - \Psi(V)$. From (5.2), $f(\alpha) \leq f_1(\alpha)$ and $f(0) = f_1(0) = 0$.

Now to estimate the decrease of the proximity during one step, we need the two successive derivatives of $f_1(\alpha)$ with respect to α. We have

$$\begin{aligned} f_1'(\alpha) &= \frac{1}{2}Tr(\psi'(V+\alpha D_X)D_X + \psi'(V+\alpha D_S)D_S), \\ f_1''(\alpha) &= \frac{1}{2}Tr(\psi''(V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_S)D_S^2). \end{aligned}$$

It is obvious that $f_1''(\alpha) > 0$, unless $D_X = D_S = 0$.

From the third equation of the system (3.2) and (4.6), we have

$$f_1'(0) = \frac{1}{2}Tr(\psi'(V)(D_X+D_S)) = \frac{1}{2}Tr(-(\psi'(V))^2) = -2\delta^2(V).$$

For notational convenience, let $\delta = \delta(V)$ and $\Psi = \Psi(V)$.

To find the default step size, we need the following lemmas.

Lemma 9. *(Lemma 5.19 in [12]) Let δ be defined as in* (4.6)*. Then we have* $f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_n(V) - 2\alpha\delta)$.

Lemma 10. *(Lemma 4.2 in [2]) If α satisfies $\psi'(\lambda_n(V) - 2\alpha\delta) + \psi'(\lambda_n(V)) \leq 2\delta$, then* $f_1'(\alpha) \leq 0$.

Lemma 11. *(Lemma 4.4 in [2]) Let ρ and $\overline{\alpha}$ be defined as in Lemma* 4 *and Lemma* 10*, respectively. Then $\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}$.*

Lemma 12. *Let ρ and $\overline{\alpha}$ be defined as in Lemma* 11. *If* $1 \leq \tau \leq \Psi(V)$, *then we have*

$$\overline{\alpha} \geq \frac{1}{2+(q+1)(4\delta+1)^{\frac{q+1}{q-1}}}.$$

Proof. Using Lemma 11, lemma 4 and (4.4) we have

$$\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))} = \frac{1}{2+\frac{1}{\rho(2\delta)^2}+q(\rho(2\delta))^{-q-1}} \geq \frac{1}{2+(4\delta+1)^{\frac{2}{q-1}}+q(4\delta+1)^{\frac{q+1}{q-1}}} \geq \frac{1}{2+(q+1)(4\delta+1)^{\frac{q+1}{q-1}}}.$$

This completes the proof. □

Define the default step size $\widetilde{\alpha}$ as follows:

$$\widetilde{\alpha} = \frac{1}{2+(q+1)(4\delta+1)^{\frac{q+1}{q-1}}}, \tag{5.3}$$

with $\widetilde{\alpha} \leq \overline{\alpha}$.

Lemma 13. *(Lemma 4.5 in [2]) If the step size α is such that $\alpha \leq \overline{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Lemma 14. *Let $\widetilde{\alpha}$ be defined as in* (5.3). *Then we have*

$$f(\widetilde{\alpha}) \leq -\frac{1}{40\sqrt{2}(q+1)}\Psi^{\frac{q-3}{2(q-1)}}.$$

Proof. Using Lemma 13 with $\alpha = \widetilde{\alpha}$ and (5.3), we have

$$\begin{aligned} f(\widetilde{\alpha}) &\leq -\widetilde{\alpha}\delta^2 = -\frac{\delta^2}{2+(q+1)(4\delta+1)^{\frac{q+1}{q-1}}} \leq -\frac{\delta^2}{2(2\delta)^{\frac{q+1}{q-1}}+(q+1)(4\delta+(2\delta))^{\frac{q+1}{q-1}}} \\ &\leq -\frac{\delta^{2-\frac{q+1}{q-1}}}{(8+(q+1)36)} \leq -\frac{\delta^{\frac{q-3}{q-1}}}{40(q+1)} \leq -\frac{\Psi^{\frac{q-3}{2(q-1)}}}{40\sqrt{2}(q+1)}. \end{aligned}$$

