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On the parallelizability of tangent bundles for 2 and 3-dimensional manifolds

by
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Abstract

We provide necessary and sufficient conditions for the parallelizability of TM, where M is a 2 or 3-manifold.

Given a 2-manifold M, its tangent bundle TM is paralelizable if and only if M is noncompact or of even Euler characteristic. We give two proofs, one using Stiefel-Whitney and Wu classes, another using obstruction theory and the classification theorem for compact surfaces.

For a 3-manifold M, its tangent bundle TM is paralelizable if and only if the cup product of the first Stiefel-Whitney class of M with itself is zero.

Key Words: Obstruction theory, tangent bundles, paralelizability, characteristic classes.
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Introduction

In the current paper we provide the necessary and sufficient conditions for the parallelizability of the total space of the tangent bundle, for 2 and 3-dimensional manifolds.

The first main theorem (Theorem 2.5) states that for a 2-dimensional manifold M its tangent space TM (viewed as a 4-dimensional manifold) is parallelizable if and only if the base manifold M is non-compact, or of even Euler characteristic. Alternatively, the manifold TM is parallelizable if and only if the second Stiefel-Whitney class of M is zero.

We present two proofs of this result. The first proof is based on a direct argument that uses Stiefel-Whitney and Wu characteristic classes. The second proof makes use of the classification theorem for surfaces to construct the obstruction explicitly.

The second main theorem (Theorem 4.2) states that for a 3-manifold M, its tangent bundle TM is paralelizable if and only if $w_1(M) \smile w_1(M) = 0$, where $w_1(M)$ is the first Stiefel-Whitney class of M. It uses both Stiefel-Whitney classes and general obstruction theory to obtain the result.

The paper is organised as follows. Theorem 1.1 is used as the starting point for both proofs. It states that for a manifold M, TM is parallelizable if and only if the Whitney sum, $TM \oplus TM \to M$, is trivial. This is important for simplifying our problem: instead of calculating the obstructions for TTM over TM, we compute them for $TM \oplus TM$ over M.

The first proof of Theorem 2.5 is a direct approach based on Stiefel-Whitney and Wu classes. It is a chain of 3 equivalencies. Let M be a surface.

Theorem 2.1 states that an n-bundle $N \to M$ over a 2-manifold M with $n > 2$ is trivial if and only if the first and second Stiefel-Whitney classes of N vanish.

Theorem 2.2 states that $TM \oplus TM \to M$ is trivial if and only if the second Stiefel-Whitney class of $TM \oplus TM \to M$ is zero. It is obtained by applying Theorem 2.1 for $N = TM \oplus TM$.

Theorem 2.4 states that the second Stiefel-Whitney class of $TM \bigoplus TM \to M$ is zero if and only if the second Stiefel-Whitney class of $TM \to M$ (the second Stiefel-Whitney class of the manifold M) is zero.

Theorem 2.3 states that the second Stiefel-Whitney class of M is zero if and only if M is noncompact or of even Euler characteristic.

Combining these with Theorem 1.1, we obtain the result for the 2-dimensional case, expressed as Theorem 2.5.

In the second proof of Theorem 2.5, we devise a method to compute obstructions to the triviality of a fiber over the connected sum of manifolds, from the obstructions of the fibers over the components, and the gluing morphism that identifies the fibers over the common boundary of the sum. This allows us to compute the obstruction for $T(N\#M) \oplus T(N\#M) \to (N\#M)$ from obstructions for $TM \oplus TM \to M$ and $TN \oplus TN \to N$. The classification theorem for compact surfaces states that every compact 2-manifold is either a sphere, a connected sum of tori, or a connected sum of projective planes. Theorems 3.2 - 3.4 explicitly compute the obstructions for the sphere, torus and projective plane. Theorem 3.5 shows how to construct the tangent bundle of a connected sum from the tangent bundles of its components. Using the above theorems, our connected-sum obstruction method, and the classification theorem for compact surfaces, we compute the obstruction for trivialising $TM \oplus TM \to M$, where M is a compact surface. This turns out to be zero except for the case where M is the connected sum of an odd number of projective planes, thus being of odd Euler characteristic. Combining with a separate analysis for the non-compact case, and Theorem 1.1, we again obtain Theorem 2.5.

Theorem 3.7 examines the question of the parallelizability of $TM \setminus M$, the tangent bundle of a surface, with the null-section removed. It turns out that $TM \setminus M$ is always parallelizable, unlike TM. The parallelizabitility of $TM \setminus M$ in the orientable case has been used by Berwald, and given an explicit construction known as the Berwald frame, [1].

The last section is concerned with obtaining analogue versions of the above results (Theorems 2.5 and 3.7, specifically), for the 3-dimensional case. We combine Stiefel-Whitney classes with general obstruction theory and elements used in the previous proofs.

Theorem 4.1 states that a bundle $N \to M$ over a 3-manifold M is trivial if and only if the restriction of N to the 2-skeleton of M is. It uses obstruction theory to arrive at this result.

Theorem 4.2 states that for a 3-manifold M, its tangent bundle TM is paralelizable if and only if $w_1(M) \smile w_1(M) = 0$. It is the analogue of Theorem 2.5. For this it combines Theorems 1.1, 4.1, and 2.1.

Theorem 4.4 shows that in the presence of a non-vanishing vector field on M, the paralelizability of $TM \setminus M$ is equivalent to the paralelizability of TM (it does this by constructing a diffeomorphism from TM to a subset of $TM \setminus M$). Finally, by noting that 3-manifolds always admits a non-vanishing vector field, we derive Theorem 4.5, which is the analogue for Theorem 3.7.

1 Preliminaries

We will use the next theorem for the proofs of our main results.

Theorem 1.1. *For an arbitrary manifold M, the manifold TM is parallelizable if and only if the vector bundle $TM \oplus TM \to M$ is trivial, where $\oplus$ denotes the Whitney sum.*

Proof. A manifold M is parallelizable if and only if the tangent bundle $TM \to M$ is trivial. Ergo, TM is parallelizable if and only if its tangent bundle $TTM \to TM$ is trivial. Using the next two lemmas it will follow that the tangent bundle $TTM \to TM$ is trivial if and only if $TM \oplus TM \to M$ is trivial that completes the proof of the theorem. □

Lemma 1.2. *The tangent bundle $TTM \to TM$ is trivial if and only if its pullback bundle $i^*(TTM) \to M$ via the inclusion morphism $i : M \to TM$ is trivial.*

Proof. If the tangent bundle $TTM \to TM$ is trivial, then so is $i^*(TTM) \to M$, its pullback bundle via the inclusion morphism.

