

An irreducibility criterion for polynomials over integers

by

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Abstract

In this article, we consider the polynomials of the form $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$, where $|a_0| = |a_1| + \cdots + |a_n|$ and $|a_0|$ is a prime. We show that these polynomials have a cyclotomic factor whenever reducible. As a consequence, we give a simple procedure for checking the irreducibility of trinomials of this form and separability criterion for certain quadrinomials.

Key Words: Irreducible polynomials, cyclotomic polynomials.

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1 Introduction

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n . The reciprocal of $f(x)$, denoted by $\tilde{f}(x)$, is defined as $\tilde{f}(x) = x^n f(x^{-1})$. A polynomial $f(x)$ is called self-reciprocal or reciprocal if $f(x) = \tilde{f}(x)$, anti-reciprocal if $f(x) = -\tilde{f}(x)$. If $f(x) \neq \pm \tilde{f}(x)$, then $f(x)$ is called non-reciprocal. The roots of reciprocal polynomials lie symmetrically with respect to the unit circle $|z| = 1$. A cyclotomic polynomial is the monic one all of whose roots lie on the unit circle. Therefore, cyclotomic polynomials are reciprocal polynomials.

Polynomials of the form $ax^n \pm b$ have cyclotomic factors if and only if $|a| = |b| \neq 0$. W. Ljunggren [5] extended the work of Selmer [10] and showed that a polynomial of the form $x^n \pm x^m \pm 1$ or $x^n \pm x^m \pm x^r \pm 1$ is either irreducible or divisible by a cyclotomic polynomial. The following result provides a criterion for the irreducibility of polynomials with integer coefficients.

Theorem 1 (L. Panaitopol, D. Ștefănescu [8]). *Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$ be a polynomial such that $|a_0| > |a_1| + \cdots + |a_n|$. If $|a_0|$ is a prime or $\sqrt{|a_0|} - \sqrt{|a_n|} < 1$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.*

In this paper we show that, if a polynomial of the form $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$, where $|a_0| = |a_1| + \cdots + |a_n|$ and $|a_0|$ is a prime is reducible, then it is divisible by a cyclotomic polynomial and has exactly one non-reciprocal factor. The condition that $|a_0|$ is a prime number is necessary. If $|a_0|$ is not a prime number, then $f(x)$ can have any number of non-reciprocal factors. For example,

$$x^4 + 3x^2 + 4 = (x^2 - x + 2)(x^2 + x + 2).$$

Every member of

$$x^{4n} - (a^2 - 1)x^{2n} - a^2 = (x^n + a)(x^n - a)(x^{2n} + 1),$$

where $a \geq 2$ is an integer and $n \geq 1$, has at least two non-reciprocal factors. By use of Capelli's theorem [2] it can be shown that an arbitrary number of non-reciprocal factors can appear for suitable choice of a .

For example, $x^{12} - 728x^6 - 729 = (x-3)(x+3)(x^2-3x+9)(x^2+3x+9)(x^6+1)$.

But if the absolute value of the product of all the roots of a non-trivial factor of $f(x)$ belongs to $(0, 1]$, then one can show that $f(x)$ has a cyclotomic factor even if $|a_0|$ is not a prime number. For example, $9x^9 + 2x^6 + 4x^3 - 15 = (x-1)(x^2+x+1)(9x^6+11x^3+15)$, $2x^9 + 13x^6 + 5x^3 - 20 = (x-1)(x^2+x+1)(2x^6+15x^3+20)$, $2x^3 + 11x^2 + 8x - 21 = (x-1)(x+3)(2x+7)$.

Theorem 2. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$, where $|a_0| = |a_1| + \cdots + |a_n|$. If $g(x) = b_0 + b_1x + \cdots + b_mx^m$ is a proper factor of $f(x)$ with $0 < |b_0| \leq |b_m|$, then $g(x)$ is a product of cyclotomic polynomials.

If $|a_0|$ is a prime number and $f(x)$ is reducible, then not only such a polynomial $g(x)$ defined in Theorem 2 exists, but also $f(x)$ has exactly one non-reciprocal factor. Further all the roots of $f(x)$ are simple zeros. Before proceeding, by removing zero coefficients if any, we can assume that $f(x) = a_{n_r}x^{n_r} + \cdots + a_{n_1}x^{n_1} + a_0$, where $a_0a_{n_1} \cdots a_{n_r} \neq 0$.

