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Strongly Gorenstein projective, injective and flat modules over formal triangular matrix rings

by

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Abstract

We explicitly describe the structures of strongly Gorenstein projective, injective and flat modules over formal triangular matrix rings.

Key Words: Formal triangular matrix ring, strongly Gorenstein projective module, strongly Gorenstein injective module, strongly Gorenstein flat module.
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1 Introduction

The origin of Gorenstein homological algebra may date back to 1960s when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring [2]. In 1990s, Enochs, Jenda and Torrecillas extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein projective, injective and flat modules over arbitrary rings, and then developed Gorenstein homological algebra [5]. Later, Bennis and Mahdou studied a particular case of Gorenstein projective, injective and flat modules, which they called respectively, strongly Gorenstein projective, injective and flat modules [3]. They proved that every projective (resp. injective, flat) module is strongly Gorenstein projective (resp. injective, flat) and every Gorenstein projective (resp. injective, flat) module is a direct summand of a strongly Gorenstein projective (resp. injective, flat) modules. So strongly Gorenstein projective modules play the role of the free modules in Gorenstein homological algebra.

The main objective of the present paper is to study strongly Gorenstein projective, injective and flat modules over formal triangular matrix rings. Let A and B be rings and U be a (B, A)-bimodule. $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is called a *formal triangular matrix ring* with usual matrix addition and multiplication. This kind of rings have been used for constructing many counterexamples in ring and module theory. Formal triangular matrix rings play an important role in ring theory and the representation theory of algebras. So the properties of formal triangular matrix rings and modules over them have deserved more and more interest. In particular, related with strongly Gorenstein modules, Gao and Zhang [7] determined all finitely generated strongly Gorenstein projective modules over a formal triangular matrix Artin algebra. Wang and Yang [16] determine all finitely generated strongly Gorenstein injective modules over a formal triangular matrix Artin algebra. Zhu, Liu and Wang [18] described strongly Gorenstein projective, injective and flat modules over a formal triangular matrix ring under the “Gorenstein regular” condition.

In this paper, we shall characterize strongly Gorenstein projective, injective and flat modules over a formal triangular matrix ring in a more general setting.

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. $pd(M)$, $id(M)$ and $fd(M)$ denote the projective, injective and flat dimensions of an R-module M respectively. $\mathrm{End}_R(M)$ means the endomorphism ring of an R-module M and the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+. $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ always stands for a formal triangular matrix ring, where A and B are rings and U is a (B, A)-bimodule. Next we recall some notions and facts needed in the sequel.

By [8, Theorem 1.5], the category T-Mod of left T-modules is equivalent to the category Ω whose objects are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\varphi^M : U \otimes_A M_1 \to M_2$ is a B-morphism, and whose morphisms from $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1), f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram

$$\begin{array}{ccc} U \otimes_A M_1 & \xrightarrow{U \otimes_A f_1} & U \otimes_A N_1 \\ {\scriptstyle \varphi^M} \downarrow & & \downarrow {\scriptstyle \varphi^N} \\ M_2 & \xrightarrow{\quad f_2 \quad} & N_2 \end{array}$$

is commutative. Given such a triple $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ in Ω, we shall denote by $\widetilde{\varphi^M}$ the morphism from M_1 to $\mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(x)(u) = \varphi^M(u \otimes x)$ for each $u \in U$ and $x \in M_1$.

Analogously, the category Mod-T of right T-modules is equivalent to the category Γ whose objects are triples $W = (W_1, W_2)_{\varphi_W}$, where $W_1 \in$ Mod-A, $W_2 \in$ Mod-B and $\varphi_W : W_2 \otimes_B U \to W_1$ is an A-morphism, and whose morphisms from $(W_1, W_2)_{\varphi_W}$ to $(X_1, X_2)_{\varphi_X}$ are pairs (g_1, g_2) such that $g_1 \in \mathrm{Hom}_A(W_1, X_1), g_2 \in \mathrm{Hom}_B(W_2, X_2)$ and $\varphi_X(g_2 \otimes_B U) = g_1 \varphi_W$. Given such a triple $W = (W_1, W_2)_{\varphi_W}$ in Γ, we shall denote by $\widetilde{\varphi_W}$ the morphism from W_2 to $\mathrm{Hom}_A(U, W_1)$ given by $\widetilde{\varphi_W}(y)(u) = \varphi_W(y \otimes u)$ for each $u \in U$ and $y \in W_2$.

In the rest of the paper we shall identify T-Mod (resp. Mod-T) with this category Ω (resp. Γ) and, whenever there is no possible confusion, we shall omit the morphism φ^M (resp. φ_W).

Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left T-module and $W = (W_1, W_2)_{\varphi_W}$ be a right T-module. By [12, Proposition 3.6.1], there is an isomorphism of abelian groups

$$W \otimes_T M \cong (W_1 \otimes_A M_1 \oplus W_2 \otimes_B M_2)/H,$$

where the subgroup H is generated by all elements of the form $(\varphi_W(w_2 \otimes u)) \otimes x_1 - w_2 \otimes \varphi^M(u \otimes x_1)$ with $x_1 \in M_1, w_2 \in W_2$ and $u \in U$.

A sequence of left T-modules $0 \to \begin{pmatrix} M_1' \\ M_2' \end{pmatrix}_{\varphi^{M'}} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} M_1'' \\ M_2'' \end{pmatrix}_{\varphi^{M''}} \to 0$ is exact if and only if both sequences $0 \to M_1' \to M_1 \to M_1'' \to 0$ and $0 \to M_2' \to M_2 \to M_2'' \to 0$ are exact.

