A new proof for the classification of simple parameterized space curve singularities
by
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Abstract

Let $K$ be an algebraically closed field of characteristic 0. The aim of the article is to give a new proof for the classification of simple parameterized space curve singularities classified by Gibson and Hobbs [6]. The proof is based on explicit computations to be a basis for the corresponding classification in characteristic $p > 0$.

Key Words: Simple singularities, space curves, parametrized curves.
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1 Introduction

The study and classification of singularities have a long history. Very important contributions go back to Zariski [14] and Arnold [1]. Most of the results were obtained over the complex numbers. Greuel and his students started a classification for hypersurface singularities in characteristic $p$ ([2],[7],[8]). Bruce and Gaffney [4] classified the simple parameterized plane curve singularities over the complex numbers. Mehmood and Pfister [12] classified the simple plane curve parametrizations in characteristic $p$. Parametrization of space curve singularities were studied by Gibson and Hobbs in characteristic zero [6]. Their proofs cannot be adapted to characteristic $p$. The aim of this paper is to give another proof of the results of Gibson and Hobbs in characteristic zero.

Let $K$ be an algebraically closed field of characteristic zero. A parametrized space curve singularity is an analytic map $(K,0) \rightarrow (K^3,0)$. Algebraically it is given by a map $f : K[[x,y,z]] \rightarrow K[[t]]$. If $f(x) = x(t)$, $f(y) = y(t)$ and $f(z) = z(t)$ then we write shortly $f = (x(t), y(t), z(t))$. The image of $f$ is the subalgebra $K[[x(t), y(t), z(t)]] \subseteq K[[t]]$ and we will always assume that

$$\dim_K K[[t]]/K[[x(t), y(t), z(t)]] < \infty.$$

The finiteness condition implies that there exist a minimal $c$ such that the ideal, called the conductor ideal, $t^cK[[t]] \subseteq K[[x(t), y(t), z(t)]]$. Two parametrized space curve singularities $f = (x(t), y(t), z(t))$ and $g = (x(t), y(t), z(t))$ are called $A$-equivalent, $f \sim g$, if there exist automorphisms,

$$\psi : K[[t]] \rightarrow K[[t]]$$

$$\varphi : K[[x,y,z]] \rightarrow K[[x,y,z]]$$
such that the following diagram commutes:

\[
\begin{array}{ccc}
K[[x, y, z]] & \xrightarrow{f} & K[[x(t), y(t), z(t)]] \subseteq K[[t]] \\
\varphi \downarrow & & \psi \\
K[[x, y, z]] & \xrightarrow{g} & K[[\hat{x}(t), \hat{y}(t), \hat{z}(t)]] \subseteq K[[t]]
\end{array}
\]

i.e.

\[(x(\psi(t)), y(\psi(t)), z(\psi(t))) = (\varphi_1(\hat{x}(t), \hat{y}(t), \hat{z}(t)), \varphi_2(\hat{x}(t), \hat{y}(t), \hat{z}(t)), \varphi_3(\hat{x}(t), \hat{y}(t), \hat{z}(t))).\]

Given a parametrization \( f = (x(t), y(t), z(t)) \), we define a semigrop as \( \Gamma_f = \{ \text{ord}_i(h) | h \in K[[x(t), y(t), z(t)]] \} \) (or \( \Gamma \) if \( f \) is fixed). If \( t^cK[[t]] \) is the conductor ideal then \( c - 1 \notin \Gamma \) and \( l \in \Gamma \) if \( l \geq c \). The integer \( c \) is called conductor of \( \Gamma \).

**Definition 1.** Let \( f = (x(t), y(t), z(t)) \in tK[[t]]^3 \) define a parametrized space curve singularity. A deformation of \( f \) is a pair \((F, m)\), \( F \in tA[[t]]^3 \) and \( m \subseteq A = K[x_1, ..., x_n]/I \) a maximal ideal, such that \( F \) mod \( mA[[t]]^3 = f \). Since the field \( K \) is algebraically closed a point \( p \in V(I) \subseteq \mathbb{K}^n \) correspond to a maximal ideal \( m_p \subseteq A \) and we will write \( F(p, t) \in K[[t]]^3 \) for \( F \) mod \( m_pA[[t]]^3 \). We will denote the point corresponding to \( m \) by \( o \).

**Definition 2.** Let \( f = (x(t), y(t), z(t)) \in tK[[t]]^3 \) define a parametrized space curve singularity. \( f \) is called simple if for any deformation \((F, m)\) of \( f \), \( F \in tA[[t]]^3 \), \( A = K[x_1, ..., x_n]/I \), there exist a Zariski open subset \( U \) of \( V(I) \subseteq \mathbb{K}^n \) containing \( o \) such that the set \( \{ F(p, t) | p \in U \} \) contains only finitely many \( A \)-equivalent classes.