Theorem 3. Let $f(x) = a_{n_r}x^{n_r} + \cdots + a_{n_1}x^{n_1} + a_0$, where $|a_{n_1}| + \cdots + |a_{n_r}| = |a_0|$ and $|a_0|$ be a prime number. If f is reducible, then $f(x) = f_c(x)f_n(x)$, where $f_n(x)$ is the irreducible non-reciprocal factor of $f(x)$ and $f_c(x) = \gcd(x^{n_r} + \operatorname{sgn}(a_0a_{n_r}), x^{n_r-1} + \operatorname{sgn}(a_0a_{n_{r-1}}), \dots, x^{n_1} + \operatorname{sgn}(a_0a_{n_1}))$, where $\operatorname{sgn}(x)$ denotes the sign of $x \in \mathbb{R}$.

The previous theorem generates several other irreducibility criteria for different families of polynomials. If n is a positive integer, we denote the *largest even part* of n as $e(n)$, i.e., if $n = 2^\alpha n_1$ with n_1 odd, then $e(n) = 2^\alpha$.

Corollary 4. Suppose $f(x) = a_{n_r}x^{n_r} + \cdots + a_{n_1}x^{n_1} + a_0 \in \mathbb{Z}_+[x]$ is a polynomial with $a_{n_1} + \cdots + a_{n_r} = a_0$ and a_0 is a prime number. Then $f(x)$ is irreducible if and only if there exist distinct i, j such that $e(n_i) \neq e(n_j)$.

Corollary 4 provides an extensive class of irreducible polynomials. The simplest among them is: If p is a prime number and $n_1, n_2, \dots, n_p$ are positive integers with $(n_1, n_2, \dots, n_p) = 1$, then $x^{n_1} + x^{n_2} + \cdots + x^{n_p} + p$ is irreducible.

On the other hand, if we have two consecutive exponents, then the irreducibility criterion for such polynomials becomes easier.

Corollary 5. Suppose $f(x) = a_{n_r}x^{n_r} + a_{n_{r-1}}x^{n_{r-1}} + \cdots + a_0$ is a polynomial with $n_r = n_{r-1} + 1$. Let $|a_{n_1}| + \cdots + |a_{n_r}| = |a_0|$ and $|a_0|$ is a prime number. Then $f(x)$ is reducible if and only if $f(1) = 0$ or $f(-1) = 0$.

The proof of this Corollary directly follows from Theorem 3 along with the fact that $(x^n \pm 1, x^{n-1} \pm 1)$ is either 1 or $x \pm 1$. The condition $n_r = n_{r-1} + 1$ can be replaced by $n_j = n_{j-1} + 1$ for $2 \leq j \leq r$.

In Section 3 we present an explicit irreducibility criterion for trinomials of the form $ax^n + b\epsilon_1x^m + p\epsilon_2$ with $a + b = p$, where $a, b, p \in \mathbb{Z}_+$, p is a prime number and $\epsilon_i \in \{-1, 1\}$. In the end, we determine the separability of quadrinomials of form $x^n + \epsilon_1x^m + \epsilon_2x^r + u\epsilon_3$, $\epsilon_i \in \{-1, 1\}$, $u \geq 1$ over $\mathbb{Q}$, the set of rational numbers.

2 Proofs

In this section, we give proofs for Theorem 3 and Corollary 4. With suitable modifications in the proof of Theorem 3, one can prove Theorem 2. Suppose n, m are positive integers. Then it is known that

$$(x^n - 1, x^m - 1) = x^{(n,m)} - 1.$$

The greatest common divisor for one or two positive signs in the above is as follows:

Lemma 6. *Suppose n, m are positive integers. Then*

1.

$$(x^n + 1, x^m + 1) = \begin{cases} x^{(n,m)} + 1 & \text{if } e(n) = e(m), \\ 1 & \text{otherwise.} \end{cases}$$

2.

$$(x^n + 1, x^m - 1) = \begin{cases} x^{(n,m/2)} + 1 & \text{if } e(m) \geq 2e(n), \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $n = 2^\alpha n_1$ and $m = 2^\beta m_1$ be prime factorizations of n and m respectively. Then the proof can be carried out from the identities: $x^n + 1 = \prod_{d|n_1} \Phi_{2^{\alpha+1}d}(x)$ and $x^m - 1 =$