Recall that the *product category* A-Mod $\times B$-Mod is defined as follows: An object of A-Mod $\times$ B-Mod is a pair (M, N) with $M \in A$-Mod and $N \in B$-Mod, a morphism from (M, N) to (M', N') is a pair (f, g) with $f \in \mathrm{Hom}_A(M, M')$ and $g \in \mathrm{Hom}_B(N, N')$. There are some functors between the category T-Mod and the product category A-Mod $\times$ B-Mod as follows:

(1) $\mathbf{p}$: A-Mod $\times$ B-Mod $\to$ T-Mod is defined as follows: for each object (M_1, M_2) of A-Mod $\times$ B-Mod, let $\mathbf{p}(M_1, M_2) = \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$ with the obvious map and for any morphism (f_1, f_2) in A-Mod $\times$ B-Mod, let $\mathbf{p}(f_1, f_2) = \begin{pmatrix} f_1 \\ (U \otimes_A f_1) \oplus f_2 \end{pmatrix}$.

(2) $\mathbf{h}$: A-Mod $\times$ B-Mod $\to$ T-Mod is defined as follows: for each object (M_1, M_2) of A-Mod $\times$ B-Mod, let $\mathbf{h}(M_1, M_2) = \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$ with the obvious map and for any morphism (f_1, f_2) in A-Mod $\times$ B-Mod, let $h(f_1, f_2) = \begin{pmatrix} f_1 \oplus \mathrm{Hom}_B(U, f_2) \\ f_2 \end{pmatrix}$.

(3) $\mathbf{q}$: T-Mod $\to$ A-Mod $\times$ B-Mod is defined, for each left T-module $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ as $\mathbf{q}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = (M_1, M_2)$, and for each morphism $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in T-Mod as $\mathbf{q}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = (f_1, f_2)$.

It is easy to see that $\mathbf{p}$ is a left adjoint of $\mathbf{q}$ and $\mathbf{h}$ is a right adjoint of $\mathbf{q}$.

2 Strongly Gorenstein projective and injective modules over formal triangular matrix rings

Recall that an exact sequence of projective left R-modules

$$\cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$$

is a *complete projective resolution* [5, 11] if $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever P is a projective left R-module. A left R-module M is said to be *strongly Gorenstein projective* [3] if there is a complete projective resolution

$$\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \ker(f)$. The complete injective resolutions and strongly Gorenstein injective modules are defined dually.

By [10, Theorem 3.1], a left T-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is projective if and only if $\varphi^M : U \otimes_A M_1 \to M_2$ is a monomorphism, M_1 is a projective left A-module and $M_2/\mathrm{im}(\varphi^M)$ is a projective left B-module.

In order to describe explicitly the structure of a strongly Gorenstein projective left T-module, we need the following lemma.

Lemma 1. *Let* $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ *with A and B rings and U be a (B, A)-bimodule.*

1. *If $fd(U_A) < \infty$ and M_1 is a strongly Gorenstein projective left A-module, then* $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}$ *is a strongly Gorenstein projective left T-module.*
2. *If $pd({}_BU) < \infty$ and M_2 is a strongly Gorenstein projective left B-module, then* $\begin{pmatrix} 0 \\ M_2 \end{pmatrix}$ *is a strongly Gorenstein projective left T-module.*

Proof. (1) There is a complete projective resolution in A-Mod

$$\Lambda : \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

with $M_1 \cong \ker(f)$. Since $fd(U_A) < \infty$, the complex $U \otimes_A \Lambda$ is exact by [4, Lemma 2.3]. So we get the exact sequence of projective left T-modules

$$\Upsilon : \cdots \xrightarrow{\begin{pmatrix} f \\ U \otimes_A f \end{pmatrix}} \begin{pmatrix} P \\ U \otimes_A P \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ U \otimes_A f \end{pmatrix}} \begin{pmatrix} P \\ U \otimes_A P \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ U \otimes_A f \end{pmatrix}} \begin{pmatrix} P \\ U \otimes_A P \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ U \otimes_A f \end{pmatrix}} \cdots$$

with $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \cong \ker\begin{pmatrix} f \\ U \otimes_A f \end{pmatrix}$. For any projective left T-module $\begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H}$, H_1 is a projective left A-module. Then the complex $\mathrm{Hom}_T(\Upsilon, \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H}) \cong \mathrm{Hom}_A(\Lambda, H_1)$ is exact by adjointness of functors $\mathbf{p}$ and $\mathbf{q}$. So $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}$ is a strongly Gorenstein projective left T-module.

(2) There is a complete projective resolution in B-Mod

$$\Theta : \cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$$

with $M_2 \cong \ker(g)$. So we get the exact sequence of projective left T-modules

$$\begin{pmatrix} 0 \\ \Theta \end{pmatrix} : \cdots \xrightarrow{\begin{pmatrix} 0 \\ g \end{pmatrix}} \begin{pmatrix} 0 \\ Q \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 \\ g \end{pmatrix}} \begin{pmatrix} 0 \\ Q \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 \\ g \end{pmatrix}} \begin{pmatrix} 0 \\ Q \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 \\ g \end{pmatrix}} \cdots$$

with $\begin{pmatrix} 0 \\ M_2 \end{pmatrix} \cong \ker\begin{pmatrix} 0 \\ g \end{pmatrix}$. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H}$ be a projective left T-module. There is the exact sequence of left B-modules

$$0 \to U \otimes_A H_1 \xrightarrow{\varphi^H} H_2 \to H_2/\mathrm{im}(\varphi^H) \to 0$$

with H_1 and $H_2/\mathrm{im}(\varphi^H)$ projective. Since $pd({}_BU) < \infty$, we have $pd(U \otimes_A H_1) < \infty$ by [15, Exercise 9.20]. Hence $pd(H_2) < \infty$ and so the complex $\mathrm{Hom}_B(\Theta, H_2)$ is exact by [11, Proposition 2.3]. Thus the complex $\mathrm{Hom}_T(\begin{pmatrix} 0 \\ \Theta \end{pmatrix}, \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}_{\varphi^H}) \cong \mathrm{Hom}_B(\Theta, H_2)$ is exact by adjointness of functors $\mathbf{p}$ and $\mathbf{q}$, and so $\begin{pmatrix} 0 \\ M_2 \end{pmatrix}$ is a strongly Gorenstein projective left T-module. ∎

Now we give a characterization of a strongly Gorenstein projective left T-module.