Assume now that the pullback bundle $i^*(TTM) \to M$ is trivial. It follows that its pullback $\pi^*\left(i^*(TTM)\right) \to TM$ via the projection morphism $\pi : TM \to M$ is trivial. Alternatively, $\pi^*\left(i^*(TTM)\right) = (i \circ \pi)^*(TTM)$ is the pullback of TTM via $i \circ \pi$. Since M is a retraction of TM, then $i \circ \pi$ is homotopic to $\mathrm{Id} : TM \to TM$, the identity morphism. Using [8, Th.11.5], the pullback of $TTM \to TM$ via $i \circ \pi$ is equivalent to the the pullback of $TTM \to TM$ via Id. The two fiber bundles $(i \circ \pi)^*(TTM) \to TM$ and $TTM \to TM$ are isomorphic, therefore $TTM \to TM$ is trivial. □

Lemma 1.3. *The pullback bundle $i^*(TTM) \to M$ is isomorphic to the Whitney sum $TM \oplus TM \to M$.*

Proof. Let $f : (A \subset \mathbb{R}^n) \to (B \subset \mathbb{R}^n)$ be a transition map for an atlas of M. Knowing that the bundle $TM \to M$ is locally isomorphic to $M \times \mathbb{R}^n$, we define $f_T : A \times \mathbb{R}^n \to B \times \mathbb{R}^n$, by $f_T(x, v) = (f(x), df(x)v)$, as the lifting of f to a transition map over TM. We also define $\pi_T(x, v) = x$ as the coordinate expression of the projection $TM \to M$ map associated to the bundle, and $i_T(x) = (x, 0)$ as the coordinate expression of the inclusion map $M \to TM$. We see that the our lifting of the transition maps from the base space to its tangent bundle is natural: $\pi_T \circ f_T = f \circ \pi_T$ and $i_T \circ f = f_T \circ i_T$. To obtain the transition maps for $TM \bigoplus TM \to M$, it is sufficient to double the fibration, getting $f_{T \bigoplus T}(x, v_1, v_2) = (f(x), df(x)v_1, df(x)v_2)$. The projection map is $\pi_{T \bigoplus T}(x, v_1, v_2) = x$ and the inclusion map $i_{T \bigoplus T}(x) = (x, 0, 0)$. We now compute the transition maps for TTM, taken as a bundle over TM. To do so we apply again the formula for constructing the atlas of tangent bundle from that of its base space, starting with TM instead of M. We have $f_{TT}(x, v, v_1, v_2) = (f(x), df(x)v, df(x)v_1, d^2f(x)v_1v + df(x)v_2)$. The resulting projection map is $\pi_{TT}(x, v, v_1, v_2) = (x, v)$, and the inclusion map is $i_{TT}(x, v) = (x, v, 0, 0)$. We must now restrict the base space TM of TTM to $M \subset TM$. We have $\pi_{TT}(i^*(TTM)) = i_1(M)$ giving $v = 0$ and the final transition map as $f_{TT}(x, 0, v_1, v_2) = (f(x), 0, df(x)v_1, df(x)v_2)$. The projection map is given by $\pi_{i^*(TT)}(x, 0, v_1, v_2) = x$ and the inclusion map is $i_{i^*(TT)}(x) = (x, 0, 0, 0)$. This has the same structure as $f_{T \bigoplus T}$. We have $(x, v_1, v_2) \to (x, 0, v_1, v_2)$ as the bundle isomorphism between $TM \oplus TM \to M$ and $i^*(TTM) \to M$, which is the restriction of $TTM \to TM$ to $M \subset TM$. □

2 First proof of Theorem 2.5

Theorem 2.1. *Let M be a surface, $N \to M$ a vector bundle of dimension $n > 2$ over that surface. Then $N \to M$ is trivial if and only if the first and second Stiefel-Whitney classes of $N \to M$ are zero.*

Proof. Given an m-manifold B and a vector bundle $E \to B$, with the fiber F of dimension n, let $k \leq m$ be a non-negative integer, and $j = n - k + 1$. We assume the existence of a fixed CW-structure for our manifold, B (such structures can always be given when B is smooth and paracompact).

We know that the obstruction to the existence of a system of k-frames over the j-skeleton of B for E is a member o_j of $H^j(B, \{\pi_{j-1}(V_{n-j+1}(F))\})$. This is the jth cohomology group with twisted coefficients in $\pi_{j-1}(V_{n-j+1}(F))$, the $j-1$ homotopy group of the Stiefel manifold $V_{n-j+1}(F)$. If j is even and less than n, the group $\pi_{j-1}(V_{n-j+1}(F))$ is cyclic of order 2, hence canonically isomorphic to $\mathbb{Z}_2$. If j is odd, or $j = n$, the group $\pi_{j-1}(V_{n-j+1}(F))$ is infinite cyclic. However, it is not canonically isomorphic with $\mathbb{Z}$, the system of coefficients $\{\pi_{j-1}(V_{n-j+1}(F))\}$ is twisted in general, [7, Ch.12, pg.140].

The j'th Stiefel-Whitney class of a vector bundle $E \to B$, w_j, is equal to the mod 2 reduction of the previously defined obstruction class, o_j [7, Th.12.1]. We are interested in the existence of a trivialization of N over the 2-skeleton of the surface M. Assume N is trivial. Then the first and second Stiefel-Whitney classes of $N \to M$ are trivially zero. For the converse, assume the second Stiefel-Whitney class o_2 is zero. This is the mod 2 reduction of the obstruction to the existence of a field of $(n-1)$-frames for the n-plane bundle N over the 2-skeleton of n. Since $j = 2$ is even and less than n, the coefficient group $\pi_{j-1}(V_{n-j+1}(F))$ is canonically isomorphic to $\mathbb{Z}_2$, hence o_2 coincides with w_2. The cancelation of o_2 implies the existence of a field of $(n-1)$-frames for N over M. Assume o_1 is also zero. Then the bundle N is orientable and hence admits a volume form. We can use that form to construct a field of n-frames from our existing field of $(n-1)$-frames, hence $o_1(N) = 0$ and $o_2(N) = 0$ implies $N \to M$ is trivial for any surface M and vector bundle N of dimension $n > 2$. □