$\prod_{i=0}^{\beta} \prod_{d|m_1} \Phi_{2^i d}(x)$. We omit the details. $\square$

Proof of Theorem 3: Suppose $f(x) = g(x)h(x)$ is a non-trivial factorization of $f(x)$. Since $|a_0| = |g(0)||h(0)|$ is a prime number, without loss of generality we can assume $|h(0)| = |a_0|$ and $h(x)$ is irreducible. Suppose $f_1(x), f_2(x), \dots, f_l(x)$, are the distinct irreducible factor of $g(x)$. Now we claim that for every i , $f_i(x)$ is a cyclotomic polynomial. It is sufficient to prove that $f_1(x)$ is a cyclotomic polynomial.

It is clear that $f_1(0) = \pm 1$. Let $\deg(f_1(x)) = s$ and $z_1, z_2, \dots, z_s$ be roots of $f_1(x)$. Then

$$\prod_{i=1}^s |z_i| = \frac{1}{|d|}, \quad (1)$$

where d is the leading coefficient of $f_1(x)$. If $|z_i| < 1$ for some i , then

$$|a_0| = |-(a_{n_1} z_i^{n_1} + a_{n_2} z_i^{n_2} + \dots + a_{n_r} z_i^{n_r})| < |a_{n_1}| + |a_{n_2}| + \dots + |a_{n_r}|$$

contradicts the hypothesis. So all of the roots of $f_1(x)$ lies in the region $|z| \geq 1$. But from Equation (1) we have $\prod_i |z_i| \leq 1$. Consequently $d = \pm 1$. From Kronecker's theorem $f_1(x)$ is a cyclotomic polynomial. Next we will prove $\prod_{i=1}^l f_i(x) = \gcd(x^{n_r} + \text{sgn}(a_0 a_{n_r}), x^{n_r-1} + \text{sgn}(a_0 a_{n_{r-1}}), \dots, x^{n_1} + \text{sgn}(a_0 a_{n_1}))$.

Let ζ be a primitive t^{th} root of unity with $f(\zeta) = 0$. Then

$$-a_0 = a_{n_1} \zeta^{n_1} + a_{n_2} \zeta^{n_2} + \dots + a_{n_r} \zeta^{n_r}. \quad (2)$$

Taking modulus on both sides

$$|a_0| = |a_{n_1}\zeta^{n_1} + a_{n_2}\zeta^{n_2} + \cdots + a_{n_r}\zeta^{n_r}| = \sum_{i=1}^r |a_{n_i}|.$$

From triangle inequality, the last two equations hold if and only if the ratio of any two parts is a positive real number. Therefore, $a_{n_r}\zeta^{n_r}/\zeta^{n_i}a_{n_i} = |a_{n_r}\zeta^{n_r}/\zeta^{n_i}a_{n_i}|$ gives $\zeta^{n_r-n_i} = \text{sgn}(a_{n_r}a_{n_i})$ for $1 \leq i \leq r-1$. From (2), we have

$$-a_0 = a_{n_i}\zeta^{n_i} \left[\left| \frac{a_{n_1}}{a_{n_i}} \right| + \cdots + \left| \frac{a_{n_{i-1}}}{a_{n_i}} \right| + 1 + \left| \frac{a_{n_{i+1}}}{a_{n_i}} \right| + \cdots + \left| \frac{a_{n_r}}{a_{n_i}} \right| \right],$$

so that $\zeta^{n_i} = -\text{sgn}(a_0a_{n_i})$ for $1 \leq i \leq r$. From $\zeta^{n_r-n_i}\zeta^{n_i}$, one gets the last equation. Remaining all the equations satisfied by ζ can be drawn from these r equations. Conversely, if ζ satisfy each of the r equations $x^{n_i} + \text{sgn}(a_0a_{n_i}) = 0$, then $f(\zeta) = 0$. It remains to show that $f_c(x) = g(x)$ that is each cyclotomic factor has multiplicity one. If not, suppose ζ satisfies the r equations $x^{n_i} + \text{sgn}(a_0a_{n_i}) = 0$, $1 \leq i \leq r$ and $f(\zeta) = 0, f'(\zeta) = 0$. Then the last relation gives

$$n_r a_{n_r} \zeta^{n_r-1} + \cdots + n_1 a_{n_1} \zeta^{n_1-1} = 0.$$

Using those r equations satisfied by ζ , we will end up with the relation $n_r|a_{n_r}| + \cdots + |a_{n_1}|n_1 = 0$, which is not possible. $\square$