This completes the proof. □

Lemma 15. *(Proposition 1.3.2 in [9]) Suppose that a sequence $t_k > 0, k = 0, 1, \ldots, K$ is satisfying the following inequality: $t_{k+1} \leq t_k - \eta t_k^{\gamma}$, $\eta > 0$, $\gamma \in [0, 1[$, $k = 0, 1, 2, \ldots, K$. Then*

$$K \leq \left[\frac{t_0^{1-\gamma}}{\eta(1-\gamma)}\right].$$

We denote the value of Ψ after $\mu-$update as Ψ_0 and the subsequent values in the same outer iteration are denoted as Ψ_l, $l = 0, 1, \ldots, K$, where K denotes the total number of inner iterations per an outer iteration. Then we have $\Psi_0 \leq \widetilde{\Psi}_0$ and $\Psi_0 \leq \overline{\Psi}_0$, where $\widetilde{\Psi}_0$ and $\overline{\Psi}_0$ are defined in (4.7). Then we have $\Psi_{K-1} > \tau$ and $0 \leq \Psi_K \leq \tau$.

Theorem 1. *Let $\widetilde{\Psi}_0$ and $\overline{\Psi}_0$ be defined as in (4.7) and let K_1 and K_2 be the total numbers of inner iterations in the outer iteration for small- and large-update methods, respectively. Then for $q \geq 1$, we have*

(i) $K_1 \leq \left[80\sqrt{2}(q-1)\widetilde{\Psi}_0^{\frac{q+1}{2(q-1)}}\right]$, *(ii)* $K_2 \leq \left[80\sqrt{2}(q-1)\overline{\Psi}_0^{\frac{q+1}{2(q-1)}}\right]$.

Proof. For (i), combining Lemma 14 and Lemma 15 with $\eta = \frac{1}{40\sqrt{2}(q+1)}$ and $\gamma = \frac{q-3}{2(q-1)}$, we have

$$K_1 \leq \left[80\sqrt{2}(q-1)\widetilde{\Psi}_0^{\frac{q+1}{2(q-1)}}\right].$$

For (ii), by the same way, we have

$$K_2 \leq \left[80\sqrt{2}(q-1)\overline{\Psi}_0^{\frac{q+1}{2(q-1)}}\right].$$

This completes the proof. ∎

The number of outer iterations is bounded above by $\left[\frac{1}{\theta}\log\frac{n}{\varepsilon}\right]$[10]. By multiplying the number of outer iterations by the number of inner iterations, then the total number of iterations for small- and large-update methods are bounded by $[80\sqrt{2}(q-1)\widetilde{\Psi}_0^{\frac{q+1}{2(q-1)}}\frac{1}{\theta}\log\frac{n}{\varepsilon}]$ and $[80\sqrt{2}(q-1)\overline{\Psi}_0^{\frac{q+1}{2(q-1)}}\frac{1}{\theta}\log\frac{n}{\varepsilon}]$ respectively.

For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we have$O((q-1)n^{\frac{q+1}{2(q-1)}}\log\frac{n}{\varepsilon})$ iteration complexity. In case of a small-update methods with $\tau = O(1)$, $\theta = \Theta(\frac{1}{\sqrt{n}})$ and $\widetilde{\Psi}_0 = O(q)$, the iteration bound becomes $O((q-1)q^{\frac{q+1}{2(q-1)}}\sqrt{n}\log\frac{n}{\varepsilon})$ iteration complexity.

Remark 4. *If q = any constant, we have $O(\sqrt{n}\log\frac{n}{\varepsilon})$ iteration complexity result for small-update method. Similarly choosing $q = 1 + \log n$, we have $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ iteration complexity for large-update method. These are the best-known complexity results for such methods.*

6 Conclusion

In this paper, we proposed a new kernel function with a logarithmic term. We have shown that the best result of iteration bounds for small- and large-update methods can be achieved, namely $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ for large-update and $O(\sqrt{n}\log\frac{n}{\varepsilon})$ for small-update methods. Future researches might extend this analysis for convex quadratic semidefinite optimization problems, complementarity and conic problems.

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