Theorem 2.2. *Let M be a surface. Then $TM \oplus TM \to M$ is trivial if and only if the second Stiefel-Whitney class of $TM \oplus TM \to M$ is zero.*

Proof. $TM \oplus TM \to M$ is always orientable, being the Whitney-sum of the bundle $TM \to M$ with itself. Hence $o_1(TM \oplus TM)$, the first Stiefel-Whitney class, is zero. Additionally, $TM \oplus TM$ is of dimension $n = 4$, $n > 2$. The theorem is obtained by applying Theorem 2.1 to the conditions given by $N = TM \oplus TM$. □

Theorem 2.3. *Let M be a surface. Then its second Stiefel-Whitney class is zero if and only if M is noncompact or of even Euler characteristic.*

Proof. Assume M is non-compact. Then by [2, Ch.VI. Cor. 7.12], its top cohomology group, $H^2(M, \mathbb{Z}_2)$, is trivial, and hence, w_2, being a member of this group, is zero.

Assume M is compact. For a manifold P let u_h be its fundamental $\mathbb{Z}_2$-homology class (the generator of the top $\mathbb{Z}_2$-homology group), and, analogously, u_c be the fundamental

$\mathbb{Z}_2$-cohomology class. From [7, Cor. 11.12] we know that when P is compact, $< w_m, u_h >= \chi(P) \bmod 2$, where m is the dimension of P and $\chi(P)$ is its Euler characteristic, therefore $w_m = (\chi(P) \bmod 2)u_c$. Applying to our compact surface M, w_2 vanishes if and only if the Euler characteristic of M is even. This completes our proof. □

Theorem 2.4. *Let M be a surface. Then the second Stiefel-Whitney class of $TM \bigoplus TM \to M$ is zero if and only if the second Stiefel-Whitney class of $TM \to M$ (the second Stiefel-Whitney class of the manifold M) is zero.*

Proof. According to the Whitney Product Theorem [7, Ch.4, Axiom 3], the Stiefel-Whitney classes of a Whitney sum can be computed from those of its components. Applying to $TM \oplus TM \to M$, we obtain:

$$w_2(TM \oplus TM) = w_2(TM) + w_1(TM) \smile w_1(TM) + w_2(TM) = w_1(TM) \smile w_1(TM)$$

When we speak of the characteristic classes of a manifold, we generally refer to the classes associated with its tangent bundle. We will now use Wu's Theorem, [7, Th.11.14],

$$w_k = \sum_{i+j=k} Sq^i(v_j),$$

and the properties of Steenrod squares [7, pg.91, (3))], to show that for a surface, the following relationship holds between its Stiefel-Whitney classes:

$$w_1 \smile w_1 = w_2.$$

The properties of Steenrod squares we shall use are as follows: for $a \in H^n(X, Y)$, $Sq^0(a) = a$, $Sq^n(a) = a \smile a$, and $Sq^m(a) = 0$ for $m > n$.

Applying the Wu formula to the first class we get $w_1 = Sq^0(v_1) + Sq^1(v_0)$, where v_n is the n'th Wu class of the manifold. This reduces to $w_1 = v_1$.

Applying to the second class, we get $w_2 = Sq^0(v_2) + Sq^1(v_1) + Sq^2(v_0)$, which reduces to $w_2 = v_2 + v_1 \smile v_1$. We know that v_k is zero when $k > n - k$, where n is the dimension of the ambient manifold [7, pg. 132]. Therefore, $v_2 = 0$, leaving us with $w_2 = v_1 \smile v_1$. Applying this to $w_1 = v_1$ gives us the desired formula.

Since for a surface M, we have $w_2(TM \oplus TM) = w_1(TM) \smile w_1(TM)$ and $w_1(TM) \smile w_1(TM) = w_2(TM)$, then $w_2(TM \oplus TM)$=0 if and only if $w_2(TM) = 0$, thus proving our theorem. □

Combining the results from the previous four theorems, we obtain the first main theorem.

Theorem 2.5. *The tangent bundle TM of a surface M is parallelizable if and only if M is non-compact, or of even Euler characteristic.*

3 Second proof of Theorem 2.5

3.1 Elements of obstruction theory

Given an n-dimensional vector bundle $\pi : E \to B$, over an m-dimensional manifold, we require the conditions under which E is trivial. A trivialization of E is a field of n-frames for E over B. For this task, we will be recalling the theory and methods of [5, Ch. 3] .

Our strategy is to construct an orthonormal trivialization of the restriction of E to the 0-skeleton of B, then extend inductively from the k-skeleton to the $(k+1)$-skeleton. E is trivial if and only if the field of frames extends to the m-skeleton [5, Ch. 3, pg. 74].

The obstruction to the extension from the k-skeleton to the $(k+1)$-skeleton takes the form of an element $o_{k+1} \in H^{k+1}(B; \pi_k(O(n)))$, where $\pi_0(O(n))$ is identified as $\mathbb{Z}_2$.

The first obstruction, o_1, is the element in $H^1(B; \mathbb{Z}_2)$. One can identify $H^1(B; \mathbb{Z}_2)$ with the homomorphisms $H_1(B) \to \mathbb{Z}_2$, and identify $H_1(B)$ as the abelianization of $\pi_1(B)$. Thus, the obstruction associates to a loop ℓ in B the element $0 \in \mathbb{Z}_2$ if the restriction of E on ℓ is orientable (trivial) and 1 if it is not. The obstruction o_1 vanishes if and only if the vector bundle E is orientable and the restriction of E to the 1-skeleton of B is trivial. Assuming o_1 vanishes, we can construct the next extensions. For this, we shall associate to E a default orientation.