Remark 7. Let $g(x)$ be the product of all cyclotomic factors of $f(x) = a_{n_r}x^{n_r} + \cdots + a_{n_1}x^{n_1} + a_0$, where $|a_{n_1}| + \cdots + |a_{n_r}| = |a_0|$. Then with the proof similar to Theorem 3, it is easy to see that $g(x) = \gcd(x^{n_r} + \text{sgn}(a_0a_{n_r}), x^{n_{r-1}} + \text{sgn}(a_0a_{n_{r-1}}), \dots, x^{n_1} + \text{sgn}(a_0a_{n_1}))$.

Proof of Corollary 4: For any $d \geq 1$ and $g(x) \in \mathbb{Z}[x]$, $g(x)$ has a cyclotomic factor if and only if $g(x^d)$ has a cyclotomic factor. Thus it is sufficient to prove the result for polynomials whose exponents are relatively prime. The result follows from Theorem 3 and Lemma 6. $\square$

3 Applications

We apply the above theory to trinomials of the form $f(x) = ax^n + b\epsilon_1 x^m + p\epsilon_2$, where a, b, p are positive integers, $a + b = p$, p is a prime and $\epsilon_i \in \{-1, 1\}$.

A. Schinzel[9] had found out the cyclotomic factors of trinomials. For the present family of polynomials, we will give the irreducibility criterion in a more simpler form. Similar studies related to trinomials can be found on [10],[11],[1],[6].

From Theorem 3, $f(x)$ is separable over $\mathbb{Q}$. However, by using the discriminant formula for trinomials, one can prove the separability of $f(x)$ over $\mathbb{Q}$, where $|f(0)|$ is prime, in a wider range. For example, it is known that

Theorem 8 (C.R. Greenfield, D. Drucker [3]). *The discriminant of the trinomial $x^n + ax^m + b$ is*

$$D = (-1)^{\binom{n}{2}} b^{m-1} \left[n^{n/d} b^{n-m/d} - (-1)^{n/d} (n-m)^{n-m/d} m^{m/d} a^{n/d} \right]^d,$$

where $d = (n, m)$.

Theorem 9. Let $a, b, p \in \mathbb{N}$, $b \leq p$ and p is a prime number. Then $f(x) = ax^n + b\epsilon_1 x^m + p\epsilon_2$, where $\epsilon_i \in \{-1, 1\}$ is separable over $\mathbb{Q}$.

Proof. From Theorem 8, the discriminant D_f of f is

$$(-1)^{\binom{n}{2}} (p\epsilon_2)^{n-m} a^{n-m-1} \left[n^{n/d} (p\epsilon_2)^{n-m/d} a^{m/d} - (-1)^{n/d} (n-m)^{n-m/d} m^{m/d} (b\epsilon_1)^{n/d} \right]^d,$$

where $d = (n, m)$. Since the polynomial $x^v - \alpha$, $0 \neq \alpha \in \mathbb{C}$, is separable over $\mathbb{Q}$, it is sufficient to consider $d = (n, m) = 1$. $f(x)$ has multiple root if and only if $D_f = 0$, i.e.,

$$n^n (p\epsilon_2)^{n-m} a^m = (-1)^n (n-m)^{n-m} m^m (b\epsilon_1)^n.$$

Let $n \geq 3$. Since $d = 1$, it is not possible to have $b = 1$. Therefore, $b > 1$ and $n|b$, $p|m$ or $p|n-m$. In that case $m, n-m < n \leq b \leq p$, gives a contradiction. If $n = 2$, then $f(x) = ax^2 + b\epsilon_1 x + p\epsilon_2$ has multiple roots if and only if $b^2 = 4ap\epsilon_1$. That is $p|b$ and $p = 4\epsilon_2 a$, contradicting the fact that p is a prime number. $\square$

To determine the cyclotomic factors of trinomials, it can readily be observed that if $\epsilon_1 = 1, \epsilon_2 = -1$, then $f(1) = 0$. In fact $f(x)/x^{(n,m)} - 1$ is an irreducible non-reciprocal polynomial by Theorem 3 and 9. We will consider the remaining cases below.

