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Finiteness properties of extension functors of ETH-cofinite modules

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Abstract

Let R be a commutative Noetherian ring, I an ideal of R, M an R-module (not necessary I-torsion) and K a finitely generated R-module with $\mathrm{Supp}_R(K) \subseteq \mathrm{V}(I)$. It is shown that if M is I-ETH-cofinite (i.e. $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated, for all $i \geq 0$) and $\dim M \leq 1$, then the R-module $\mathrm{Ext}^n_R(M, K)$ is finitely generated, for all $n \geq 0$. As a consequence it is shown that if M is I-ETH-cofinite and $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian), then the R-module $\mathrm{Ext}^n_R(M, K)$ is finitely generated, for all $n \geq 0$ which removes I-torsion condition of M from [3, Corollary 3.11] and [20, Theorem 2.8]. As an application to local cohomology, let Φ be a system of ideals of R and $I \in \Phi$, if $\dim M/\mathfrak{a}M \leq 1$ (e.g., $\dim R/\mathfrak{a} \leq 1$) for all $\mathfrak{a} \in \Phi$, then the R-modules $\mathrm{Ext}^j_R(\mathrm{H}^i_\Phi(M), K)$ are finitely generated, for all $i \geq 0$ and $j \geq 0$. A similar result is true for local cohomology modules defined by a pair of ideals.

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1 Introduction

Throughout this paper R is a commutative Noetherian ring with non-zero identity and I and J two ideals of R. For an R-module M, the i^{th} local cohomology module M with respect to ideal I is defined as

$$\mathrm{H}^i_I(M) \cong \varinjlim_n \mathrm{Ext}^i_R(R/I^n, M).$$

Hartshorne in [17] defined a module M to be *I–cofinite* if $\mathrm{Supp}_R(M) \subseteq \mathrm{V}(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. and asked the following question:

Question 1.1. *Let M be a finite R–module and I be an ideal of R. When are $\mathrm{H}^i_I(M)$ I–cofinite for all $i \geq 0$?*

This question was studied by several authors in [17, 13, 18, 14, 23, 22, 7] and [3].

The study of cofinite modules arises the following natural question:

Question 1.2. *Let I be an ideal of a Noetherian ring R and M an R-module. When are the R-modules $\mathrm{Ext}^n_R(M, R/I)$ finitely generated for all integers $n \geq 0$.*

Irani and the second author proved that when M is I-cofinite and $\dim M \leq 1$, then for any finitely generated R-module K with $\mathrm{Supp}_R(K) \subseteq \mathrm{V}(I)$, instead of R/I, the R-modules $\mathrm{Ext}^n_R(M, K)$ are finitely generated for all integers $n \geq 0$. Here we will prove that the same answer is true without I-torsion condition on M. To do this, recall that an R-module M is called *I-ETH-cofinite* if $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all integers $i \geq 0$. This class introduced in [1, Definition 2.2]. More precisely, we shall show that:

Theorem 1.3. *Let R be a Noetherian local ring, I a proper non-zero ideal of R and K be a finitely generated R-module with $\mathrm{Supp}_R(K) \subseteq V(I)$. Also, let M be an I-ETH-cofinite R-module (e.g., $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all integers $i \geq 0$) and $\dim(M) \leq 1$. Then the R-module $\mathrm{Ext}^n_R(M, K)$ is finitely generated, for all integers $n \geq 0$.*

As a special case of [26, Definition 2.1] and generalization of FSF modules (see [19, Definition 2.1]), in [3, Definition 2.1] the authors of this paper introduced the class of $\mathrm{FD}_{\leq \mathrm{n}}$ modules. A module M is said to be $\mathrm{FD}_{\leq \mathrm{n}}$ module, if there exist a finitely generated submodule N of M such that $\dim M/N \leq n$. For more details about properties of this class see [3, Lemma 2.3]. Recall that an R-module M is called *weakly Laskerian* if $\mathrm{Ass}_R(M/N)$ is a finite set for each submodule N of M. The class of weakly Laskerian modules introduced in [15]. Bahmanpour in [6, Theorem 3.3] proved that over Noetherian rings, an R-module M is weakly Laskerian if and only if M is FSF module. Thus the class of weakly Laskerian modules is contained in the class of $\mathrm{FD}_{\leq 1}$ modules. Using the class of $\mathrm{FD}_{\leq 1}$, we will generalize Theorem 1.3 and [3, Corollary 3.11] as below:

Corollary 1.4. *Let R be a Noetherian ring and I be an ideal of R. Let M be an $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) and I-ETH-cofinite R-module. Then, the R-modules $\mathrm{Ext}^n_R(M, K)$ are finitely generated, for all finitely generated R-modules K with $\mathrm{Supp}_R(K) \subseteq V(I)$ and all integers $n \geq 0$.*