**Remark 1.** Given parametrizations \((x(t), y(t), z(t)), (\hat{x}(t), \hat{y}(t), \hat{z}(t))\) and assume the \((x(t), y(t), z(t))\) is not simple, if \((x(t), y(t), z(t))\) is \( A \)-equivalent to a parametrization in a deformation of \((\hat{x}(t), \hat{y}(t), \hat{z}(t))\) then \((\hat{x}(t), \hat{y}(t), \hat{z}(t))\) is not simple.

**Theorem 1.** Let \( f \in tK[[t]]^3 \) be a simple parametrized space curve singularity then \( f \) is \( A \)-equivalent to a parametrized space curve singularity in the following table:
Normal Forms

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<table>
<thead>
<tr>
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<tr>
<td>$(t, 0, 0)$</td>
<td>1 ≤ $k$ ≤ $s &lt; 2k - 1$</td>
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<tr>
<td>$(t^2, t^{2k+1}, 0)$</td>
<td>1 ≤ $k$ ≤ $s &lt; 2k$</td>
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<td>$(t^3, t^{3k+1}, 0)$</td>
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<td>$(t^3, t^{3k+1} + t^{3s+2}, 0)$</td>
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<td>$(t^3, t^{3k+2}, 0)$</td>
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<td>$(t^3, t^{3k+2} + t^{3s+1}, 0)$</td>
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<tr>
<td>$(t^3, t^{3k+1} + t^{3r+2}, t^{3s+2})$</td>
<td>1 ≤ $k$ ≤ $s &lt; 2k, r &lt; s$</td>
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<tr>
<td>$(t^3, t^{3k+2}, t^{3s+1})$</td>
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<td>$(t^3, t^{3k+2} + t^{3r+1}, t^{3s+2})$</td>
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<td>$(t^4, t^5, 0)$</td>
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<td>$(t^4, t^5 + t^6)$</td>
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<td>$(t^4, t^5, t^7)$</td>
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<td>$(t^4, t^5, t^{11})$</td>
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<td>$(t^4, t^6 + t^{2k+1}, 0)$</td>
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<td>$(t^4, t^6, t^{2k+1})$</td>
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<td>$(t^4, t^6 + t^{2k-1}, t^{2k+1})$</td>
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<td>$(t^4, t^6 + t^{2k-3}, t^{2k+1})$</td>
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<td>$(t^4, t^6 + t^{2k-7}, t^{2k+1})$</td>
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<td>$(t^4, t^7 + t^{13}, 0)$</td>
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<td>$(t^4, t^7 + t^{13} + t^{17})$</td>
<td>k ≥ 7</td>
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The idea to prove Theorem 1 is the following. We may assume that for a given parametrized space curve singularity $f = (x(t), y(t), z(t))$ that $ord_x(x(t) < ord_y(t) < ord_z(t)$. We will prove that $f$ is not simple if $ord_x(x(t) ≥ 5$ or $ord_x(x(t) = 4$ and $ord_y(t) ≥ 8$ (Corollary 1). For the other cases we give normal forms not depending on parameters and the property $ord_x(x(t) ≤ 4$ and $ord_y(t) ≤ 7$ is kept under deformations. A basis for the classification is the following theorem of Zariski [14].

**Theorem 2.**

1. Given a parametrization $(t^i + \Sigma_{i>k} a_i t^i, t^m + \Sigma_{i>m} b_i t^i, t^n + \Sigma_{i>n} c_i t^i)$ and $k \in \Gamma$ then there exist an equivalent parameterization $(t^i + \Sigma_{i>k} a_i t^i, t^m + \Sigma_{i>m} b_i t^i, t^n + \Sigma_{i>n} c_i t^i)$ with $a_i = a_i, b_i = b_i, c_i = c_i$ if $i < k$ and $a_k = b_k = c_k = 0, a_s = b_s = c_s = 0, for all s \geq c$.

2. Given a parametrization $(t^i, t^m + \Sigma_{i>m} b_i t^i, t^n + \Sigma_{i>n} c_i t^i)$ and an integer $k$ such that
If \( k+l-m \in \Gamma \) then there exist an equivalent parametrization \((t^i, t^m + \Sigma_{l>m} \hat{b}_i t^i, t^n + \Sigma_{i>n} \hat{c}_i t^i)\). such that \( b_i = \hat{b}_i, c_i = \hat{c}_i \) if \( i < k \) and \( \hat{b}_k = 0 \).

**Theorem 3.**

(i) \((t^5, t^6, 0)\) and \((t^5, t^6, t^7)\) are not simple.

(ii) \((t^4, t^9, 0)\) and \((t^4, t^9, t^{10})\) are not simple.