Theorem 10. Let $f(x) = ax^n + b\epsilon_1 x^m + p\epsilon_2$, where $a, b, p \in \mathbb{N}$, p be a prime number, $\epsilon_i \in \{-1, 1\}$ and $a + b = p$. Then $f(x)$ is reducible if and only if any of the following holds:

1. $\epsilon_1 = -1, \epsilon_2 = 1; e(n) < e(m)$,
2. $\epsilon_1 = \epsilon_2 = -1; e(n) > e(m)$,
3. $\epsilon_1 = \epsilon_2 = 1; e(n) = e(m)$,

and in each of the above cases $f_c(x) = x^{(n,m)} + 1$.

Proof. Suppose $f(x) = ax^n - bx^m + p$ be reducible. From Theorem 3 the cyclotomic factors of $f(x)$ divide the equations $x^n + 1 = 0$ and $x^m - 1 = 0$. From Lemma 6, we have,

$$f_c(x) = (x^n + 1, x^m - 1) = \begin{cases} x^{(n,m/2)} + 1 & \text{if } e(m) \geq 2e(n), \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, by Theorem 3 $f(x)$ is reducible if and only if $e(n) < e(m)$. Also, if $e(n) < e(m)$ then $(n, m/2) = (n, m)$. Conversely, let ζ be a root of $x^d + 1$, where $d = (n, m)$. Then $n = dn', m = dm'$ with $(n', m') = 1$ and $e(n) < e(m)$ implies n' odd, m' even. Therefore,

$$f(\zeta) = a\zeta^n - b\zeta^m + p = a(-1)^{n'} - b(-1)^{m'} + p = 0.$$

From Theorem 3 and Lemma 6, the polynomial $g(x) = ax^{n'} - bx^{m'} + p$ has cyclotomic part $g_c(x) = x + 1$ and hence $f_c(x) = x^{(n,m)} + 1$. In a similar fashion, part 2, 3 can be proved. We omit the details. $\square$

W. Ljunggren[5] has proved that the quadrinomials of the form $g(x) = x^n + \epsilon_1 x^m + \epsilon_2 x^r + \epsilon_3$ can have at most one irreducible non-reciprocal factor apart from its cyclotomic factors. However, Mills [7] proved that Ljunggren's result is true except for four families in which the corresponding quadrinomials factor as a product of two non-reciprocal irreducible polynomials apart from its cyclotomic factors. For example,

$$x^{8k} - x^{7k} - x^k - 1 = (x^{2k} + 1)(x^{3k} - x^k - 1)(x^{3k} - x^{2k} - 1), \text{ for all } k \in \mathbb{N}.$$

Therefore, in either case, determination of the separability of quadrinomials is same as to check separability of its cyclotomic parts. There are quadrinomials which are not separable. For example, $x^4 + x^3 + x + 1 = (x + 1)^2(x^2 - x + 1)$. Jankauskas[4] studied quadrinomials of the form $f(x) = x^n + x^m + x^r + 4$ such that $f(x)$ is irreducible but $f(x^k)$ is reducible for some $k \geq 2$. From his results, it easily follows that this polynomials are separable over $\mathbb{Q}$. However, for arbitrary $u \geq 1$ the separability of $f(x) = x^n + \epsilon_1 x^m + \epsilon_2 x^r + u\epsilon_3$ is still unknown.

Theorem 11. *Let $f(x) = x^n + \epsilon_1 x^m + \epsilon_2 x^r + u\epsilon_3$ be a polynomial of degree n , $\epsilon_i \in \{1, -1\}$ and $u \geq 1$. Then $f(x)$ is separable in the following cases:*

1. $u \geq 2$ and $(r, u) = 1$.
2. $u = 1$ and $f(1)f(-1) \neq 0$.

