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Rings related to $\mathcal{S}$-projective modules

by
Zhanmin Zhu

Abstract

Let R be a ring and $\mathcal{S}$ be a class of some finitely generated left R-modules. Then a left R-module M is called $\mathcal{S}$-projective if for every homomorphism $f : S \to M$, where $S \in \mathcal{S}$, there exist homomorphisms $g : S \to F$ and $h : F \to M$ such that $f = hg$, where F is a free module . In this paper, we give some characterizations of the following four classes of rings: (1). every injective left R-module is $\mathcal{S}$-projective; (2). every left R-module has a monic $\mathcal{S}$-projective preenvelope; (3). every submodule of a projective left R-module is $\mathcal{S}$-projective; (4). every left R-module has an epic $\mathcal{S}$-projective envelope. Some applications are given.

Key Words: $\mathcal{S}$-projective modules, $\mathcal{S}$-Π-coherent rings, $\mathcal{S}$-F rings, $\mathcal{S}$-semihereditary rings.
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1 Introduction

Recall that a ring R is called right coherent if every finitely generated right ideal of R is finitely presented, or equivalently, every finitely generated submodule of a free right R-module is finitely presented. By [4, Theorem 2.1], a ring R is right coherent if and only if any direct product of ${}_RR$ is flat; a left R-module M is called torsionless if M can be embedded into some direct product of ${}_RR$, or equivalently, if the natural map $i : M \to M^{**}$ is monic, where M^* denotes $\mathrm{Hom}_R(M, R)$; a ring R is called *right* Π-*coherent* [2] if every finitely generated torsionless right R-module is finitely presented. Clearly, a right Π-coherent ring is right coherent, so, in [13], Π- coherent rings are also called strongly coherent rings. Π-coherent rings and their generalizations have been studied by a series of authors (see, for example, [2, 5, 8, 9, 13, 14, 15, 16, 21]). It is well known that a ring R is right Π-coherent if and only if any direct product of ${}_RR$ is finitely projective [13] if and only if every finitely generated left R-module has a projective preenvelope.

We recall also that a left R-module M is called *finitely projective (resp., singly projective, simple projective)* [1, 15] if for every epimorphism $f : N \to M$ and any homomorphism $g : C \to M$ with C finitely generated (resp., cyclic, simple) right R-module, there exists $h : C \to N$ such that $g = fh$. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory for R-Mod. Then according to [8], a left R-module M is called τ-*finitely generated* (or τ-*FG* for short) if there exists a finitely generated submodule N such that $M/N \in \mathcal{T}$; a left R-module A is called τ-*finitely presented* (or τ-*FP* for short) if there exists an exact sequence of left R-modules $0 \to K \to F \to A \to 0$ with F finitely generated free and K τ-finitely generated;

a left R-module M is called τ-flat if every homomorphism from a τ-finitely presented left R-module A to M factors through a finitely generated free left R-module F, that is, there exist $g : A \to F$ and $h : F \to M$ such that $f = hg$. It is easy to see that a left R-module M is τ-flat if and only if for every epimorphism $f : N \to M$ and any homomorphism $g : C \to M$ with C τ-finitely presented left R-module, there exists $h : C \to N$ such that $g = fh$.

In 2011, Parra and Rada extended the concept of finitely projective modules and introduced the concept of $\mathcal{S}$-*projective* modules. Let $\mathcal{S}$ be a class of some finitely generated left R-modules. Then according to [19], a left R-module M is called $\mathcal{S}$-projective if for every homomorphism $f : S \to M$, there exist homomorphisms $g : S \to F$ and $h : F \to M$ such that $f = hg$, where F is a free module . Moreover, in [19, Theorem 3.1], they proved that any direct product of copies of $_RR$ is $\mathcal{S}$-projective if and only if every left R-module has an $\mathcal{S}$-projective preenvelope. For convenience, we call this class of rings $\mathcal{S}$-Π-*coherent*. Let $\mathcal{S}$ be the class of all finitely generated (resp., finitely presented) left R-modules, then it is easy to see that R is $\mathcal{S}$-Π-coherent if and only if R is right Π-coherent (resp., right coherent).

In section 2, we give a series of examples, characterizations and properties of $\mathcal{S}$-projective modules.