**Proof.** The cases \((t^5, t^6, 0)\) and \((t^4, t^9, 0)\) follow from the classification of plane parametrizations [13].

We will now prove that 

\[
(t^5, t^6 + t^8 + at^9, t^7) \sim (t^5, t^6 + t^8 + bt^9, t^7)
\]

implies \( a = b \) or \( a = -b \) and 

\[
(t^4, t^9 + t^{11}, t^{10} + at^{11}) \sim (t^4, t^9 + t^{11}, t^{10} + bt^{11})
\]

implies \( a = b \) or \( a = -b \).

This will prove the theorem since for different \( a \) the parametrizations are in different classes.

This gives infinitely many different classes since the field is algebraically closed.

To see this we make the following ansatz:

\[
\psi(t) = t + \sum_{i>1} a_i t^i
\]

\[
\varphi(x, y, z) = (\varphi_1, \varphi_2, \varphi_3) ; \varphi_j = \Sigma_{k,l,m} b_{j,k,l,m} x^k y^l z^m
\]

and assume

\[
\psi^5 = \varphi_1(t^5, t^6 + t^8 + at^9, t^7)
\]

\[
\psi^6 + b\psi^9 = \varphi_2(t^5, t^6 + t^8 + at^9, t^7)
\]

\[
\psi^7 = \varphi_3(t^5, t^6 + t^8 + at^9, t^7).
\]

This is the condition for \((t^4, t^9 + t^{11}, t^{10} + at^{11}) \sim (t^4, t^9 + t^{11}, t^{10} + bt^{11})\) according to Definition 2. Writing down this explicitly we see that \( a_2 = ... = a_5 = 0 \) and
\[ \varphi_1 = x + 5a_6x^2 + 5a_7xy + b_{020}y^2 + (5a_8 - b_{020})xz + b_{011}yz + b_{002}z^2 + \text{ terms of order } \geq 3, \]
\[ \varphi_2 = y + 6a_6xy + b_{020}y^2 + (6a_7 - b_{020})xz + b_{011}yz + b_{002}z^2 + \text{ terms of order } \geq 3, \]
\[ \varphi_3 = z + b_{020}y^2 + (7a_6 - b_{020})xz + b_{011}yz + b_{002}z^2 + \text{ terms of order } \geq 3. \]

\[ \psi^5 = \varphi_1(t^5, t^6 + t^8 + at^9, t^7) \mod t^{10} \]
\[ \psi^6 + \psi^8 + b\psi^9 = \varphi_2(t^5, t^6 + t^8 + at^9, t^7) + (b - a)t^9 \mod t^{10} \]
\[ \psi^7 = \varphi_3(t^5, t^6 + at^9, t^7) \mod t^{10} \]

This implies \( a = b \).

If we consider the \( K^* \)-action, i.e. \( \psi(t) = at \) with \( \alpha \neq 0 \).

We obtain
\[ (t^5, t^6 + t^8 + at^9, t^7) \sim (t^5, t^6 + \alpha^2 t^8 + \alpha^2 at^9, t^7). \]

This implies \( \alpha^2 = 1 \) and
\[ (t^5, t^6 + t^8 + at^9, t^7) \sim (t^5, t^6 + t^8 - at^9, t^7). \]

The case \( (t^4, t^9 + t^{11}, t^{10} + at^{11}) \) can be treated similarly.

The computation can be done using SINGULAR.

The corresponding code for \( (t^4, t^9 + t^{11}, t^{10} + t^{11}) \) look as follows:

```plaintext
ing ring R=(0,a,b,c,d,e,f,g,h,i,j,k,n,m,u1,u2,u3,u4,u5,u6,u7,u8,u9,u10,v1,v2,v3,
v4,v5,v6,v7,v8,v9,v10,w1,w2,w3,w4,w5,w6,w7,w8,w9,w10) ,(x,y,z,t) ,ds;
poly phi=t+a*t2+b*t3+c*t4+d*t5+e*t6+f*t7+g*t8+h*t9+i*t10+j*t11+k*t12;
poly Hx=u1*x+u2*y+u3*z+u4*x2+u5*xy+u6*xz+u7*x3;
poly Hy=v1*x+v2*y+v3*z+v4*x2+v5*xy+v6*xz+v7*x3;
poly Hz=w1*x+w2*y+w3*z+w4*x2+w5*xy+w6*xz+w7*x3;
jet(phi^4-subs t(Hx,x,t4,y,t9+t11,z,t10+m*t11) ,11);
jet(phi^9+phi^11-subs t(Hy,x,t4,y,t9+t11,z,t10+m*t11), 11);
jet(phi^10+n*phi^11-subs t(Hz,x,t4,y,t9+t11,z, t10+m*t11), 11);
```

Using Remark 1, we obtain the following Corollary.