Proof. Proof of Part 1. Let $u \geq 2$, $(r, u) = 1$ and $h(x) = (f(x), f'(x))$. We required to prove that $h(x) = 1$. On the contrary assume that $h(x)$ is a non-constant polynomial. Since $h(x)$ divides $nf(x) - xf'(x) = (n - m)\epsilon_1 x^m + (n - r)\epsilon_2 x^r + nu\epsilon_3$ and $u \geq 2$, it follows that all the roots of $nf(x) - xf'(x)$ lies outside the unit circle. For if z be a root of $nf(x) - xf'(x)$, where $|z| \leq 1$, then

$$nu = |(n - m)\epsilon_1 z^m + (n - r)\epsilon_2 z^r| \leq 2n - m - r$$

which is a contradiction for $u \geq 2$. Consequently $|h(0)| > 1$. Let $f'(x) = x^{r-1}g(x)$. Since $h(x)|f'(x)$, we have $h(x)|g(x)$. In particular, $h(0)$ divides both $f(0)$ and $g(0)$. This contradicts the fact that $(r, u) = 1$. Therefore $h(x) = 1$.

Proof of Part 2. Let $u = 1$ and $h(x) = (f(x), f'(x))$. If $f(x)$ is irreducible, then $h(x) = 1$. If $f(x)$ is reducible, then $f(x) = f_c(x)f_n(x)$, where $f_c(x)$ is the product of all cyclotomic polynomials which divide $f(x)$ and $f_n(x)$ is the product of non-reciprocal factors of $f(x)$ which is either irreducible or product of two distinct irreducible non-reciprocal factors of $f(x)$ (see [7]). Consequently $f(x)$ is separable if and only if $f_c(x)$ is separable. Thus $h(x)|f_c(x)$. Without loss of generality, we can take $n \leq m + r$. Otherwise one can replace $f(x)$ by $\epsilon_3 x^n f(x^{-1})$.

$n < m + r$: Similar to Part 1 one can show that all the roots of $nf(x) - xf'(x)$ lies in the region $|z| > 1$ which contradicts the fact that $h(x)|f_c(x)$. Hence $h(x) = 1$.

$n = m + r$: Suppose $h(x)$ is non-constant polynomial. Before proceeding, note that for any $a(x) \in \mathbb{Z}[x]$, if $a(0) \neq 0$, then $a(x)$ is separable if and only if $a(x^d)$ is separable for any $d \geq 1$. Therefore, it is sufficient to consider $(m, r) = 1$. Thus from Remark 7, we have either $h(x) = x + 1$ or $h(x) = x - 1$. Hence if $f(x)$ is non-separable, then $f(1)f(-1) = 0$.

□

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References

- [1] A. BREMNER, On trinomials of type $x^n + Ax^m + 1$, *Math. Scand.*, **49**, 145–155 (1981).
- [2] A. CAPELLI, Sulla riduttibilita delle equazioni algebriche, Nota prima, *Red. Accad. Fis. Mat. Soc. Napoli*, **3**, 243–252 (1897).
- [3] C.R. GREENFIELD, D. DRUCKER, On the Discriminant of a Trinomial, *Linear Algebra Appl.*, **62**, 105–112 (1984).
- [4] J. JANKAUSKAS, On the reducibility of certain quadrinomials, *Glas. Mat. Ser. III*, **45** (65), 31–41 (2010).
- [5] W. LJUNGGREN, On the irreducibility of certain trinomials and quadrinomials, *Math. Scand.*, **8**, 65–70 (1960).
- [6] J. MIKUSINSKI, A. SCHINZEL, Sur l a réductibilité de certains trinômes, *Acta Arith.*, **9**, 91–95 (1964).
- [7] W.H. MILLS, The factorization of certain quadrinomials, *Math. Scand.*, **57**, 44–50 (1985).
- [8] L. PANAITOPOL, D. ȘTEFĂNESCU, Some criteria for irreducibility of polynomials, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* (N. S.), **29** (77), 69–74 (1985).
- [9] A. SCHINZEL, On the reducibility of polynomials and in particular of trinomials, *Acta Arith.*, **11**, 1–34 (1965).
- [10] ERNST S. SELMER, On the irreducibility on certain trinomials, *Math.Scand.*, **4**, 287–302 (1956).
- [11] H. TVERBERG, On the irreducibility of the trinomials $x^n \pm x^m \pm 1$, *Math.Scand.*, **8**, 121–126 (1960).

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