In Section 3, we call a ring R $\mathcal{S}$-*F ring* if every $S \in \mathcal{S}$ embeds in a free module. $\mathcal{S}$-F rings are characterized by $\mathcal{S}$-projective modules. As corollaries, characterizations of left FGF rings and left CF rings as well as left IF rings are given. Furthermore, $\mathcal{S}$-Π-coherent $\mathcal{S}$-F rings are investigated. It is shown that R is an $\mathcal{S}$-Π-coherent and $\mathcal{S}$-F ring if and only if every left R-module has a monic $\mathcal{S}$-projective preenvelope.

In Section 4, we call a ring R $\mathcal{S}$-semihereditary if, for any $S \in \mathcal{S}$ and any projective module P, every homomorphic image of S to P is $\mathcal{S}$-projective. It is shown that a ring R is $\mathcal{S}$-semihereditary if and only if every submodule of a projective left R-module is $\mathcal{S}$-projective. $\mathcal{S}$-Π-coherent $\mathcal{S}$-semihereditary rings are characterized , it is shown that a ring R is $\mathcal{S}$-Π-coherent and $\mathcal{S}$-semihereditary if and only if every left R-module has an epic $\mathcal{S}$-projective envelope.

Throughout this paper, R is an associative ring with identity , all modules considered are unitary, $\mathcal{S}$ is a class of some finitely generated left R-modules, R-Mod denotes the class of all left R-modules.

2 $\mathcal{S}$-projective modules

Let P and M be left R-modules. There is a natural homomorphism

$$\sigma_{P,M} : \mathrm{Hom}_R(P, R) \otimes M \to \mathrm{Hom}_R(P, M)$$

defined via $\sigma_{P,M}(f \otimes m)(p) = f(p)m$ for $f \in \mathrm{Hom}_R(P, R), m \in M, p \in P$.

Theorem 1. *Let $\mathcal{S}$ be a class of some finitely generated left R-modules. Then the following statements are equivalent for a left R-module M:*

(1) M is $\mathcal{S}$-projective.

(2) For any $S \in \mathcal{S}$ and any homomorphism $f : S \to M, f$ factors through a finitely generated free left R-module F, that is, there exist $g : S \to F$ and $h : F \to M$ such that $f = hg$.

(3) For any $S \in \mathcal{S}$ and any homomorphism $f : S \to M, f$ factors through a projective module P.

(4) For every epimorphism $f : N \to M$ and any homomorphism $g : S \to M$ with $S \in \mathcal{S}$, there exists $h : S \to N$ such that $g = fh$.

(5) For any $S \in \mathcal{S}, \sigma_{S,M}$ is an epimorphism.

(6) For any $S \in \mathcal{S}$ and any homomorphism $f : S \to M, f$ factors through an $\mathcal{S}$-projective module P.

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6). It is obvious.

(3) $\Rightarrow$ (1). Let $f : S \to M$ be any homomorphism, where $S \in \mathcal{S}$. Then by (3), there exist a projective left R-module P, a homomorphism $\alpha : S \to P$ and a homomorphism $\beta : P \to M$ such that $f = \beta\alpha$. Let $\pi : F \to P$ be an epimorphism , where F is free. Since P is projective, there exists a homomorphism $g : S \to F$ such that $\alpha = \pi g$. Now write $h = \beta\pi$. Then $h \in \mathrm{Hom}_R(F, M)$ and $f = hg$.

(2) $\Rightarrow$ (4). Let $f : N \to M$ be an epimorphism and $g : S \to M$ be any homomorphism, where $S \in \mathcal{S}$. By (2), g factors through a finitely generated free left R-module F, i.e., there exist $\varphi : S \to F$ and $\psi : F \to M$ such that $g = \psi\varphi$. Since F is projective, there exists a homomorphism $\theta : F \to N$ such that $\psi = f\theta$. Now write $h = \theta\varphi$, then h is a homomorphism from S to N, and $g = \psi\varphi = f(\theta\varphi) = fh$. And so (4) follows.

(4) $\Rightarrow$ (2). Let F_1 be a free module and $\pi : F_1 \to M$ be an epimorphism. By (4), there exists a homomorphism $g : S \to F_1$ such that $f = \pi g$. Note that $\mathrm{Im}(g)$ is finitely generated , so there is a finitely generated free module F such that $\mathrm{Im}(g) \subseteq F \subseteq F_1$. Let $\iota : F \to F_1$ be the inclusion map and $h = \pi\iota$. Then h is a homomorphism from F to M and $f = hg$.