Theorem 1. *Let $fd(U_A) < \infty$ and $pd(_BU) < \infty$. Then $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is a strongly Gorenstein projective left T-module if and only if the following conditions hold:*

1. *M_1 is a strongly Gorenstein projective left A-module, i.e., there exists a complete projective resolution $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ with $M_1 \cong \ker(f)$.*
2. *$\varphi^M : U \otimes_A M_1 \to M_2$ is a monomorphism and $M_2/\mathrm{im}(\varphi^M)$ is a strongly Gorenstein projective left B-module, i.e., there exists a complete projective resolution $\cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$ with $M_2/\mathrm{im}(\varphi^M) \cong \mathrm{im}(g)$.*
3. *There exist $\mu : M_2 \to U \otimes_A P$ and $\nu : Q \to M_2$ such that $\mu\varphi^M = U \otimes_A \iota$, $\rho\nu = \omega$ and $\ker\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix} = \mathrm{im}\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix}$, where $\iota : M_1 \to P$ is the obvious monomorphism, $\omega : Q \to M_2/\mathrm{im}(\varphi^M)$ and $\rho : M_2 \to M_2/\mathrm{im}(\varphi^M)$ are the obvious epimorphisms and $\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix} \in End_B((U \otimes_A P) \oplus Q)$.*

Proof. " $\Rightarrow$ " (1) There is a complete projective resolution in T-Mod

$$\Delta : \cdots \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \cdots$$

with P and Q projective and $M \cong \ker\begin{pmatrix} f \\ h \end{pmatrix}$. So we get the exact sequence

$$\Lambda : \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

of projective left A-modules with $M_1 \cong \ker(f)$.

For any projective left A-module N, there exists the exact sequence in T-Mod

$$0 \to \begin{pmatrix} 0 \\ U \otimes_A N \end{pmatrix} \to \begin{pmatrix} N \\ U \otimes_A N \end{pmatrix} \to \begin{pmatrix} N \\ 0 \end{pmatrix} \to 0,$$

which induces the exact sequence of complexes

$$0 \to \mathrm{Hom}_T(\Delta, \begin{pmatrix} 0 \\ U \otimes_A N \end{pmatrix}) \to \mathrm{Hom}_T(\Delta, \begin{pmatrix} N \\ U \otimes_A N \end{pmatrix}) \to \mathrm{Hom}_T(\Delta, \begin{pmatrix} N \\ 0 \end{pmatrix}) \to 0.$$

Since $\begin{pmatrix} N \\ U \otimes_A N \end{pmatrix}$ is projective, the complex $\mathrm{Hom}_T(\Delta, \begin{pmatrix} N \\ U \otimes_A N \end{pmatrix})$ is exact. Since $pd(_BU) < \infty$, $pd(U \otimes_A N) < \infty$ by [15, Exercise 9.20] and so $pd\begin{pmatrix} 0 \\ U \otimes_A N \end{pmatrix} < \infty$. Thus the complex $\mathrm{Hom}_T(\Delta, \begin{pmatrix} 0 \\ U \otimes_A N \end{pmatrix})$ is exact by [11, Proposition 2.3] and so the complex $\mathrm{Hom}_T(\Delta, \begin{pmatrix} N \\ 0 \end{pmatrix})$

is exact by [15, Theorem 6.3]. By adjointness of functors $\mathbf{p}$ and $\mathbf{q}$, the complex $\mathrm{Hom}_A(\Lambda, N) \cong \mathrm{Hom}_T(\Delta, \begin{pmatrix} N \\ 0 \end{pmatrix})$ is exact. Thus M_1 is a strongly Gorenstein projective left A-module.

(2) Let $\lambda_1 : U \otimes_A P \to (U \otimes_A P) \oplus Q$ and $\lambda_2 : Q \to (U \otimes_A P) \oplus Q$ be the injections, $\pi_1 : (U \otimes_A P) \oplus Q \to U \otimes_A P$ and $\pi_2 : (U \otimes_A P) \oplus Q \to Q$ be the projections.

There exist an epimorphism $\begin{pmatrix} \xi \\ \psi \end{pmatrix} : \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ and a monomorphism $\begin{pmatrix} \iota \\ \delta \end{pmatrix} : \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix}$ such that $\begin{pmatrix} \iota \\ \delta \end{pmatrix}\begin{pmatrix} \xi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$. Then we get the following commutative diagram with exact rows in B-Mod:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U \otimes_A \xi} & & \downarrow{\scriptstyle \psi} & & \vdots{\scriptstyle \omega} & & \\
 & & U \otimes_A M_1 & \xrightarrow{\varphi^M} & M_2 & \xrightarrow{\rho} & M_2/\mathrm{im}(\varphi^M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U \otimes_A \iota} & & \downarrow{\scriptstyle \delta} & & \vdots{\scriptstyle i} & & \\
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0.
\end{array}
$$

Since $fd(U_A) < \infty$, the complex $U \otimes_A \Lambda$ is exact by [4, Lemma 2.3]. Since $U \otimes_A \xi$ is an epimorphism, $U \otimes_A \iota$ is a monomorphism and so $\delta\varphi^M = \lambda_1(U \otimes_A \iota)$ is a monomorphism. Hence φ^M is a monomorphism.