For the inductive step, assume o_{k+1} is zero for $k < d$. Our task is to construct $o_{d+1} \in H^{d+1}(B; \pi_d(O(n)))$. Given that $o_d = 0$, we can select a trivialization t_d of E on the d-skeleton, that respects the chosen orientation of the bundle. The $(d+1)$-skeleton is defined as the union of the d-skeleton of B with the $(d+1)$-cells of B. Each $(d+1)$-cell C is contractible, so we can associate an orientation-preserving trivialization of E restricted to C, t_C. On the boundary of each $(d+1)$-cell C we now have two trivializations, one being the restriction of t_d to ∂C, the other being the restriction of t_C. We can associate to each point p in the boundary of C, the linear transformation $q \in O(n)$ that takes the t_C frame at p and transforms it to the t_d frame. Thus we obtain for each $(d+1)$-cell C a map $m_C : \partial C \to O(n)$. Composing with the cell's characteristic map, we obtain a map $n_C : \mathbb{S}^d \to O(n)$. This corresponds to an element in $\pi_d(O(n))$. Thus, we get a cellular co-chain with coefficients in $\pi_d(O(n))$. This turns out to be a cocycle, defining an element $o_{d+1} \in H^{d+1}(B; \pi_d(O(n)))$. It can be proved that o_{d+1} is well defined, as it does not depend on our choice of triangulation and trivializations [5, Ch. 3, pg. 75].

In order to exemplify obstruction theory, we will compute explicitly the obstructions for some basic examples, before using them for our main theorem. This will show how the obstructions to the parallelizability of a manifold differ from those of its tangent space.

We will do the computations for $\mathbb{S}^2$, $T\mathbb{S}^2$, $T\mathbb{T}^2$ and $T\mathbb{P}^2$.

Theorem 3.1. *The manifold $\mathbb{S}^2$ is not parallelizable, equivalently $T\mathbb{S}^2 \to \mathbb{S}^2$ is not trivial. The obstruction to the parallelizability of $\mathbb{S}^2$ is o_2, a non-zero cohomology class of $H^2(\mathbb{S}^2; \pi_1(O(2)))$.*

Proof. Using the procedure described above, we will now construct the obstruction to the parallelizability of $\mathbb{S}^2$, hence the triviality of $T\mathbb{S}^2 \to \mathbb{S}^2$, explicitly. We require a CW-decomposition of the sphere with non-degenerate boundary maps.
Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}$, with

$$
\begin{aligned}
&0 - \text{cells } C_{0.1} = \{(1,0,0)\}, C_{0.2} = \{(-1,0,0)\},\\
&1 - \text{cells } C_{1.1} = \{(x,y,0) \in \mathbb{S}^2, y \geq 0\}, C_{1.2} = \{(x,y,0) \in \mathbb{S}^2, y \leq 0\},\\
&2 - \text{cells } C_{2.1} = \{(x,y,z) \in \mathbb{S}^2, z \geq 0\}, C_{2.2} = \{(x,y,z) \in \mathbb{S}^2, z \leq 0\}
\end{aligned}
$$

and characteristic maps

$$m_{C_{1.1}} : [0,\pi] \to C_{1.1} \subset \mathbb{S}^2, m_{C_{1.1}}(t) = (\cos(t), \sin(t), 0),$$
$$m_{C_{1.2}} : [\pi, 2\pi] \to C_{1.2} \subset \mathbb{S}^2, m_{C_{1.2}}(t) = (\cos(t), \sin(t), 0),$$
$$m_{C_{2.1}} : D\,(0,1) \to C_{2.1} \subset \mathbb{S}^2, m_{C_{2.1}}(x,y) = \frac{1}{1+r^2}\left(2x, 2y, 1-r^2\right),$$
$$m_{C_{2.2}} : D\,(0,1) \to C_{2.2} \subset \mathbb{S}^2, m_{C_{2.2}}(x,y) = \frac{1}{1+r^2}\left(2x, 2y, r^2-1\right),$$

where $r = (x^2+y^2)^{1/2}$.

Since $H^1(\mathbb{S}^2, \mathbb{Z})$ is trivial, the first obstruction, o_1, vanishes. The restriction of the tangent bundle to the 1-skeleton of $\mathbb{S}^2$ admits a trivialization. We shall use the 2-frame field

$$r(t) = \begin{pmatrix} \sin(t) & -\cos(t) & 0 \\ 0 & 0 & -1 \end{pmatrix}, t \in [0, 2\pi],$$

along the points $m_{C_{1.s}}(t), s \in \{1,2\}$, given by the 1-cell attaching maps.

Now we must construct a trivialization of the bundle for each 2-cell. $T\mathbb{S}^2$ pulls back along the characteristic map $m_{C_{2.1}}$ to $TD(0,1)$. We require an isomorphism between TD (with the induced metric from the sphere) and $D \times \mathbb{R}^2$. One such isomorphism is given by the field of orthonormal 2-frames

$$f(x,y) = \frac{r^2+1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

on $TD \to D$, where $r = (x^2+y^2)^{1/2}$. This corresponds to the pull-back through $m_{C_{2.1}}$ of the frame-field

$$s(x,y,z) = \begin{pmatrix} 1-\frac{x^2}{z+1} & \frac{-xy}{z+1} & x \\ \frac{-xy}{z+1} & 1-\frac{y^2}{z+1} & y \end{pmatrix}$$

over the upper hemisphere of $\mathbb{S}^2$.

Parameterising the inverse image of $C_{2.1}$'s boundary by its characteristic map by $g : [0,2\pi] \to \partial D(0,1)$, $g(t) = (\cos(t), \sin(t))$, we obtain the following result. The pullback of the trivialization of the 1-skeleton by $m_{C_{2.1}} \circ g$ gives us the frame-field

$$m_1(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

while the pullback of the trivialization of $C_{2.1}$ by $m_{C_{2.1}} \circ g$ (or equivalently, the pullback of f through g), gives us the frame-field

$$m_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using these two trivializations of $T\mathbb{S}^2$ over the 2-cell $C_{2.1}$'s boundary, we define $m_s : [0,2\pi] \to SO(2)$, $m_s(t) = m_1(t)m_2^{-1}(t)$,

$$m_s(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

as the transformation field that takes m_2 into m_1.