(2) $\Rightarrow$ (5). Let $S \in \mathcal{S}$ and $f \in \mathrm{Hom}(S, M)$. Then by (2), there exists a positive integer n, two homomorphisms $g : S \to R^n$ and $h : R^n \to M$ such that $f = hg$. Let $\pi_i : R^n \to R$ be the ith projection and $\iota_i : R \to R^n$ be the ith injection, $g_i = \pi_i g, m_i = h(\iota_i(1))$. Then $g_i \in S^*$ and $m_i \in M, i = 1, 2, \cdots, n$. Write $e_i = \iota_i(1), i = 1, 2, \cdots, n$. Then for any $s \in S$, let $g(s) = (r_1, r_2, \cdots, r_n)$. We have $\sigma_{S,M}(\sum_{i=1}^{n} g_i \otimes m_i)(s) = \sum_{i=1}^{n} g_i(s)m_i = \sum_{i=1}^{n}(\pi_i g(s))m_i = \sum_{i=1}^{n} \pi_i g(s)h(\iota_i(1)) = \sum_{i=1}^{n} r_i h(e_i) = \sum_{i=1}^{n} h(r_i e_i) = h(\sum_{i=1}^{n} r_i e_i) = hg(s) = f(s)$. Thus, $f = \sigma_{S,M}(\sum_{i=1}^{n} g_i \otimes m_i)$, and so $\sigma_{S,M}$ is an epimorphism.

(5) $\Rightarrow$ (2). Let $S \in \mathcal{S}$ and $f \in \mathrm{Hom}(S, M)$. By (5), $f = \sigma_{S,M}(\sum_{i=1}^{n} f_i \otimes m_i), f_i \in S^*, m_i \in M$, so $f(s) = \sum_{i=1}^{n} f_i(s)m_i$ for each $s \in S$. Define $g : S \to R^n$ by $g(s) = (f_1(s), \cdots, f_n(s)), h : R^n \to M$ by $h(r_1, \cdots, r_n) = \sum_{i=1}^{n} r_i m_i$. Then $f = hg$ and (2) follows.

(6) $\Rightarrow$ (1). Let $f : S \to M$ be any homomorphism, where $S \in \mathcal{S}$. Then by (6), there exist an $\mathcal{S}$-projective left R-module P, a homomorphism $\alpha : S \to P$ and a homomorphism $\beta : P \to M$ such that $f = \beta\alpha$. Let $\pi : F \to P$ be an epimorphism , where F is free. Since P is $\mathcal{S}$-projective, by the equivalence of (1) and (4), there exists a homomorphism $g : S \to F$ such that $\alpha = \pi g$. Now write $h = \beta\pi$. Then $h \in \mathrm{Hom}_R(F, M)$ and $f = hg$. ∎

Example 1. *Let $\mathcal{S}$ be the class of all finitely generated (resp., cyclic , simple, finitely presented, τ-finitely presented) left R-modules. Then by Theorem 1, a left R-module M is $\mathcal{S}$-projective if and only if M is finitely projective (resp., singly projective, simple projective, flat, τ-flat).*

Let m and n be two fixed positive integers. Then according to [22], a right R-module A is called (m,n)*-presented* in case there exists an exact sequence of left R-modules $0 \to K \to R^m \to M \to 0$, where K is n-generated; a left R-module M is called (m,n)*-flat* [22] in case $i_I \otimes 1_M : I \otimes M \to R^m \otimes M$ is a monomorphism for all n-generated submodule I of the right R-module R^m. $(1,1)$-flat modules are also call *1-flat* [20] or *P-flat* [6]. A left R-module M is called *min-flat* [14] if $\mathrm{Tor}_R^1(R/I, M) = 0$ for every minimal left ideal I.