Let $g = i\omega$. Then we get the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U \otimes_A f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U \otimes_A f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Since the first column and the second column above are exact, we get the exact sequence

$$\Xi : \cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$$

of projective left B-modules by [15, Theorem 6.3]. Note that ω is an epimorphism and so i is a monomorphism by the exactness of Ξ. Thus $M_2/\mathrm{im}(\varphi^M) \cong \mathrm{im}(g)$.

Let G be a projective left B-module. Then $\begin{pmatrix} 0 \\ G \end{pmatrix}$ is a projective left T-module, so the complex $\mathrm{Hom}_T(\Delta, \begin{pmatrix} 0 \\ G \end{pmatrix})$ is exact. Thus the complex $\mathrm{Hom}_B(\Xi, G) \cong \mathrm{Hom}_T(\Delta, \begin{pmatrix} 0 \\ G \end{pmatrix})$ is exact by adjointness of functors $\mathbf{p}$ and $\mathbf{q}$. Hence $M_2/\mathrm{im}(\varphi^M)$ is a strongly Gorenstein projective left B-module.

(3) Let $\mu = \pi_1\delta$ and $\nu = \psi\lambda_2$. Then

$$\mu\varphi^M = \pi_1\delta\varphi^M = \pi_1\lambda_1(U \otimes_A \iota) = U \otimes_A \iota$$

and

$$\rho\nu = \rho\psi\lambda_2 = \omega\pi_2\lambda_2 = \omega.$$

Note that $\pi_1 h\lambda_1 = \pi_1\lambda_1(U \otimes_A f) = U \otimes_A f$, $\pi_1 h\lambda_2 = \pi_1\delta\psi\lambda_2 = \mu\nu$, $\pi_2 h\lambda_1 = g\pi_2\lambda_1 = 0$, $\pi_2 h\lambda_2 = g\pi_2\lambda_2 = g$. Thus $h = \begin{pmatrix} \pi_1 h\lambda_1 & \pi_1 h\lambda_2 \\ \pi_2 h\lambda_1 & \pi_2 h\lambda_2 \end{pmatrix} = \begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix}$ and so $\ker\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix} = \mathrm{im}\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix}$.

"$\Leftarrow$" By (1), there is a complete projective resolution in A-Mod

$$\Lambda : \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

with $M_1 \cong \ker(f) = \mathrm{im}(f)$. Write $\iota : M_1 \to P$ to be the obvious monomorphism and $\xi : P \to M_1$ to be the obvious epimorphism such that $f = \iota\xi$.

By (2), φ^M is a monomorphism and there is a complete projective resolution

$$\cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$$

with $M_2/\mathrm{im}(\varphi^M) \cong \mathrm{im}(g) = \ker(g)$. Write $i : M_2/\mathrm{im}(\varphi^M) \to Q$ to be the obvious monomorphism and $\omega : Q \to M_2/\mathrm{im}(\varphi^M)$ to be the obvious epimorphism such that $g = i\omega$.

By (3), there exist $\mu : M_2 \to U \otimes_A P$ and $\nu : Q \to M_2$ such that $\mu\varphi^M = U \otimes_A \iota$, $\rho\nu = \omega$ and $\ker\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix} = \mathrm{im}\begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix}$.

Define $\eta : (U \otimes_A P) \oplus Q \to M_2$ by

$$\eta(x, y) = \varphi^M(U \otimes_A \xi)(x) + \nu(y), x \in U \otimes_A P, y \in Q$$

and define $\theta : M_2 \to (U \otimes_A P) \oplus Q$ by

$$\theta(z) = (\mu(z), i\rho(z)), z \in M_2.$$

Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U \otimes_A \xi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \omega} & & \\
0 & \longrightarrow & U \otimes_A M_1 & \xrightarrow{\varphi^M} & M_2 & \xrightarrow{\rho} & M_2/\mathrm{im}(\varphi^M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U \otimes_A \iota} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & U \otimes_A P & \xrightarrow{\lambda_1} & (U \otimes_A P) \oplus Q & \xrightarrow{\pi_2} & Q & \longrightarrow & 0.
\end{array}$$

Since $U \otimes_A \xi$ and ω are epimorphisms, η is an epimorphism by the diagram above. Since $fd(U_A) < \infty$, the complex $U \otimes_A \Lambda$ is exact by [4, Lemma 2.3]. So $U \otimes_A \iota$ is a monomorphism. Since i is a monomorphism, θ is a monomorphism by the diagram above.