**Corollary 1.**
\((x(t), y(t), z(t))\) is not simple if
\( \text{ (i) } 5 \leq \text{ord}_t x(t) \leq \text{ord}_t y(t) \leq \text{ord}_t z(t) \)
\( \text{ (ii) ord}_t x(t) = 4, 8 \leq \text{ord}_t y(t) \leq \text{ord}_t z(t). \)
Let \((\bar{x}(t), \bar{y}(t), \bar{z}(t))\) be a parametrized space curve singularity. Then there exists a parametrization \((x(t), y(t), z(t))\), \(\text{ord}_t x(t) = l\), \(\text{ord}_t y(t) = m\) and \(\text{ord}_t z(t) = n\) such that\(^1\)

- \(l < m < n\)
- \(l\) does not divide \(m\)
- \(n \not< l, m \geq \{ul + vm|u, v \in \mathbb{Z}_{\geq 0}\}\).

**Proof.** We prove the lemma using induction on the orderings. Let \(x(t) = \sum_{i \geq l} a_i t^i, a_l \neq 0, y(t) = \sum_{i \geq m} b_i t^i, b_m \neq 0\) and \(z(t) = \sum_{i \geq n} c_i t^i, c_n \neq 0\).

Permuting \(x(t), y(t), z(t)\) if necessary, we may assume \(l \leq m \leq n\).

- If \(l = m\) then we replace \(y(t)\) by \(y(t) - \frac{b_l}{a_l} x(t)\), then we obtain that \(\text{ord}_t y(t) > l\).

Similarly we arrange \(m < n\).

- We need to show that \(l\) does not divides \(m\). If \(m = l \cdot k\) then similarly we replace \(y(t)\) by \(y(t) - \frac{b_m}{a_m} x(t)^k\) then we obtain that \(\text{ord}_t y(t) > l\).

- Assume \(n < l, m > n\). We need to show that \(l\) does not divides \(m\). If \(m = l \cdot k\) then similarly we replace \(y(t)\) by \(y(t) - \frac{b_m}{a_m} x(t)^k\) then we obtain that \(\text{ord}_t y(t) > l\). The resulting expression has higher order than \(n\).

\[\square\]

**Lemma 2.** Let \((x(t), y(t), z(t))\) be a parametrized space curve singularity with \(\text{ord}_t x(t) = l\). Then there exists an equivalent parametrization \((t^l, \bar{y}(t), \bar{z}(t))\) with \(\text{ord}_t \bar{y} = \text{ord}_t y\) and \(\text{ord}_t \bar{z} = \text{ord}_t z\).

**Proof.** Let \(x(t) = \sum_{i \geq l} a_i t^i; a_l \neq 0\) then \(u(t) := \sum_{i \geq l} a_i t^{i-1}\) is a unit.

By implicit function theorem, there exist a \(\phi: \mathbb{K}[[t]] \to \mathbb{K}[[t]], \phi(\omega(t) \cdot t) = t\). We obtain \((x(t), y(t), z(t)) \sim (t^l, \phi(y(t)), \phi(z(t)))\).

\[\square\]

**Theorem 4.** Let \((x(t), y(t), z(t))\) be a parametrized space curve singularity and \(\text{ord}_t x(t) = 2\). Then for a suitable odd \(k\), \((x(t), y(t), z(t)) \sim (t^2, t^k, 0)\).

**Proof.** We may assume that \(x(t) = t^2\). If \(y(t) \in \mathbb{K}[[t^2]]\) then \((x(t), y(t), z(t))\) is equivalent to \((t^2, z(t), 0)\).

If \(y(t) \notin \mathbb{K}[[t^2]]\), \(y = \sum_{i \geq k} b_i t^i, k\) minimal such that \(k\) odd, \(b_k \neq 0\). We obtain

\[(t^2, y(t), z(t)) \sim (t^2, \sum_{i \geq k} b_i t^i; \sum_{i > k} c_i t^i).\]

Since the conductor of the semigroup is less than or equal to \(k - 1\), we obtain using Zariski’s Theorem that \((t^2, \sum_{i \geq k} b_i t^i, 0) \sim (t^2, t^k, 0)\).

\[\square\]

\(^1\)The case \(n = \infty\), i.e. \(z(t) = 0\) is included. Here \(\text{ord}_t x(t)\) denotes the order of the power series \(x(t)\), i.e. \(x(t) = \text{unit} \cdot t^{\text{ord}_t x(t)}\).
Theorem 5. Let \((x(t), y(t), z(t))\) be a parametrized space curve singularities and \(\text{ord } x(t) = 3\) then \((x(t), y(t), z(t))\) is equivalent to one of the following parametrizations:

1. \((t^3, t^{3k+1}, 0)\)
2. \((t^3, t^{3k+1} + t^{3s+2}, 0) ; 1 \leq k \leq s < 2k - 1\)
3. \((t^3, t^{3k+2}, 0)\)
4. \((t^3, t^{3k+2} + t^{3s+1}, 0) ; 1 \leq k \leq s \leq 2k\)
5. \((t^3, t^{3k+1}, t^{3s+2}) ; 1 \leq k \leq s < 2k - 1\)
6. \((t^3, t^{3k+1} + t^{3r+2}, t^{3s+2}) ; 1 \leq k \leq s < 2k, r < s\)
7. \((t^3, t^{3k+2}, t^{3s+1}) ; 1 \leq k < s \leq 2k\)
8. \((t^3, t^{3k+2} + t^{3r+1}, t^{3s+2}) ; 1 \leq k < s \leq 2k, r < s\)

Proof. We may assume that \(x(t) = t^3\). If \((y(t) \in K[[t^3]])\), we obtain \((t^3, y(t), z(t))\) is equivalent to \((t^3, z(t), 0)\) and using the classification of plane curves, we obtain (1),(2),(3) or (4).

We may assume \(y(t) \not\in K[[t^3]]\), \(y(t) = \sum_{i \geq b} b_i t^i\), \(s\) minimal such that \(3\) does not divides \(s\) and \(s_i \neq 0\). We first consider the case \(s = 3k + 1\). If \((z(t) \in K[[t^3, y(t)]])\), we have \((t^3, y(t), z(t))\) is equivalent to \((t^3, y(t), 0)\). Then we obtain the cases (1) or (2) due to plane curve classification.

If \(z(t) \not\in K[[t^3, y(t)]]\), then we may assume \(\text{ord } z(t) = 3r + 2\). Let \(\Gamma_0 :=< 3, 3k + 1 >= \{0, 3, 6, 9, ..., 3k, 3k + 1, ..., 6k - 2, 6k, 6k + 1, 6k + 2, ...\}\). The conductor of \(\Gamma_0\) is \(6k\). The semigroup is \(\Gamma :=< 3, 3k + 1, 3r + 2 >\). If \(r \geq 2k\) then all the terms of \(z(t)\) are in \(K[[t^3, y(t)]]\).

This problem reduces to the case of plane curves and we have \((t^3, y(t), z(t))\) is equivalent to \((t^3, y(t), 0)\).

If \(k \leq r < 2k + 1\) then we obtain \((t^3, y(t), z(t)) \sim (t^3, y(t), t^{3r+2})\), since for any \(a \geq 3r + 3\), \(t^5 \in K[[t^3, y(t), z(t)]]\). The plane curve classification implies that we obtain one of the cases (5) or (8) since the conductor of \(\Gamma\) is \(3r\) and automorphisms bringing \((t^3, y(t))\) to the normal form may only produce terms above \(t^{3r+2}\).

Now we consider the case \(s = 3k + 2\). If \((z(t) \in K[[t^3, y(t)]])\), we have \((t^3, y(t), z(t))\) is equivalent to \((t^3, y(t), 0)\). Then we obtain the cases (3) or (4) due to plane curve classification.

If \(z(t) \not\in K[[t^3, y(t)]]\), then we may assume \(\text{ord } z(t) = 3r + 2\). Let \(\Gamma_0 :=< 3, 3k + 2 >= \{0, 3, 6, 9, ..., 3k, 3k + 2, ..., 6k - 3, 6k - 2, 6k, 6k + 2, ...\}\). The conductor of \(\Gamma_0\) is \(6k + 2\). The semigroup is \(\Gamma :=< 3, 3k + 2, 3r + 1 >\). If \(r \geq 2k + 1\) then all the terms of \(z(t)\) are in \(K[[t^3, y(t)]]\). This problem reduces to the case of plane curves and we have \((t^3, y(t), z(t))\) is equivalent to \((t^3, y(t), 0)\).

If \(k \leq r \leq 2k\) then we obtain \((t^3, y(t), z(t)) \sim (t^3, y(t), t^{3r+1})\), since for any \(a \geq 3r + 1\), \(t^4 \in K[[t^3, y(t), z(t)]]\). The plane curve classification implies that we obtain one of the cases (6) or (7) since the conductor of \(\Gamma\) is \(3r\) and as above automorphisms bringing \((t^3, y(t))\) to the normal form may only produce terms above \(t^{3r+1}\). 