Example 2. *(1). Let $\mathcal{S}$ be the class of all (n,m)-presented left R-modules. Then by [22, Theorem 4.3(6)] and Theorem 1, a left R-module M is $\mathcal{S}$-projective if and only if M is (m,n)-flat. In particular, let $\mathcal{S}$ be the class of all (1,1)-presented left R-modules. Then a left R-module M is $\mathcal{S}$-projective if and only if M is P-flat.*

(2). Let $\mathcal{S} = \{R/Ra : aR$ be a minimal right ideal$\}$. Then by [14, Lemma 3.2] and Theorem 1, a left R-module M is $\mathcal{S}$-projective if and only if M is min-flat.

Remark 1. *As corollaries of Theorem 1, we can obtain a series of characterizations of finitely (resp., singly, simple) projective modules and flat (resp., τ-flat, (m,n)-flat, P-flat, min-flat) modules.*

Let $\mathcal{A}$ be a class of left R-modules and M a left R-module. Following [10], we say that a homomorphism $\varphi : M \to A$ where $A \in \mathcal{A}$ is an $\mathcal{A}$-preenvelope of M if for any morphism $f : M \to A'$ with $A' \in \mathcal{A}$, there is a $g : A \to A'$ such that $g\varphi = f$. An $\mathcal{A}$-preenvelope $\varphi : M \to A$ is said to be an $\mathcal{A}$-envelope if every endomorphism $g : A \to A$ such that $g\varphi = \varphi$ is an isomorphism. It is easy to see that an epic $\mathcal{A}$-preenvelope is an $\mathcal{A}$-envelope.

Proposition 1. *Let $S \in \mathcal{S}$. Then*

(1) Every finitely projective preenvelope of S is an $\mathcal{S}$-projective preenvelope of S.

(2) Every projective preenvelope of S is a finitely projective preenvelope of S.

Proof. (1). Let $f : S \to P$ be a finitely projective preenvelope of S. Then P is clearly $\mathcal{S}$-projective. And for any $\mathcal{S}$-projective left R-module P' and any homomorphism $g : S \to P'$, by Theorem 1, g factors through a finitely generated free left R-module F, that is, there exist $\alpha : S \to F$ and $\beta : F \to P'$ such that $g = \beta\alpha$. Since $f : S \to P$ is a finitely projective preenvelope of S, there exists a homomorphism $\gamma : P \to F$ such that $\alpha = \gamma f$. Now let $h = \beta\gamma$. Then $g = hf$. So f is an $\mathcal{S}$-projective preenvelope of S.

(2). It is similar to the proof of (1). □

Corollary 1. *(1). If M is a finitely generated left R-module, then every projective preenvelope of M is a finitely projective preenvelope of M.*

(2). If M is a cyclic (resp., simple, finitely presented, (n,m)-presented, (1,1)-presented , τ-finitely presented) left R-module, then every finitely projective preenvelope of M is a singly projective (resp., simple projective, flat, (m,n)-flat, P-flat , τ-flat) preenvelope of M.

(3). If $M = R/Ra$, where aR is a minimal right ideal, then every finitely projective preenvelope of M is a min-flat preenvelope of M.

3 $\mathcal{S}$-F-rings

Recall that a ring R is called *left CF* [18] if every cyclic left R-module embeds in a free module; a ring R is called *left FGF* [9] if every finitely generated left R-module embeds in a free module; a ring R is called a *left IF ring* [11] if every injective left R-module is flat, by [7, Theorem 1], a ring R is a left IF ring if and only if every finitely presented left R-module embeds in a free module; a ring R is called *left Kasch* [18] if every simple left R-module embeds in R, or equivalently, if every simple left R-module embeds in a free module; A ring R is called a *left IF-(m,n) ring* [23] if every injective left R-module is (m,n)-flat, by [23, Theorem 2.5], a ring R is a left IF-(m,n) ring if and only if every (n,m)-presented left R-module embeds in a free module. Now we extend these concepts as follows.

Definition 1. *Let $\mathcal{S}$ be a class of some finitely generated left R-modules. Then R is called an $\mathcal{S}$-F ring if every $S \in \mathcal{S}$ embeds in a free module.*

Example 3. *Let $\mathcal{S}$ be the class of all finitely generated (resp., cyclic , finitely presented, (n,m)-presented) left R-modules. Then R is an $\mathcal{S}$-F-ring if and only if R is left FGF (resp., CF, IF, IF-(m,n)).*

Theorem 2. *The following statements are equivalent for a ring R:*

(1) R is an $\mathcal{S}$-F ring.