For any $x \in U \otimes_A P$ and $y \in Q$, we have
$\theta\eta(x, y) = (\mu(\varphi^M(U \otimes_A \xi)(x) + \nu(y)), i\rho(\varphi^M(U \otimes_A \xi)(x) + \nu(y)))$
$= ((U \otimes_A \iota)(U \otimes_A \xi)(x) + \mu\nu(y), i\rho\varphi^M(U \otimes_A \xi)(x) + i\omega(y))$
$= ((U \otimes_A f)(x) + \mu\nu(y), g(y)) = \begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Hence $\theta\eta = \begin{pmatrix} U \otimes_A f & \mu\nu \\ 0 & g \end{pmatrix}$. Then $\mathrm{im}(\theta\eta) = \ker(\theta\eta)$ by (3) and so $\mathrm{im}\begin{pmatrix} f \\ \theta\eta \end{pmatrix} = \ker\begin{pmatrix} f \\ \theta\eta \end{pmatrix}$. Thus we get the exact sequence in T-Mod

$$\Upsilon : \cdots \overset{\begin{pmatrix} f \\ \theta\eta \end{pmatrix}}{\to} \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \overset{\begin{pmatrix} f \\ \theta\eta \end{pmatrix}}{\to} \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \overset{\begin{pmatrix} f \\ \theta\eta \end{pmatrix}}{\to} \begin{pmatrix} P \\ (U \otimes_A P) \oplus Q \end{pmatrix} \overset{\begin{pmatrix} f \\ \theta\eta \end{pmatrix}}{\to} \cdots$$

with $\mathrm{im}\begin{pmatrix} f \\ \theta\eta \end{pmatrix} \cong \begin{pmatrix} M_1 \\ \theta(M_2) \end{pmatrix} \cong \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} = M$.

On the other hand, there exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} 0 \\ M_2/\mathrm{im}(\varphi^M) \end{pmatrix} \to 0.$$

For any projective left T-module H, we get the induced exact sequence

$$\mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ M_2/\mathrm{im}(\varphi^M) \end{pmatrix}, H) \to \mathrm{Ext}^1_T(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}, H) \to \mathrm{Ext}^1_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, H).$$

By Lemma 1, $\begin{pmatrix} 0 \\ M_2/\mathrm{im}(\varphi^M) \end{pmatrix}$ and $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}$ are strongly Gorenstein projective. So $\mathrm{Ext}^1_T(\begin{pmatrix} 0 \\ M_2/\mathrm{im}(\varphi^M) \end{pmatrix}, H) = 0$ and $\mathrm{Ext}^1_T(\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, H) = 0$ by [5, Remark 10.2.2]. Thus $\mathrm{Ext}^1_T(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}, H) = 0$ and hence Υ is a complete projective resolution in T-Mod. So M is a strongly Gorenstein projective left T-module. □

By [9, Proposition 5.1] and [1, p.956], $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is an injective left T-module if and only if $\widetilde{\varphi^M} : M_1 \to \mathrm{Hom}_B(U, M_2)$ is an epimorphism, $\ker(\widetilde{\varphi^M})$ is an injective left A-module and M_2 is an injective left B-module. Using this fact, one can obtain the following result by a proof dual to that of Theorem 1.

Theorem 2. *Let $fd(U_A) < \infty$, either $pd({}_BU) < \infty$ or $id({}_BU) < \infty$. Then $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is a strongly Gorenstein injective left T-module if and only if the following conditions hold:*

1. *M_2 is a strongly Gorenstein injective left B-module, i.e., there exists a complete injective resolution* $\cdots \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \cdots$ *with* $M_2 \cong \mathrm{im}(f)$.

2. $\widetilde{\varphi^M} : M_1 \to \mathrm{Hom}_B(U, M_2)$ *is an epimorphism and* $\ker(\widetilde{\varphi^M})$ *is a strongly Gorenstein injective left A-module, i.e., there exists a complete injective resolution* $\cdots \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} I \xrightarrow{g} \cdots$ *with* $\ker(\widetilde{\varphi^M}) \cong \ker(g)$.

3. *There exist* $\mu : \mathrm{Hom}_B(U, E) \to M_1$ *and* $\nu : M_1 \to I$ *such that* $\widetilde{\varphi^M}\mu = \mathrm{Hom}_B(U, \theta)$, $\nu\lambda = i$ *and* $\ker\begin{pmatrix} g & \nu\mu \\ 0 & \mathrm{Hom}_B(U, f) \end{pmatrix} = \mathrm{im}\begin{pmatrix} g & \nu\mu \\ 0 & \mathrm{Hom}_B(U, f) \end{pmatrix}$, *where* $\theta : E \to M_2$ *is the obvious epimorphism,* $i : \ker(\widetilde{\varphi^M}) \to I$ *and* $\lambda : \ker(\widetilde{\varphi^M}) \to M_1$ *are the obvious monomorphisms and* $\begin{pmatrix} g & \nu\mu \\ 0 & \mathrm{Hom}_B(U, f) \end{pmatrix} \in End_A(I \oplus \mathrm{Hom}_B(U, E))$.

As an immediate consequence of Theorems 1 and 2, we have

Corollary 1. *Let R be a ring,* $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ *and* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *be a left $T(R)$-module.*

1. *If M is a strongly Gorenstein projective left $T(R)$-module, then M_1 and $M_2/\mathrm{im}(\varphi^M)$ are strongly Gorenstein projective left R-modules, and φ^M is a monomorphism.*

2. *If M is a strongly Gorenstein injective left $T(R)$-module, then M_2 and $\ker(\widetilde{\varphi^M})$ are strongly Gorenstein injective left R-modules, and $\widetilde{\varphi^M}$ is an epimorphism.*

3 Strongly Gorenstein flat modules over formal triangular matrix rings

Recall that an exact sequence of flat left R-modules

$$\cdots \to F^{-1} \to F^0 \to F^1 \to F^2 \to \cdots$$

is a *complete flat resolution* [11] if $E \otimes_R -$ leaves the sequence exact whenever E is an injective right R-module. A left R-module N is called *strongly Gorenstein flat* [3] if there exists a complete flat resolution

$$\cdots \xrightarrow{h} L \xrightarrow{h} L \xrightarrow{h} L \xrightarrow{h} \cdots$$

with $N \cong \ker(h)$.