Theorem 6. Let \((x(t), y(t), z(t))\) and \(\text{ord } x(t) = 4\) and \(\text{ord } y(t) = 5\) then \((x(t), y(t), z(t))\) is equivalent to one of the following parametrizations:

1. \((t^4, t^5, 0)\)
2. \((t^4, t^5 + t^7, 0)\)
3. \((t^4, t^5, t^8)\)
4. \((t^4, t^5, t^7)\)
\[(5) \ (t^4, t^5, t^{11})
\]
\[(6) \ (t^4, t^5 + t^7, t^{11})
\]

**Proof.** Let \( y = \sum_{i \geq 5} b_it^i \) and \( b_5 \neq 0 \). From the classification of plane curve singularities, it follows that

\[ (x(t), y(t)) \sim (t^4, t^5) \text{ or } (t^4, t^5 + t^7). \]

We assume that \( x(t) = t^4, \ y(t) = t^5 + \beta t^7 \) with \( \beta = 0 \) or \( \beta = 1 \) and \( z(t) = 0 \) or \( z(t) = \sum_{i \in \mathbb{C}, \beta > 0} c_it^i = \alpha t^6 + c_7t^7 + c_{11}t^{11} \). First we consider the case \( c_6 \neq 0 \). We obtain as semigroup \( \Gamma = <4, 5, 6, 8, \ldots> \).

So by Zariski’s theorem \((x(t), y(t), z(t)) \sim (t^4, t^5 + \beta t^7, t^6 + \alpha t^7)\). We map \( t \) to \( t - \frac{\alpha}{6}t^2 \), and obtain \( ((t - \frac{\alpha}{6}t^2)^4, (t - \frac{\alpha}{6}t^2)^5 + \beta(t - \frac{\alpha}{6}t^2)^7, t^6) \mod t^8 \).

It is equivalent to \((t^4 + \lambda t^7, t^5 + \gamma t^7, t^6)\). The map \( t \to t - \frac{\gamma}{4}t^3 \) gives \((t^4 + \lambda t^7, t^5, t^6) \mod t^8 \).

Now we perform the map \( t \to t - \frac{6}{4}t^4 \) to obtain \((t^4, t^5, t^6)\).

Assume now \( c_6 = 0 \) and \( c_7 \neq 0 \) then the semigroup is \( \Gamma = <4, 5, 7, \ldots> \).

It implies \((t^4, y(t), z(t)) \sim (t^4, t^5 + \alpha t^6, t^7)\). As before we obtain \((t^4, t^5, t^7)\).

Now we consider the case \( c_6 = c_7 = 0 \) and \( c_{11} \neq 0 \) then the semigroup is \( \Gamma = <4, 5, 11, \ldots> \). So we have

\[ (t^4, y(t), z(t)) \sim (t^4, t^5 + \alpha t^6 + \beta t^7, t^{11}). \]

Then as before, we obtain \((t^4, t^5, t^{11})\) or \((t^4, t^5 + t^7, t^{11})\).

\[ \Box \]

**Theorem 7.** Let \((x(t), y(t), z(t))\) be a parametrization of space curve singularities. Assume that \( \text{ord } z(t) = 4 \), \( \text{ord } y(t) = 6 \) then \((x(t), y(t), z(t))\) is equivalent to one of the following parameterizations:

\[(1) \ (t^4, t^6 + t^{2k+1}, 0) ; k \geq 3 \]
\[(2) \ (t^4, t^6, t^{2k+1}) ; k \geq 3 \]
\[(3) \ (t^4, t^6 + t^{2k-1}, t^{2k+1}) ; k \geq 4 \]
\[(4) \ (t^4, t^6 + t^{2k-3}, t^{2k+1}) ; k \geq 5 \]
\[(5) \ (t^4, t^6 + t^{2k-7}, t^{2k+1}) ; k \geq 7. \]

**Proof.** We may assume \( x(t) = t^4 \).

**Case 1:** \( y(t) \in K[[t^2]] \). We may assume \( \text{ord } z(t) = 2k + 1, k \geq 3 \) and \( y(t) = t^6 \).

Using \( t^4, t^6 \) we can kill the even terms of \( z(t) \) and the terms greater than or equal to \( 2k + 5 \). We obtain \((t^4, t^6, t^{2k+1} + at^{2k+3})\). Using the map \( t \to t - \frac{\alpha}{2k+1}t^3 \), we obtain \((t^4 + \alpha t^6 + \alpha t^8 + \ldots, t^6 + \beta t^8 + \beta t^{10} + \ldots, t^{2k+1} + \gamma t^{2k+4} + \ldots)\).

The higher order terms of \( t^4 + \alpha t^6 + \alpha t^8 + \ldots \) can be killed using \( t^4 + \alpha t^6 + \alpha t^8 + \ldots \) and \( t^6 + \beta t^8 + \beta t^{10} + \ldots \), since all the exponents are even. The same holds for \( t^6 + \beta t^8 + \beta t^{10} + \ldots \). The higher order terms of \( t^{2k+1} + \gamma t^{2k+4} + \ldots \) can be killed using Zariski’s theorem since the conductor of the semigroup is \( 2k + 4 \). This implies that \((x(t), y(t), z(t)) \sim (t^4, t^6, t^{2k+1})\).