(2) Every injective left R-module is $\mathcal{S}$-projective.

(3) The injective envelope of any $S \in \mathcal{S}$ is $\mathcal{S}$-projective.

(4) For every free right R-module F, $F^+ = \mathrm{Hom}_R(F, \mathbb{Q}/\mathbb{Z})$ is $\mathcal{S}$-projective.

Proof. (1) $\Rightarrow$ (2) . Let E be an injective left R-module. Then for every epimorphism $f : N \to E$ and any homomorphism $g : S \to E$ with $S \in \mathcal{S}$. By (1), there exists a free module F and a monomorphism $\iota : S \to F$. So there exists a $h : F \to E$ such that $g = h\iota$, and hence there exists a $\varphi : F \to N$ such that $h = f\varphi$. Thus, $\varphi\iota$ is a homomorphism from S to N and $g = f(\varphi\iota)$. Therefore, E is $\mathcal{S}$-projective.

(2) $\Rightarrow$ (3) . It is clear.

(3) $\Rightarrow$ (1) . Let $S \in \mathcal{S}$, $\iota : S \to E(S)$ be the inclusion map and $\pi : F \to E(S)$ be an epimorphism, where F is a free module. Then since $E(S)$ is $\mathcal{S}$-projective, there exists a homomorphism $f : S \to F$ such that $\iota = \pi f$. It is easy to see that f is monic, and so (1) follows.

(2) $\Rightarrow$ (4) . Since F is a free right R-module, F^+ is injective and hence $\mathcal{S}$-projective by (2).

(4) $\Rightarrow$ (2) . Let E be an injective left R-module. There is a free right R-module F and an epimorphism $F \to E^+$, which gives a monomorphism $E^{++} \to F^+$. Since F^+ is $\mathcal{S}$-projective and $E \subseteq E^{++}$, E is a direct summand of F^+ and hence E is $\mathcal{S}$-projective by [19, Proposition 2.5]. □

Corollary 2. *The following statements are equivalent for a ring R:*

(1) R is a left CF (resp., FGF, Kasch) ring.

(2) Every injective left R-module is singly (resp., finitely, simple) projective.

(3) The injective envelope of any cyclic (resp., finitely generated , simple) left R-module is singly (resp., finitely, simple) projective.

(4) For every free right R-module F, F^+ is singly (resp., finitely, simple) projective.

Corollary 3. *The following statements are equivalent for a ring R:*

(1) R is a left IF (resp., IF-(m,n)) ring.

(2) Every injective left R-module is flat (resp., (m,n)-flat).

(3) The injective envelope of any finitely presented (resp., (n,m)-presented) left R-module is flat (resp., (m,n)-flat).

(4) For every free right R-module F, F^+ is flat(resp., (m,n)-flat).

We note that part of the results in Corollary 2 and Corollary 3 appeared in [13, Theorem 2.10], [16, Lemma 3.6], [7, Theorem 1] and [23, Theorem 2.5], respectively.

Recall that a ring R is called *right AFG* [16] if any direct product of copies of $_RR$ is singly projective.

Example 4. *Let $\mathcal{S}$ be the class of all finitely generated (resp., finitely presented, (n,m)-presented, (1,1)-presented, τ-finitely presented , cyclic) left R-modules. Then R is $\mathcal{S}$-Π-coherent if and only if R is right Π-coherent (resp., coherent, (m,n)-cohernt, P-coherent, τ-coherent, AFG).*

Theorem 3. *The following statements are equivalent for ring R:*

(1) R is an $\mathcal{S}$-Π-coherent $\mathcal{S}$-F ring.

(2) Every left R-module has a monic $\mathcal{S}$-projective preenvelope.

(3) Every $S \in \mathcal{S}$ has a monic $\mathcal{S}$-projective preenvelope.

(4) Every $S \in \mathcal{S}$ has a monic projective preenvelope.

(5) Every $S \in \mathcal{S}$ has a monic finitely projective preenvelope.