Recall that R is a *right coherent ring* [13] if every finitely generated right ideal is finitely presented.

By [6, Proposition 1.14], $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is a flat left T-module if and only if $\varphi^M : U \otimes_A M_1 \to M_2$ is a monomorphism, M_1 is a flat left A-module and $M_2/\mathrm{im}(\varphi^M)$ is a flat left B-module.

Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left T-module. Then the character module M^+ of M may be identified with $(M_1^+, M_2^+)_{\varphi_{M^+}}$ (see [12, p.67]), where $\varphi_{M^+} : M_2^+ \otimes_B U \to M_1^+$ is defined by

$$\varphi_{M^+}(f \otimes u)(x) = f(\varphi^M(u \otimes x))$$

for any $f \in M_2^+$, $u \in U$ and $x \in M_1$.

Next we give a characterization of a strongly Gorenstein flat left T-module.

Theorem 3. *Let T be a right coherent ring, $fd(_BU) < \infty$, either $pd(U_A) < \infty$ or $id(U_A) < \infty$. Then $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is a strongly Gorenstein flat left T-module if and only if the following conditions hold:*

1. *M_1 is a strongly Gorenstein flat left A-module, i.e., there exists a complete flat resolution $\cdots \xrightarrow{f} F_1 \xrightarrow{f} F_1 \xrightarrow{f} F_1 \xrightarrow{f} \cdots$ with $M_1 \cong \ker(f)$.*
2. *$\varphi^M : U \otimes_A M_1 \to M_2$ is a monomorphism and $M_2/\mathrm{im}(\varphi^M)$ is a strongly Gorenstein flat left B-module, i.e., there exists a complete flat resolution $\cdots \xrightarrow{g} G \xrightarrow{g} G \xrightarrow{g} G \xrightarrow{g} \cdots$ with $M_2/\mathrm{im}(\varphi^M) \cong \mathrm{im}(g)$.*
3. *There exist $\mu : \mathrm{Hom}_A(U, F_1^+) \to M_2^+$ and $\nu : M_2^+ \to G^+$ such that $\widetilde{\varphi_{M^+}}\mu = \mathrm{Hom}_A(U, \iota_1^+)$, $\nu\rho^+ = \omega^+$ and $\ker\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U, f^+) \end{pmatrix} = \mathrm{im}\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U, f^+) \end{pmatrix}$, where $\iota_1 : M_1 \to F_1$ is the obvious monomorphism, $\omega : G \to M_2/\mathrm{im}(\varphi^M)$ and $\rho : M_2 \to M_2/\mathrm{im}(\varphi^M)$ are the obvious epimorphisms and $\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U, f^+) \end{pmatrix} \in End_B(G^+ \oplus \mathrm{Hom}_A(U, F_1^+))$.*

Proof. " $\Rightarrow$ " (1) There is a complete flat resolution in T-Mod

$$\Delta : \cdots \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}_{\varphi^F} \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}_{\varphi^F} \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}_{\varphi^F} \xrightarrow{\begin{pmatrix} f \\ h \end{pmatrix}} \cdots$$

with $M \cong \ker\begin{pmatrix} f \\ h \end{pmatrix}$. So we get the exact sequence in A-Mod

$$\Lambda : \cdots \xrightarrow{f} F_1 \xrightarrow{f} F_1 \xrightarrow{f} F_1 \xrightarrow{f} \cdots$$

with F_1 flat and $M_1 \cong \ker(f) = \mathrm{im}(f)$. For any injective right A-module E, the exact sequence of right T-modules

$$0 \to (E, 0) \to (E, \mathrm{Hom}_A(U, E)) \to (0, \mathrm{Hom}_A(U, E)) \to 0$$

induces the exact sequence of complexes

$$0 \to (E, 0) \otimes_T \Delta \to (E, \mathrm{Hom}_A(U, E)) \otimes_T \Delta \to (0, \mathrm{Hom}_A(U, E)) \otimes_T \Delta \to 0.$$

Since $(E, \mathrm{Hom}_A(U,E))$ is an injective right T-module, the complex $(E, \mathrm{Hom}_A(U,E)) \otimes_T \Delta$ is exact. Since $fd(_B U) < \infty$, we have $id(\mathrm{Hom}_A(U,E)) < \infty$ by [14, Lemma 2.2]. So $id(0, \mathrm{Hom}_A(U,E)) < \infty$. Thus the complex $(0, \mathrm{Hom}_A(U,E)) \otimes_T \Delta$ is exact by [14, Lemma 2.1]. Hence the complex $E \otimes_A \Lambda \cong (E,0) \otimes_T \Delta$ is exact by [15, Theorem 6.3]. It follows that M_1 is a strongly Gorenstein flat left A-module.

(2) There exist an epimorphism $\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} : \begin{pmatrix}F_1\\ F_2\end{pmatrix}_{\varphi^F} \to \begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$ and a monomorphism $\begin{pmatrix}\iota_1\\ \iota_2\end{pmatrix} : \begin{pmatrix}M_1\\ M_2\end{pmatrix} \to \begin{pmatrix}F_1\\ F_2\end{pmatrix}$ such that $\begin{pmatrix}\iota_1\\ \iota_2\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}f\\ h\end{pmatrix}$. So we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U\otimes_A F_1 & \xrightarrow{\varphi^F} & F_2 & \xrightarrow{\eta} & F_2/\mathrm{im}(\varphi^F) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U\otimes_A \xi_1} & & \downarrow{\scriptstyle \xi_2} & & \vdots{\scriptstyle \omega} & & \\
 & & U\otimes_A M_1 & \xrightarrow{\varphi^M} & M_2 & \xrightarrow{\rho} & M_2/\mathrm{im}(\varphi^M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U\otimes_A \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \vdots{\scriptstyle i} & & \\
0 & \longrightarrow & U\otimes_A F_1 & \xrightarrow{\varphi^F} & F_2 & \xrightarrow{\eta} & F_2/\mathrm{im}(\varphi^F) & \longrightarrow & 0.
\end{array}$$

Since $pd(U_A) < \infty$ or $id(U_A) < \infty$, the complex $U \otimes_A \Lambda$ is exact by [4, Lemma 2.3] or [14, Lemma 2.1]. Since $U \otimes_A \xi_1$ is an epimorphism, $U \otimes_A \iota_1$ is a monomorphism and so $\iota_2 \varphi^M = \varphi^F (U \otimes_A \iota_1)$ is a monomorphism. Hence φ^M is a monomorphism.