By factoring through the equivalence relationship $x \sim x + 2\pi$, we may identify $[0, 2\pi]$ with $\mathbb{S}^1 \cong \partial C_{2.1}$. This gives us the required element $e_1 \in \pi_1(SO(2))$, associated to $C_{2.1}$ by the cocycle. Using an analogous computation for $C_{2.2}$, we obtain that the element associated to $\mathbb{S}^2$ (viewed as the chain $C_{2.1} + C_{2.2}$), by the cohomology class o_2, is $e \in \pi_1(SO(2))$,

$$e(t) = \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix},$$

completing our calculation. □

Theorem 3.2. *Given a 2-dimentional vector bundle $F \to \mathbb{S}^2$ over the sphere, its Whitney sum $F \oplus F \to \mathbb{S}^2$ is trivial. In particular, $T\mathbb{S}^2 \oplus T\mathbb{S}^2 \to \mathbb{S}^2$ is trivial, thus $T\mathbb{S}^2$ is parallelizable.*

Proof. Since $H^1(\mathbb{S}^2; \mathbb{Z}_2)$ is trivial, o_1 vanishes, both for F and $F \oplus F$.

The only remaining obstruction for trivialising $F \oplus F \to \mathbb{S}^2$ is o_2. This can be computed from the o_2 obstruction of $F \to \mathbb{S}^2$.

For $F \to \mathbb{S}^2$, o_2 associates to $\mathbb{S}^2$ the homotopy class of an element

$$e_1(t) = \begin{pmatrix} x(t) & y(t) \\ -y(t) & x(t) \end{pmatrix}, \; x(t)^2 + y(t)^2 = 1$$

where $e : (\mathbb{S}^1 \cong [0, 2\pi]/\{0, 2\pi\}) \to SO(2)$. Thus, in our case, (as our bundle is the Whitney sum of F with itself) o_2 associates to S_2 the class of

$$e_2(t) = \begin{pmatrix} e_1(t) & 0 \\ 0 & e_1(t) \end{pmatrix},$$

where $e_2 : (\mathbb{S}^1 \cong \mathbb{R}/[0, 2\pi]) \to SO(4)$. We shall now prove that e_2 is null-homotopic. Let

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

g is constant, therefore null-homotopic. e_2 is homotopic with g via the transition morphism $f : [0, \frac{\pi}{2}] \times \mathbb{S}^1 \to SO(4)$, $f(\theta, t) = \cos\theta e_2(t) + \sin\theta g$, therefore e_2 is null-homotopic and the obstruction o_2 vanishes. Thus, for any 2-bundle F over the sphere, $F \bigoplus F \to \mathbb{S}^2$ is trivial. In particular, $T\mathbb{S}^2 \bigoplus T\mathbb{S}^2 \to \mathbb{S}^2$ is trivial, therefore $T\mathbb{S}^2$ is parallelizable. □

Theorem 3.3. *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. $T\mathbb{T}^2 \oplus T\mathbb{T}^2 \to \mathbb{T}^2$ is trivial and $T\mathbb{T}^2$ is parallelizable.*

Proof. We know that $\mathbb{S}^1$ is paralelizable and $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Thus $\mathbb{T}^2$ is parallelizable, $T\mathbb{T}^2 \to \mathbb{T}^2$ is trivial. Therefore $T\mathbb{T}^2 \oplus T\mathbb{T}^2 \to \mathbb{T}^2$ is trivial. □

Theorem 3.4. *Let $\mathbb{P}^2$ denote the real projective plane. $T\mathbb{P}^2 \oplus T\mathbb{P}^2 \to \mathbb{P}^2$ is not trivial, the obstruction being $o_2 \in H^2(\mathbb{P}^2, \pi_1(SO(4)))$, the cohomology class which associates to $\mathbb{P}^2$ the generator of $\pi_1(SO(4)) \cong \mathbb{Z}_2$.*

Proof. We know that $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$ ([4, Ch. 13.2, Prop. 13.10]). We now explicitly construct representatives for the homotopy classes. Let $m : (\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1) \to SO(2)$, $m(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$.
We choose $m_0, m_1 : \mathbb{S}^1 \to SO(n)$, with $m_0(t) = I_n$, $m_1(t) = m(t) \oplus I_{n-2}$ as representatives for the classes of $\pi_1(SO(n)) \cong \mathbb{Z}_2$, with $[m_0] = 0$, and $[m_1] = 1$.

We can model $\mathbb{P}^2$ as a quotient of $\mathbb{S}^2$ by identifying antipodal points. $\mathbb{S}^2$ is the double cover of $\mathbb{P}^2$ via the covering map $p : \mathbb{S}^2 \to \mathbb{P}^2$. This map gives an equivalence relation for our chosen triangulation of $\mathbb{S}^2$ (see proof of Theorem 3.1), with $C_{n.1} \cong C_{n.2}$, $n \in \{0, 1, 2\}$. From this we have a triangulation of $\mathbb{P}^2$, and can begin to construct the obstruction tower. The Whitney sum of a bundle with itself is always orientable, thus the first obstruction o_1 vanishes.

Defining $\mathbb{P}^2_{C_2} : D(0,1) \to \mathbb{P}^2$, $\mathbb{P}^2_{C_2} = p \circ m_{C_{2.1}}$ as the characteristic map of the 2-cell of the projective plane, we must pull back a trivialization of the bundle $T\mathbb{P}^2 \oplus T\mathbb{P}^2 \to \mathbb{P}^2$ over the 1-skeleton of $\mathbb{P}^2$, to obtain a frame-field over ∂D for $TD \oplus TD \to D$. One such field is

$$n(x,y) = \begin{pmatrix} xx & xy & yx & yy \\ -y & x & 0 & 0 \\ -yx & -yy & xx & xy \\ 0 & 0 & -y & x \end{pmatrix},$$

$(x, y) \in \partial D$. It is constructed so as to factor trough $P^2_{C_2}$ to a field of orthonormal 4-frames over the 1-skeleton of $\mathbb{P}^2$.

Now we need a trivialization of $TD \oplus TD \to D$. We obtain one by doubling f, our trivialization of $TD \to D$, as used in the proof of Theorem 3.1. We define $f_2(x,y) = \begin{pmatrix} f(x,y) & 0 \\ 0 & f(x,y) \end{pmatrix}$.