Proof. (1) $\Rightarrow$ (2). Let M be any left R-module. Since R is $\mathcal{S}$-Π-coherent, M has an $\mathcal{S}$-projective preenvelope $f : M \to P$. Since R is an $\mathcal{S}$-F ring, by Theorem 2, $E(M)$ is $\mathcal{S}$-projective. Let $\iota : M \to E(M)$ be the inclusion map. Then there exists a homomorphism $g : P \to E(M)$ such that $\iota = gf$, and hence f is monic.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Let $S \in \mathcal{S}$. Then S has a monic $\mathcal{S}$-projective preenvelope $f : S \to P$ by (3). Thus, by Theorem 1(2), there exist a finitely generated free left R-module F, a monomorphism $g : S \to F$ and a homomorphism $h : F \to P$ such that $f = hg$. Now let P' be a projective left R-module and φ be a homomorphism from S to P'. Then there exists a homomorphism $\theta : P \to P'$ such that $\varphi = \theta f$. Thus, θh is a homomorphism from F to P' and $\varphi = (\theta h)g$. Therefore, $g : S \to F$ is a monic projective preenvelope of S.

(4) $\Rightarrow$ (1). Assume (4) holds. Then it is easy to see that R is an $\mathcal{S}$-F ring. Moreover, by [19, Theorem 3.1], R is $\mathcal{S}$-Π-coherent.

(4) $\Rightarrow$ (5) $\Rightarrow$ (3). It follows from Proposition 1. □

Corollary 4. *[16, Theorem 3.7] The following are equivalent for ring R:*

(1) R is a right AFG left CF ring.

(2) Every left R-module has a monic singly projective preenvelope.

(3) Every cyclic left R-module has a monic singly projective preenvelope.

(4) Every cyclic left R-module has a monic projective preenvelope.

(5) Every cyclic left R-module has a monic finitely projective preenvelope.

Corollary 5. *[8, Theorem 4.1] Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory for R-Mod. Then the following are equivalent:*

(1) R is right τ-coherent and every injective left R-module is τ-flat.
(2) Every left R-module has a monic τ-flat preenvelope.
(3) Every τ-finitely presented left R-module has a monic τ-flat preenvelope.
(4) Every τ-finitely presented left R-module has a monic projective preenvelope.
(5) Every τ-finitely presented left R-module has a monic finitely projective preenvelope.

4 $\mathcal{S}$-semihereditary rings

Recall that a ring R is called *left semihereditary* ([3], p.14) if every finitely generated left ideal of R is projective; a ring R is called *left PP* [12] if every principal left ideal of R is projective. In this section, we generalize the concepts of left semihereditary rings and left PP rings, and we introduce the concept of $\mathcal{S}$-semihereditary rings.

Definition 2. *Let $\mathcal{S}$ be a class of some finitely generated left R-modules. Then R is called $\mathcal{S}$-semihereditary if, for any $S \in \mathcal{S}$ and any projective module P, every homomorphic image of S to P is $\mathcal{S}$-projective.*

Recall that a ring R is called *left PS* [17] if every minimal left ideal of R is projective. It is easy to see that a ring R is left PS if and only if every simple submodule of a projective left R-module is projective. Observing that every $\mathcal{S}$-projective module in $\mathcal{S}$ is projective,we have

Example 5. *Let $\mathcal{S}$ be the class of all finitely generated (resp., all cyclic , all simple) left R-modules. Then R is $\mathcal{S}$-semihereditary if and only if R is left semihereditary(resp., left PP, left PS).*

Theorem 4. *Let $\mathcal{S}$ be a class of some finitely generated left R-modules. Then the following statements are equivalent:*

(1) R is $\mathcal{S}$-semihereditary.
(2) Every submodule of an $\mathcal{S}$-projective left R-module is $\mathcal{S}$-projective.
(3) Every submodule of a projective left R-module is $\mathcal{S}$-projective.

Proof. (1) $\Rightarrow$ (2). Let K be a submodule of an $\mathcal{S}$-projective module M and $\lambda : K \to M$ be the inclusion map. For any $S \in \mathcal{S}$ and any $f \in \mathrm{Hom}(S, K)$. Since M is $\mathcal{S}$-projective, by Theorem 1(2), λf factors through a finitely generated free left R-module F, i.e., there are homomorphisms $g : S \to F$ and $h : F \to M$ such that $\lambda f = hg$, which shows that $\mathrm{Ker} g \subseteq \mathrm{Ker} f$. Write $P = \mathrm{Im} g$. Then P is $\mathcal{S}$-projective by (1). Now we define $\varphi : P \to K$ by $\varphi(g(s)) = f(s)$ for $s \in S$. Then φ is a well-defined homomorphism and $f = \varphi g$. This shows that f factors through the $\mathcal{S}$-projective module P. Therefore, K is $\mathcal{S}$-projective by Theorem 1(6).