**Case 2:** \( y(t) \not\in K[[t^2]] \).

Using the plane curve classification we may assume that \((t^4, y(t))\) is equivalent to \((t^4, t^6 + t^{2l+1})\) such that \((t^4, y(t), z(t)) \sim (t^4, t^6 + t^{2l+1}, \bar{z}(t))\), \( l \geq 3 \), \( \text{ord } z(t) \geq 7 \). Reducing \( \bar{z}(t) \) with \( t^4, t^6 + t^{2l+1} \), we may obtain \((x(t), y(t), z(t)) \sim (t^4, t^6 + t^{2l+1}, 0)\).
Assume now $\tilde{z}(t) = t^{2k+1} + \text{terms of higher order degree }, k \geq 3$.

2.1 If $l \geq k$, we will see that $(x(t), y(t), z(t)) \sim (t^4, t^6, t^{2k+1})$.

If $l = k$ then we can kill the term $t^{2l+1}$ in $y(t)$ using $z(t)$.

If $l = k + 1$ we map $t$ to $t - \frac{1}{6}t^{2k-2}$ to obtain $(t^4 + \alpha_1 t^{2k+1} + \alpha_2 t^{4k-2} + ... , t^6 + \beta_1 t^{4k} + ... , t^{2k+1} + ... )$, since the term $\alpha_1 t^{2k+1}$ of $(t^4 + \alpha_1 t^{2k+1} + ... )$ can be killed using $t^{2k+1} + ...$ and the remaining higher order terms using Zariski’s Theorem (the conductor of the semigroup is $2k+4$). The same holds for $t^6 + \beta_1 t^{4k} + ... , t^{2k+1} + ...$). We obtain that $(x(t), y(t), z(t)) \sim (t^4, t^6, t^{2k+1} + \gamma t^{2k+3})$. As above we can see that this is equivalent to $(t^4, t^6, t^{2k+1})$.

If $l \geq k + 2$, we can reduce $t^6 + t^{2l+1}$ to $t^6$ using $x(t), y(t), z(t)$ since the conductor of the semigroup is $2k + 4$ we obtain $(x(t), y(t), z(t)) \sim (t^4, t^6, t^{2l+1}, 0)$, since $\Gamma =< 4, 6, 2l + 7 >$ has conductor $2l + 10 \leq 2k$ and the terms of $z(t)$ are strictly above the conductor.  

2.2 If $l \leq k - 5$ then $(x(t), y(t), z(t)) \sim (t^4, t^6 + t^{2l+1}, 0)$, since $\Gamma =< 4, 6, 2l + 7 >$ has conductor $2l + 10 \leq 2k$ and the terms of $z(t)$ are strictly above the conductor.

2.3 If $l = k - 3$, we can reduce the case to $l \leq k - 4$, since $2k+1 = 2l + 7 \in \Gamma$, the semigroup of $(t^4, t^6 + t^{2l+1})$. We can reduce $z(t)$ to $\alpha_1 t^{2l+9} + \alpha_2 t^{2l+10} + ...$.

If $\alpha_1 \neq 0$ we are in the case to $l = k - 4$. If $\alpha_1 = 0$ the odd terms of $z(t)$ are strictly above the conductor and we obtain (1).

2.4 The cases $l = k - 4, l = k - 2, l = k - 1$ are the cases (5), (4) and (3). Since the conductor of $\Gamma$ is less than or equal to $2l + 6$, it implies $(x(t), y(t), z(t)) \sim (t^4, t^6 + t^{2l+1}, t^{2k+1} + \alpha t^{2k+3}), a = 0$ if $l < k - 1$. We can use the map $t$ to $t - \frac{a}{2k+1} t^3$ to obtain $(t^4, t^6 + t^{2l+1}, t^{2k+1})$ in the case $l = k - 1$.

\[ \square \]

**Theorem 8.** Given $(x(t), y(t), z(t))$ a parametrization of a space curve and ord $x(t) = 4$ and ord $y(t) = 7$ then $(x(t), y(t), z(t))$ is equivalent to one of the following parameterizations:

1. $(t^4, t^7, 0)$
2. $(t^4, t^7 + t^9, 0)$
3. $(t^4, t^7 + t^{13}, 0)$
4. $(t^4, t^7, t^9)$
5. $(t^4, t^7, t^9 + t^{10})$
6. $(t^4, t^7, t^{10})$
7. $(t^4, t^7, t^9, t^{10})$
8. $(t^4, t^7, t^{13})$
9. $(t^4, t^7 + t^9, t^{13})$
10. $(t^4, t^7, t^{17})$
11. $(t^4, t^7 + t^9, t^{17})$
12. $(t^4, t^7 + t^{13}, t^{17})$

**Proof.** We may assume that $x(t) = t^4$. Let $y = \sum_{i \geq 7} b_i t^i$ and $b_7 \neq 0$. From the classification of plane curve singularities it follows that

$(x(t), y(t)) \sim (t^4, t^7)$ or $(t^4, t^7 + t^9)$ or $(t^4, t^7 + t^{13})$.