By restricting f_2 to ∂D, we obtain a second frame-field over the boundary of our 2-cell. The transformation between them gives us the member $e \in \pi_1(SO(4))$ associated with the cell. As in Theorem 3.1, we parameterise the boundary of D by $g : [0, 2\pi] \to \partial D(0,1)$, $g(t) = (\cos(t), \sin(t))$. This gives us $e : [0, 2\pi] \cong \mathbb{S}^1 \to SO(4)$, $e(t) = n(g(t)) f_2^{-1}(g(t))$ as:

$$e(t) = \begin{pmatrix} \cos^2(t) & \cos(t)\sin(t) & \sin(t)\cos(t) & \sin^2(t) \\ -\sin(t) & \cos(t) & 0 & 0 \\ -\sin(t)\cos(t) & -\sin^2(t) & \cos^2(t) & \cos(t)\sin(t) \\ 0 & 0 & -\sin(t) & \cos(t) \end{pmatrix}$$

We maintain the notation $x = \cos(t)$, $y = \sin(t)$, for the sake of brevity. Consider the morphism $j : [0, 2\pi]/\{0, 2\pi\} \simeq \mathbb{S}^1 \to SO(4)$,

$$j(t) = \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix}.$$

It is of the form $j(t) = \begin{pmatrix} e(t) & 0 \\ 0 & e(t) \end{pmatrix}$, where $e(t) \in SO(2)$. According to Theorem 3.2, j is null-homotopic. Thus, we have $e = e \circ I \simeq e \circ j$. Explicitly:

$$e(t) = \begin{pmatrix} xx & xy & yx & yy \\ -y & x & 0 & 0 \\ -yx & -yy & xx & xy \\ 0 & 0 & -y & x \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e(t) \simeq \begin{pmatrix} xx & xy & yx & yy \\ -y & x & 0 & 0 \\ -yx & -yy & xx & xy \\ 0 & 0 & -y & x \end{pmatrix} \circ \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix},$$

$$e(t) \simeq \begin{pmatrix} x & 0 & y & 0 \\ 0 & 1 & 0 & 0 \\ -y & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Further manipulation gives us the form:

$$e(t) \simeq \begin{pmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus e turns out to be homotopic to m_1, proving our theorem. $T\mathbb{P}^2 \oplus T\mathbb{P}^2 \to \mathbb{P}^2$ is not trivial, $T\mathbb{P}^2$ is not parallelizable. □

3.2 Obstructions to the triviality of a bundle over a connected sum

Let $F_1 \to M_1$, $F_2 \to M_2$ be n-dimensional vector bundles over m-dimensional manifolds. Defining $\#$ as the connected sum operation, we are interested in the bundles $F_s \to M_1 \# M_2$, with the property that the restriction of F_s to M_1 is isomorphic to F_1 and the restriction to M_2 is isomorphic to F_2. $M_1 \# M_2$ is built by selecting two balls $B_1 \subset M_1$, $B_2 \subset M_2$, deleting their interiors, and identifying the boundaries via a homeomorphism $f : \partial B_1 \to \partial B_2$. For the construction of $F_s \to M_1 \# M_2$, it is necessary to select a specific identification of the fibers of F_1 with the fibers of F_2 along the common boundary. This gives a function $l : \mathbb{S}^{m-1} \to O(n)$, corresponding to an element $L \in \pi_{n-1}(SO(n))$. The element L determines F_s by specifying the way in which F_1 and F_2 are glued along the common boundary. Let us denote the obtained bundle as $F_1 \#_L F_2 \to M_1 \# M_2$.

The Mayer-Vietoris sequence [6, Ch. 2, §2.2] can be used to compute the cohomology of $M_1 \# M_2$, starting from the cohomology of components. Using this, we can determine the obstruction to the triviality of $F_1 \#_L F_2$ (as an element of the cohomology of the connected sum), from the obstructions determined by the individual bundles over the components. We obtain the following cases:

1) Assume $F_1 \to M_1$ has an obstruction $o_{k,1} \in H^k(M_1, \pi_{k-1}(O(n)))$ of rank k, and $F_2 \to M_2$ has an obstruction $o_{p,2} \in H^p(M_2, \pi_{p-1}(O(n)))$ of rank p, where $k < p$. Knowing that $H^t(M_1 \# M_2, G) \cong H^t(M_1, G) \times H^t(M_2, G)$ for all $t < m$, and lower-rank obstructions take "priority" over higher-ones (all lower-rank obstructions must vanish before a higher one can be constructed), the obstruction to the trivialization of $F_1 \#_L F_2 \to M_1 \# M_2$ will be of rank k, given by $o_z \in H^k(M_1 \# M_2, \pi_{k-1}(O(n))) \cong H^k(M_1, \pi_{k-1}(O(n))) \times H^k(M_2, \pi_{k-1}(O(n)))$, with $o_z \cong (o_{k,1}, 0)$.

2) Assume both obstructions $o_{k,1}$ and $o_{p,2}$ have the same rank, and that this rank is lower than m. Then, by similar reasoning to the previous case, the obstruction to the sum will be $o_z \cong (o_{k,1}, o_{p,2})$.

3) Assume both obstructions $o_{k,1}$ and $o_{p,2}$ have rank m. Given that an n-cochain is a function from n-cells to a group G, and $H^n(M, G)$ is constructed by an equivalence class over closed chains, we can characterise o_z with the following equation:

$$o_z(M_1 \# M_2) \cong o_{k,1}(M_1) \circ o_{p,2}(M_2) \circ L,$$

where '$o_k(M)$' denotes the application of the obstruction o_k to the chain given by M, and '$\circ$' denotes the group operation for $\pi_{m-1}(SO(n))$.

Theorem 3.5. *The tangent bundle of a connected sum is isomorphic to a connected sum of tangent bundles $T(M_1 \# M_2) \cong TM_1 \#_L TM_2$, with gluing morphism L equal to $o_m(\mathbb{S}^m)^{-1}$, where o_m is the obstruction to trivialising $T\mathbb{S}^m \to \mathbb{S}^m$.*

Proof. Denote $M_{k,s} = M_k \cap (M_1 \# M_2)$, $k \in \{1, 2\}$. The restriction to $M_{k,s}$ of $T(M_1 \# M_2) \to (M_1 \# M_2)$ is trivially isomorphic to the restriction to $M_{k,s}$ of $TM_k \to M_k$. Therefore the tangent bundle of a connected sum can indeed be written as a connected sum of tangent bundles. What remains is to compute the gluing morphism. Let $B_{k,s} = M_k - M_{k,s}$. In other words, let $B_{1,s}$ and $B_{2,s}$ be n-balls in M_1 and M_2, such that $M_1 \# M_2$ is obtained by selecting a bijection from $\overline{B}_{1,s}$ to $\overline{B}_{2,s}$ and identifying $\partial B_{1,s}$ from $M_1 - B_{1,s}$ with $\partial B_{2,s}$ from $M_1 - B_{2,s}$ via said bijection. Denote ∂B_s to be the common boundary obtained by this identification. Let F_k, $k \in 1, 2$ be trivializations of TM_k over $\overline{B}_{k,s} \subset M_k$. Our gluing morphism L is the transformation morphism between the two restrictions of those trivializations to the common boundary ∂B_s (we assume the trivializations are chosen such as they posses the same orientation along B_s). We shall compute L using the methods of obstruction theory.