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) is clear. $\square$

Corollary 6. *(1). A ring R is left semihereditary if and only if every submodule of a finitely projective left R-module is finitely projective if and only if every submodule of a projective left R-module is finitely projective.*

(2). A ring R is left PP if and only if every submodule of a singly projective left R-module is singly projective if and only if every submodule of a projective left R-module is singly projective.

(3). A ring R is left PS if and only if every submodule of a simple projective left R-module is simple projective if and only if every submodule of a projective left R-module is simple projective.

Recall that a ring R is called a *left FGTF ring* [9] if every finitely generated torsionless left R-module embeds in a free module. We call R an *$\mathcal{S}$-TF ring* if every torsionless module in $\mathcal{S}$ embeds in a free module, and we call R a *left CTF ring* if every cyclic torsionless left R-module embeds in a free module.

Theorem 5. *Let $\mathcal{S}$ be a class of some finitely generated left R-modules. Then the following conditions are equivalent:*

(1) R is $\mathcal{S}$-Π-coherent and $\mathcal{S}$-semihereditary.

(2) Every left R-module has an epic $\mathcal{S}$-projective envelope.

(3) Every $S \in \mathcal{S}$ has an epic $\mathcal{S}$-projective envelope.

(4) Every $S \in \mathcal{S}$ has an epic projective envelope.

(5) Every $S \in \mathcal{S}$ has an epic finitely projective envelope.

(6) Every torsionless left R-module is $\mathcal{S}$-projective.

Moreover, if $\mathcal{S}$ is closed under homomorphic images, then these conditions are equivalent to:

(7) Every torsionless module in $\mathcal{S}$ is projective.

(8) R is $\mathcal{S}$-semihereditary and $\mathcal{S}$-TF.

Proof. (1) $\Rightarrow$ (2). Let M be any left R-module. Since R is $\mathcal{S}$-Π-coherent, M has an $\mathcal{S}$-projective preenvelope $f : M \to P$. Since R is $\mathcal{S}$-semihereditary, by Theorem 4, $\mathrm{Im} f$ is $\mathcal{S}$-projective, so $M \to \mathrm{Im} f$ is an epic $\mathcal{S}$-projective preenvelope. Note that for any class of left R-modules $\mathcal{A}$, each epic $\mathcal{A}$-preenvelope is an $\mathcal{A}$-envelope, we have (2).

(2) $\Rightarrow$ (3), and (6) $\Rightarrow$ (7) are trivial.

(3) $\Rightarrow$ (4). Let $S \in \mathcal{S}$. Then by (3), S has an epic $\mathcal{S}$-projective envelope $f : S \to P$. By Theorem 1(2), f factors through a finitely generated free left R-module F, that is, there exist $g : S \to F$ and $h : F \to P$ such that $f = hg$. Since F is $\mathcal{S}$-projective, there exists $\varphi : P \to F$ such that $g = \varphi f$. So $f = (h\varphi)f$, and hence $h\varphi = 1_P$ since f is epic. Hence, P is isomorphic to a direct summand of F, and thus P is projective.

(4) $\Rightarrow$ (5) by Proposition 1(2).

(5) $\Rightarrow$ (4). Let $S \in \mathcal{S}$. Then by (5), S has an epic finitely projective envelope $f : S \to P$. Note that every finitely generated finitely projective module is projective, so P is projective, and hence $f : S \to P$ is an epic projective envelope.