Let $g = i\omega$. Then we get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U\otimes_A F_1 & \xrightarrow{\varphi^F} & F_2 & \xrightarrow{\eta} & F_2/\mathrm{im}(\varphi^F) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U\otimes_A f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & U\otimes_A F_1 & \xrightarrow{\varphi^F} & F_2 & \xrightarrow{\eta} & F_2/\mathrm{im}(\varphi^F) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U\otimes_A f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & U\otimes_A F_1 & \xrightarrow{\varphi^F} & F_2 & \xrightarrow{\eta} & F_2/\mathrm{im}(\varphi^F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since the first column and the second column above are exact, we get the exact sequence

$$\Xi : \cdots \xrightarrow{g} F_2/\mathrm{im}(\varphi^F) \xrightarrow{g} F_2/\mathrm{im}(\varphi^F) \xrightarrow{g} F_2/\mathrm{im}(\varphi^F) \xrightarrow{g} \cdots$$

with $F_2/\mathrm{im}(\varphi^F)$ flat and $M_2/\mathrm{im}(\varphi^M) \cong \ker(g) = \mathrm{im}(g)$.

Let H be an injective right B-module. Then the exact sequence

$$0 \to U \otimes_A F_1 \stackrel{\varphi^F}{\to} F_2 \to F_2/\mathrm{im}(\varphi^F) \to 0$$

induces the exact sequence

$$H \otimes_B U \otimes_A F_1 \stackrel{H\otimes\varphi^F}{\to} H \otimes_B F_2 \to H \otimes_B (F_2/\mathrm{im}(\varphi^F)) \to 0.$$

So we have

$$H \otimes_B (F_2/\mathrm{im}(\varphi^F)) \cong (H \otimes_B F_2)/\mathrm{im}(H \otimes \varphi^F) \cong (0, H) \otimes_T \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}_{\varphi^F}.$$

Since $(0, H)$ is an injective right T-module, the complex $H \otimes_B \Xi \cong (0, H) \otimes_T \Delta$ is exact. It follows that $M_2/\mathrm{im}(\varphi^M)$ is a strongly Gorenstein flat left B-module.

(3) Since $0 \to U \otimes F_1 \stackrel{\varphi^F}{\to} F_2 \stackrel{\eta}{\to} F_2/\mathrm{im}(\varphi^F) \to 0$ is a pure exact sequence, we obtain the split exact sequence

$$0 \to (F_2/\mathrm{im}(\varphi^F))^+ \stackrel{\eta^+}{\to} F_2^+ \stackrel{\widetilde{\varphi_{F^+}}}{\to} \mathrm{Hom}_A(U, F_1^+) \to 0.$$

Let $\lambda_1 : (F_2/\mathrm{im}(\varphi^F))^+ \to (F_2/\mathrm{im}(\varphi^F))^+ \oplus \mathrm{Hom}_A(U, F_1^+)$ and $\lambda_2 : \mathrm{Hom}_A(U, F_1^+) \to (F_2/\mathrm{im}(\varphi^F))^+ \oplus \mathrm{Hom}_A(U, F_1^+)$ be the injections, $\pi_1 : (F_2/\mathrm{im}(\varphi^F))^+ \oplus \mathrm{Hom}_A(U, F_1^+) \to (F_2/\mathrm{im}(\varphi^F))^+$ and $\pi_2 : (F_2/\mathrm{im}(\varphi^F))^+ \oplus \mathrm{Hom}_A(U, F_1^+) \to \mathrm{Hom}_A(U, F_1^+)$ be the projections.

There is an isomorphism $\chi : F_2^+ \to (F_2/\mathrm{im}(\varphi^F))^+ \oplus \mathrm{Hom}_A(U, F_1^+)$ such that $\lambda_1 = \chi\eta^+$ and $\widetilde{\varphi_{F^+}} = \pi_2\chi$. Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (F_2/\mathrm{im}(\varphi^F))^+ & \stackrel{\eta^+}{\longrightarrow} & F_2^+ & \stackrel{\widetilde{\varphi_{F^+}}}{\longrightarrow} & \mathrm{Hom}_A(U, F_1^+) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^+} & & \downarrow{\scriptstyle \iota_2^+} & & \downarrow{\scriptstyle \mathrm{Hom}_A(U,\iota_1^+)} & & \\
0 & \longrightarrow & (M_2/\mathrm{im}(\varphi^M))^+ & \stackrel{\rho^+}{\longrightarrow} & M_2^+ & \stackrel{\widetilde{\varphi_{M^+}}}{\longrightarrow} & \mathrm{Hom}_A(U, M_1^+) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \omega^+} & & \downarrow{\scriptstyle \xi_2^+} & & \downarrow{\scriptstyle \mathrm{Hom}_A(U,\xi_1^+)} & & \\
0 & \longrightarrow & (F_2/\mathrm{im}(\varphi^F))^+ & \stackrel{\eta^+}{\longrightarrow} & F_2^+ & \stackrel{\widetilde{\varphi_{F^+}}}{\longrightarrow} & \mathrm{Hom}_A(U, F_1^+) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \chi} & & \| & & \\
0 & \longrightarrow & (F_2/\mathrm{im}(\varphi^F))^+ & \longrightarrow & (F_2/\mathrm{im}(\varphi^F))^+ \oplus \mathrm{Hom}_A(U, F_1^+) & \longrightarrow & \mathrm{Hom}_A(U, F_1^+) & \longrightarrow & 0.
\end{array}$$