Let $B_{k,l} \supset B_{k,s}$, $k \in 1, 2$ be open m-balls in M_k. We can extend the trivializations F_k to $B_{k,l}$. Let $T = B_{1,o} \cup B_{2,o}$, where $B_{k,o} = B_{k,l} - B_{k,s}$. It is a manifold with boundary, and can be considered as a "tube" gluing the two manifolds to form their connected sum. We have $\partial T = \partial B_{1,l} \cup \partial B_{2,l}$. Thus, we can define F, a trivialization of the bundle on ∂T, as the union of the restriction of F_1 to $\partial B_{1,l}$ with the restriction of F_2 to $\partial B_{2,l}$. L can be considered as $o_F(T)$, where $o_F \in H^m(T, \partial T; \pi_{k-1}(O(m)))$ is the obstruction to extending F from ∂T to T. Fix a trivialization of B_s.

To illustrate the geometric nature of the property we are about to use, consider a local embedding of $B_{1,l}, B_{2,l}$ in $\mathbb{R}^{m+1}$, such that $B_{1,s}$ is the upper-half of an m-sphere, $B_{2.s}$ is the lower-half, $B_{1,o}$ is a lower-cylinder capped by $B_{1,l}$, $B_{2,l}$ is a higher-cylinder capped by $B_{2,s}$, and T is the tube resulting from the union of those two cylinders. $o_F(B_{k,o})$ gives the obstruction to extending our trivialization from $\partial B_{k,o}$ to $B_{k,o}$ (this depends on our choice of trivialization for B_s). Since the trivialization can always be extended from $\partial B_{k,l}$ to $B_{k,l}$

(as it was constructed from F_k), we have $o_F(B_{k,l}) = 0$. Writing $B_{k,l} = B_{k,s} + B_{k,o}$ (as a chain complex), we have

$$\begin{aligned} o_F(B_{k,o}) &= o_F(B_{k,l} - B_{k,s}) = o_F(B_{k,l}) - o_F(B_{k,s}) = -o_F(B_{k,s}), \\ o_F(T) &= o_F(B_{1,o} + B_{2,o}) = o_F(B_{1,o}) + o_F(B_{2,o}) \\ &= -(o_F(B_{1,s}) + o_F(B_{2,s})) = -o_F(B_{1,s} + B_{2,s}). \end{aligned}$$

Since (as illustrated in the above geometric description) $B_{1,s} + B_{2,s}$ glue to form an m-sphere, and their tangent bundles glue to form the tangent bundle over the m-sphere, $o_F(B_{1,s}) + o_F(B_{2,s}) = o_m(\mathbb{S}^m)^{-1}$, where o_m is the obstruction to the parallelizability of an m-sphere. Hence the gluing morphism, $L = o_F(T) = -o_F(B_{1,s}) + o_F(B_{2,s})$, is its inverse, thus proving our theorem. □

An intuitive corollary of the theorem and obstruction-rules for the connected sum is that $M\#\mathbb{S}^m$ has the same tangent bundle and obstructions as M. Taking the connected-sum with an m-sphere is an identity operation for n-manifolds (up to isomorphism). Analogously, the gluing morphism for the Whitney sum of the tangent bundle is $o_m(\mathbb{S}^m)^{-1}$, where o_m is the obstruction to trivialising $T\mathbb{S}^m \bigoplus T\mathbb{S}^m \to \mathbb{S}^m$.

Knowing the obstructions to trivialising $TM \bigoplus TM \to M$ for the case when M is the 2-sphere, the torus, or projective plane, and how to compute the obstruction of a connected sum from its components, we will now use the classification theorem for surfaces [3] to explicitly construct the obstruction for an arbitrary compact surface, using the above instances as building blocks. Thus we determine the compact surfaces M for which the vector bundle $TM \bigoplus TM \to M$ is trivial.

As the Whitney sum of a bundle with itself is always orientable, the first obstruction o_1 always vanishes. We are interested in the second obstruction. For a compact surface M, denote o_{2M} to mean the second obstruction of $TM \bigoplus TM \to M$. As $T\mathbb{S}^2 \bigoplus T\mathbb{S}^2 \to \mathbb{S}^2$ is trivial, the equation for the second obstruction is $o_{2(A\#B)}(A\#B) = o_{2(A)}(A) \circ o_{2(B)}(B)$. Every compact surface is either a sphere, a connected sum of n tori, or a connected sum of n projective planes. If it is a connected sum of tori, the obstruction will vanish. Denote P_n to be the connected sum of n projective planes. The obstruction satisfies the equation $o_{2(P_n)}(P_n) = n \cdot o_{2(\mathbb{P}^2)}(\mathbb{P}^2)$. Since we are working in $\mathbb{Z}_2$, the second obstruction to the triviality of $TP_n \bigoplus TP_n \to P_n$ vanishes if and only if n is even. Therefore, in the compact case, $TM \bigoplus TM \to M$ is trivial for a surface M if and only if M is of even Euler characteristic. In the non-compact case, the top cohomology group, $H^2(M, \mathbb{Z}_2)$, is trivial [2, Ch.VI. Cor. 7.12], therefore w_2 vanishes. $TM \bigoplus TM \to M$ is always trivial for a non-compact surface M. Combining this with Theorem 1.1, we obtain the desired result. This completes the constructive version of the proof of Theorem 2.5.