We assume that $x(t) = t^4$, $y(t) = t^7 + \beta t^i$ with $\beta = 0$ or $\beta = 1$ and $i = 9$ or $i = 13$ and $z(t) = 0$ or $z(t) = \sum_{i \geq 4, 7 >} c_i t^i = c_9 t^9 + c_{10} t^{10} + c_{13} t^{13} + c_{17} t^{17}$. First we consider the case $c_9 \neq 0$. We obtain as semigroup $\Gamma =< 4, 7, 9 >$, since the conductor of the semigroup is 11. By Zariski’s Theorem, we have $(x(t), y(t), z(t)) \sim (t^4, t^7 + \beta t^{10}, t^9 + \alpha t^{10})$. We map $t$ to $t - \frac{\beta}{7} t^4$ and obtain that $(x(t), y(t), z(t)) \sim (t^4, t^7, t^9 + \alpha t^{10})$. If $\alpha \neq 0$
we can use the map $t$ to $\frac{1}{α}t$ to obtain $(t^4, t^7, t^9 + t^{10})$.

Assume $c_9 = 0$ and $c_{10} \neq 0$, then the semigroup is $Γ =< 4, 7, 10 >= \{0, 4, 7, 8, 10, 12, 14, ...\}$.

We have

$$ (x(t), y(t), z(t)) \sim (t^4, t^7 + αt^9 + α_2t^{13}, t^{10} + αt^{13}). $$

We map $t$ to $t - \frac{α}{10}t^4$ and obtain $(t^4 + a_1t^7 + a_2t^{10} + a_3t^{13} + ..., t^7 + b_1t^9 + b_2t^{10} + b_3t^{13} + b_4t^{13} + ..., t^{10} + c_1t^{15} + ...).$ This is equivalent to $(t^4 - a_1b_1t^9 + (a_2 - a_1b_2)t^{10} + a_1b_3t^{12} + (a_3 - a_1b_4)t^{13} + ..., t^7 + ..., t^{10} + ...).$

Now we use the transformation $t$ to $t + \frac{α}{6}t^6$ to kill the term $-a_1b_1t^9$. Using Zariski’s theorem, we obtain as equivalent parametrization $(t^4 + αt^{13}, t^7 + b_1t^9 + b_2t^{13}, t^{10})$. We map $t$ to $t - \frac{α}{7}t^4$ then $y(t)$ will reduce to $t^7 + αt^{13}$. As above we obtain parametrization $(x(t), y(t), z(t))$ is equivalent to (6) or (7).

Now we consider the next case $c_9 = c_{10} = 0$ and $c_{13} \neq 0$, the semigroup is $Γ =< 4, 7, 13 >= \{0, 4, 7, 8, 11, 13, ...\}$. By Zariski’s Theorem $(x(t), y(t), z(t))$ will be equivalent to the $(t^4, y(t), t^{13})$ where $y(t) = t^7 + αt^9 + βt^{10}$. By mapping $t$ to $t - \frac{β}{7}t^4$, we obtain $(x(t), y(t), z(t)) \sim (t^4, t^7, t^{13})$ or $(t^4, t^7 + t^9, t^{13})$. We consider now the case $c_9 = c_{10} = c_{13} = 0$ and $c_{17} \neq 0$. The semigroup is $Γ =< 4, 7, 17 >= \{0, 4, 7, 8, 11, 12, 14, ...\}$. As before we obtain (10), (11) or (12).

Now we are ready to prove Theorem 1:

**Proof.** We fix the order of $x(t)$ and assume $ord_4x(t) < ord_4y(t) < ord_4z(t)$. If $ord_4x(t) = 1$ is one we obtain $(t, 0, 0)$. If $ord_4x(t) = 2$ we obtain $(t^2, t^k, 0)$, $k \geq 3$, odd. If $ord_4x(t) = 3$ the classification is given by Theorem 4. Theorem 5, 6 and 7 treat the cases $ord_4x(t) = 4$ and $ord_4y(t) = 5$, 6 and 7. From Corollary 1, we know that the classification is completed since parametrized space curve singularities with $5 \leq ord_4x(t)$ respectively $ord_4x(t) = 4$ and $ord_4y(t) \geq 8$ are not simple and the properly $ord_4x(t) \leq 4$, $ord_4y(t) \leq 7$ is kept under deformation.

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