(4) $\Rightarrow$ (1). Assume (4). Then it is easy to see that R is $\mathcal{S}$-Π-coherent by [19, Theorem 3.1]. Let K be a submodule of an $\mathcal{S}$-projective module M and $\lambda : K \to M$ be the inclusion map. For any $S \in \mathcal{S}$ and any $f \in \mathrm{Hom}(S, K)$. Since M is $\mathcal{S}$-projective, by Theorem 1(2), λf factors through a finitely generated free left R-module F, i.e., there are $g : S \to F$ and $h : F \to M$ such that $\lambda f = hg$. By (4), S has an epic projective envelope $\alpha : S \to P$, then there exists $\beta : P \to F$ such that $g = \beta\alpha$. Thus $\lambda f = hg = (h\beta)\alpha$, and hence $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker} f$. Now we define $\varphi : P \to K$ by $\varphi(\alpha(s)) = f(s)$ for $s \in S$. Then φ is a well-defined homomorphism and $f = \varphi\alpha$. This shows that f factors through the projective module P. Therefore, K is $\mathcal{S}$-projective by Theorem 1(3).

(2) ⇒ (6). Let M be a torsionless left R-module. Then there is a monomorphism $i : M \to {}_RR^I$ for some index set I. Since every left R-module has an $\mathcal{S}$-projective preenvelope, R is $\mathcal{S}$-Π-coherent , and so ${}_RR^I$ is $\mathcal{S}$-projective. Let $f : M \to P$ be an epic $\mathcal{S}$-projective envelope. Then there exists a homomorphism $g : P \to {}_RR^I$ such that $i = gf$. Thus f is an isomorphism, and so M is $\mathcal{S}$-projective.

Now, suppose that $\mathcal{S}$ is closed under homomorphic images, then:

(7) ⇒ (1). Let $S \in \mathcal{S}$ and $f : S \to {}_RR^I$ be any left R-homomorphism. Since $\mathcal{S}$ is closed under homomorphic images, by (7), Imf is projective. So, by Theorem 1(3), ${}_RR^I$ is $\mathcal{S}$-projective, it shows that R is $\mathcal{S}$-Π-coherent. Moreover, it is easy to see that R is $\mathcal{S}$-semihereditary.

(7) ⇔ (8). It is easy. □

Corollary 7. *[8, Theorem 5.1] Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory for R-Mod. Then the following statements are equivalent:*

(1) R is right τ-coherent and submodules of τ-flat left R-modules are τ-flat.

(2) Every left R-module has an epic τ-flat envelope.

(3) Every τ-finitely presented left R-module has an epic τ-flat envelope.

(4) Every τ-finitely presented left R-module has an epic projective envelope.

(5) Every τ-finitely presented left R-module has an epic finitely projective envelope.

Our following Corollary 8 and Corollary 9 partially improve the results of [8, Corollary 5.3] and [16, Theorem 3.9] respectively.

Corollary 8. *The following statements are equivalent:*

(1) R is right Π*-coherent and left semihereditary.*

(2) Every left R-module has an epic finitely projective envelope.

(3) Every finitely generated left R-module has an epic projective envelope.

(4) Every finitely generated left R-module has an epic finitely projective envelope.

(5) Every torsionless left R-module is finitely projective.

(6) Every finitely generated torsionless left R-module is projective.

(7) R is left FGTF and left semihereditary.

Corollary 9. *The following statements are equivalent:*

(1) R is right AFG and left PP.

(2) Every left R-module has an epic singly projective envelope.

(3) Every cyclic left R-module has an epic singly projective envelope.

(4) Every cyclic left R-module has an epic projective envelope.

(5) Every cyclic left R-module has an epic finitely projective envelope.

(6) Every torsionless left R-module is singly projective.

(7) Every cyclic torsionless left R-module is projective.

(8) R is left CTF and left PP.

Recall that a ring R is called *left SPP* [15] if any direct product of copies of ${}_RR$ is simple-projective. By [15, Remark 3.3(4)], every left PS ring is left SPP. Let $\mathcal{S}$ be the class of all simple and zero left R-modules. Then by Theorem 5 and Corollary 6(3), we have

Corollary 10. *[15, Theorem 3.7]* The following statements are equivalent:

(1) R is a left PS ring.
(2) Every left R-module has an epic simple-projective envelope.
(3) Every simple left R-module has an epic simple-projective envelope.
(4) Every simple left R-module has an epic projective envelope.
(5) Every simple left R-module has an epic finitely projective envelope.
(6) Every torsionless left R-module is simple-projective.
(7) Every submodule of a simpe-projective left R-module is simple-projective .
(8) Every submodule of a projective left R-module is simple-projective .

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Department of Mathematics
Jiaxing University, P.R.China
E-mail: zhuzhanminzjxu@hotmail.com