Lemma 3.6. *For any manifold M, its tangent space TM is orientable*

Proof. The orientability of TM is equivalent to $TTM \to TM$ being orientable. If $TTM \to TM$ is orientable, then so is $i^*(TTM) \to M$, where i is the inclusion morphism. As $i \circ \pi$ is homotopic to the identity, then if $i^*(TTM) \to M$ is orientable, so is $TTM \to TM$ [8, Th.11.5]. From Lemma 1.3, $i^*(TTM) \to M$ is equivalent to $TM \oplus TM \to M$. Orientability is measured by the first Stiefel-Whitney class, which can be computed using the Whitney product formula: $w_1(TM \oplus TM) = w_1(TM) + w_1(TM) = 0$. This completes our proof. □

Theorem 3.7. *For any 2-manifold M, $TM \setminus M$ is parallelizable.*

Proof. We will now examine the question of the parallelizability of $TM \setminus M$, the tangent bundle of a 2-manifold M with its zero section removed. Assuming M is noncompact, then, due to the previous result, TM is parallelizable, and so is $TM \setminus M$.

Assume M is compact. Since TM is orientable (Lemma 3.6), so is $TM \setminus M$. $TM \setminus M$ is isomorphic as a manifold to $S(TM) \times I$, where $I = (0, \infty)$, $S(TM)$ is the total space of the unit tangent bundle (the subset of the tangent bundle where the vector component is of unit length). The isomorphism assigns to each non-zero element in a fiber of $TM \to M$ the pair given by its normalization and its norm. $S(TM)$ is an orientable, compact, three dimensional manifold, therefore it is parallelizable [8, Remark 39.9] (it can be proven that any orientable 3-manifold is parallelizable). Since I is parallelizable, so is $I \times S(TM) \cong TM \setminus M$. Therefore, for any 2-manifold M, $TM \setminus M$ is parallelizable. □

4 The three-dimensional case

Here we examine the paralelizability of TM, where M is a 3-manifold.

Theorem 4.1. *Let $N \to M$ be an n-bundle over a 3-manifold. Then N is trivial if and only if the restriction of N to M_2, the 2-skeleton of M, is trivial.*

Proof. Assume $N \to M$ is trivial. Then $N \to M_2 \subset M$ is trivial. Conversely, assume $N \to M_2 \subset M$ is trivial. Then, using obstruction theory discussed in section 3, the obstruction to the extension of the trivialization of N from M_2 to M is $o_3 \in H^3(B; \pi_2(O(n)))$. Using the Bott periodicity theorem [6, pg.384] and some known values, we obtain than $\pi_2(O(n)) = 0$. The coefficient being trivial gives $o_3 = 0$, so there is no obstruction to the extension of a trivialization of N from M_2 to M . This completes our proof. □

Theorem 4.2. *The tangent bundle of a 3-manifold M is paralelizable if and only if $w_1(M) \smile w_1(M) = 0$, where $w_1(M)$ is the first Stiefel-Whitney class of M.*

Proof. From Theorem 1.1, TM is paralelizable if and only if $TM \oplus TM \to M$ is trivial. From Theorem 4.1, $TM \oplus TM \to M$ is trivial if and only if $TM \oplus TM \to M_2$ is trivial, where M_2 is the 2-skeleton of M. Applying Theorem 2.1, $TM \oplus TM \to M_2$ is trivial if and only if the first and second Stiefel-Whitney classes of $TM \oplus TM \to M_2$ are zero. From the Whitney product formula, $w_{1(TM \oplus TM \to M_2)} = 2w_{1(TM \to M_2)} = 0$, therefore TM is paralelizable if and only if $w_{2(TM \oplus TM)} \to M_2 = 0$. The inclusion morphism $i : M_2 \to M$, induces a morphism $i^* : H^k(M, Z_2) \to H^k(M_2, Z_2)$ which is bijective for $k < 2$, injective for $k = 2$, and zero for $k > 2$.

As the n-th Steifel Whitney class of a bundle depends only on the restriction of that bundle to the n-skeleton of the base space, we have

$$w_{2(TM \oplus TM \to M_2)} = i^*(w_{2(TM \oplus TM \to M)}),$$
$$w_{2(TM \oplus TM \to M_2)} = 0 \text{ if and only if } w_{2(TM \oplus TM \to M)} = 0.$$

Therefore, TM is paralelizable if and only if $w_{2(TM\oplus TM\to M)} = 0$.
Aplying the Whitney-Product formula again, we have $w_{2(TM\oplus TM\to M)} = w_{1(TM\to M)} \smile w_{1(TM\to M)} = w_1(M) \smile w_1(M)$, thus completing our proof. TM for a 3-manifold is paralelizable if and only if $w_1(M) \smile w_1(M) = 0$. □

Lemma 4.3. *An m-manifold M admits a global non-vanishing vector field if and only if M is non-compact or $\chi(M) = 0$.*

Proof. The obstruction to the existence of a field of 1-frames on $TM \to M$ is $o_m \in H^m(B, \{\pi_{m-1}(V_1(F))\})$ [7, pg.140]. However, in the non-compact case, the top cohomology group is trivial [2, Cor. 7.12], hence o_m vanishes. The compact case is done in [8, Cor. 39.8.]. □

Theorem 4.4. *Let M be a manifold that is non-compact or of vanishing Euler characteristic. Then $TM \setminus M$ is paralelizable if and only if TM is paralelizable.*

Proof. Assume M is given a riemannian metric. The above conditions characterise the existence of a non-vanishing vector field $X : M \to TM$ (Lemma 4.3). Define $f : TM \to TM$ by $f(x, v) = (x, \frac{|X(x)|v}{\sqrt{|X(x)|^2+|v|^2}} + 2X(x))$. Restricting the codomain of f to its image gives us a diffeomorphism. Additionally, $f(TM) \subset TM \setminus M$. We know TM being paralelizable implies $TM \setminus M \subset TM$ being paralelizable. Conversely, $TM \setminus M$ being paralelizable implies $f(TM) \subset TM \setminus M$ being paralelizable. Since f is a diffeomorphism from TM to $f(TM)$, this implies TM being paralelizable. This completes our proof. □

Theorem 4.5. *When M is an odd-dimensional manifold, $TM \setminus M$ is paralelizable iff TM is paralelizable. In particular, removing the zero section of the tangent bundle of a 3-manifold does not remove the obstruction to its paralelizability, unlike the 2-dimensional case (comp. Theorem 3.7.).*

Proof. We know from [6, Cor. 3.37] that a closed manifold of odd dimension has Euler characteristic zero. Applying Theorem 4.4. to M we get the desired conclusion □

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