Let $\mu = \iota_2^+\chi^{-1}\lambda_2$ and $\nu = \pi_1\chi\xi_2^+$. Then

$$\widetilde{\varphi_{M^+}}\mu = \widetilde{\varphi_{M^+}}\iota_2^+\chi^{-1}\lambda_2 = \mathrm{Hom}_A(U, \iota_1^+)\widetilde{\varphi_{F^+}}\chi^{-1}\lambda_2 = \mathrm{Hom}_A(U, \iota_1^+)\pi_2\chi\chi^{-1}\lambda_2 = \mathrm{Hom}_A(U, \iota_1^+)$$

and

$$\nu\rho^+ = \pi_1\chi\xi_2^+\rho^+ = \pi_1\chi\eta^+\omega^+ = \pi_1\lambda_1\omega^+ = \omega^+.$$

Since $g\eta = \eta h$, we have $\eta^+g^+ = h^+\eta^+$. So

$$g^+ = \pi_1\lambda_1 g^+ = \pi_1\chi\eta^+g^+ = \pi_1\chi h^+\eta^+.$$

For any $t \in F_2^+$, we have
$\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U,f^+) \end{pmatrix}\chi(t) = \begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U,f^+) \end{pmatrix}\begin{pmatrix} \pi_1\chi(t) \\ \pi_2\chi(t) \end{pmatrix}$
$= (g^+\pi_1\chi(t) + \nu\mu\pi_2\chi(t), \mathrm{Hom}_A(U,f^+)\pi_2\chi(t))$
$= (\pi_1\chi h^+\eta^+\pi_1\chi(t) + \pi_1\chi\xi_2^+\iota_2^+\chi^{-1}\lambda_2\pi_2\chi(t), \mathrm{Hom}_A(U,f^+)\widetilde{\varphi_{F^+}}(t))$
$= (\pi_1\chi h^+\chi^{-1}\lambda_1\pi_1\chi(t) + \pi_1\chi h^+\chi^{-1}\lambda_2\pi_2\chi(t), \widetilde{\varphi_{F^+}}h^+(t))$
$= (\pi_1\chi h^+\chi^{-1}(\lambda_1\pi_1 + \lambda_2\pi_2)\chi(t), \widetilde{\varphi_{F^+}}h^+(t))$
$= (\pi_1\chi h^+(t), \pi_2\chi h^+(t)) = \chi h^+(t)$.
Thus $\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U,f^+) \end{pmatrix}\chi = \chi h^+$.
Since χ is an isomorphism and $\ker(h^+) = \mathrm{im}(h^+)$, we have

$$\ker\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U,f^+) \end{pmatrix} = \mathrm{im}\begin{pmatrix} g^+ & \nu\mu \\ 0 & \mathrm{Hom}_A(U,f^+) \end{pmatrix}.$$

" $\Leftarrow$ " Since there exists a complete flat resolution $\cdots \xrightarrow{f} F_1 \xrightarrow{f} F_1 \xrightarrow{f} F_1 \xrightarrow{f} \cdots$ with $M_1 \cong \ker(f)$ by (1), we get the complete injective resolution

$$\cdots \xrightarrow{f^+} F_1^+ \xrightarrow{f^+} F_1^+ \xrightarrow{f^+} F_1^+ \xrightarrow{f^+} \cdots$$

with $M_1^+ \cong (\ker(f))^+ \cong \ker(f^+) = \mathrm{im}(f^+)$. Thus M_1^+ is a strongly Gorenstein injective right A-module.

By (2), $\varphi^M : U \otimes_A M_1 \to M_2$ is a monomorphism, so $\widetilde{\varphi_{M^+}} : M_2^+ \to \mathrm{Hom}_A(U, M_1^+) \cong (U \otimes_A M_1)^+$ is an epimorphism. Note that

$$\cdots \xrightarrow{g^+} G^+ \xrightarrow{g^+} G^+ \xrightarrow{g^+} G^+ \xrightarrow{g^+} \cdots$$

a complete injective resolution with $(M_2/\mathrm{im}(\varphi^M))^+ \cong (\mathrm{im}(g))^+ \cong \mathrm{im}(g^+) = \ker(g^+)$. Hence $\ker(\widetilde{\varphi_{M^+}}) \cong (M_2/\mathrm{im}(\varphi^M))^+$ is a strongly Gorenstein injective right A-module.

By (3) and the right module version of Theorem 2, $M^+ = (M_1^+, M_2^+)_{\varphi_{M^+}}$ is a strongly Gorenstein injective right T-module. Thus M is a strongly Gorenstein flat left T-module by [17, Theorem 2.4] since T is a right coherent ring. ∎

From the proof of Theorem 3, we have

Corollary 2. *Let R be a ring, $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left $T(R)$-module. If M is a strongly Gorenstein flat left $T(R)$-module, then M_1 and $M_2/\mathrm{im}(\varphi^M)$ are strongly Gorenstein flat left R-modules, and φ^M is a monomorphism.*

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