There is in [9], a generalization of ordinary local cohomology modules defined by Bijan-Zadeh. Let Φ be a non-empty set of ideals of R. We call Φ a *system of ideals* of R if, whenever $I_1, I_2 \in \Phi$, then there is an ideal $J \in \Phi$ such that $J \subseteq I_1 I_2$. For such a system, for every R-module M, one can define

$$\Gamma_\Phi(M) = \{\, x \in M \mid Ix = 0 \;\text{ for some } I \in \Phi\}.$$

Then $\Gamma_\Phi(-)$ is a functor from $\mathscr{C}(R)$ to itself (where $\mathscr{C}(R)$ denotes the category of all R-modules and all R-homomorphisms). The functor $\Gamma_\Phi(-)$ is additive, covariant, R-linear and left exact. In [10], $\Gamma_\Phi(-)$ is denoted by $L_\Phi(-)$ and is called the "general local cohomology functor with respect to Φ". For each $i \geq 0$, the i-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $\mathrm{H}^i_\Phi(-)$. The functor $\mathrm{H}^i_\Phi(-)$ and $\varinjlim_{I\in\Phi} \mathrm{H}^i_I(-)$ (from $\mathscr{C}(R)$ to itself) are naturally equivalent (see [9]). For an ideal I of R, if $\Phi = \{I^n | n \in \mathbb{N}_0\}$, then the functor $\mathrm{H}^i_\Phi(-)$ coincides with the ordinary local cohomology functor $\mathrm{H}^i_I(-)$. It is shown that, the study of torsion theory over R is equivalent to study the general local cohomology theory (see [10]).

As a special case of general local cohomology and generalization of ordinary local cohomology modules, R. Takahashi, Y. Yoshino, and T. Yoshizawa [24], introduced local cohomology modules with respect to a pair of ideals. The (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M is a submodule of M consists of all elements x of M with $\mathrm{Supp}(Rx) \subseteq W(I, J)$, in which

$$W(I, J) = \{\, \mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \;\text{ for an integer } n \geq 1\}.$$

For an integer i, the i-th local cohomology functor $\mathrm{H}^i_{I,J}$ with respect to (I, J) is the i-th right derived functor of $\Gamma_{I,J}$. The R-module $\mathrm{H}^i_{I,J}(M)$ is called the i-th local cohomology module of M with respect to (I, J). In the case $J = 0$, $\mathrm{H}^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $\mathrm{H}^i_I(-)$. Also, we are concerned with the following set of ideals of R:

$$\tilde{W}(I, J) = \{\, \mathfrak{a} \trianglelefteq R \mid I^n \subseteq \mathfrak{a} + J \;\text{ for an integer } n \geq 0\}.$$

As an application to local cohomology, we prove the following corollaries:

Corollary 1.5. *Let $I \in \Phi$ be an ideal of a Noetherian ring R, M a non-zero finite R-module such that $\mathrm{H}^i_\Phi(M)$ are $\mathrm{FD}_{\leq 1}$(or weakly Laskerian) R-modules for all $i \geq 0$. Then for each finite R-module K with $\mathrm{Supp}_R(K) \subseteq V(I)$, the R-modules $\mathrm{Ext}^j_R(\mathrm{H}^i_\Phi(M), K)$ are finitely generated for all $i \geq 0$ and $j \geq 0$.*

Corollary 1.6. *Let Φ be a system of ideals of R and $I \in \Phi$. If $\dim M/\mathfrak{a}M \leq 1$ (e.g., $\dim R/\mathfrak{a} \leq 1$) for all $\mathfrak{a} \in \Phi$, then for each finite R-module K with $\mathrm{Supp}_R(K) \subseteq V(I)$, the*

R-modules $\mathrm{Ext}^j_R(\mathrm{H}^i_\Phi(M), K)$ *are finitely generated for all* $i \geq 0$ *and* $j \geq 0$.

Similar corollaries are true for local cohomology modules defined by a pair of ideals because it is a special case of local cohomology with respect to a system of ideals.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$. For any unexplained notation and terminology we refer the reader to [11], [12] and [21].

2 Main results

The following lemma is needed in the proof of Lemma 2.4.

Lemma 2.1. *Let I be an ideal of a Noetherian ring R and M be an R-module such that $M = IM$. Let K be a finitely generated R-module with* $\mathrm{Supp}_R(K) \subseteq V(I)$. *Then we have* $\mathrm{Hom}_R(M, K) = 0$.

Proof. Since $\mathrm{Supp}_R(K) \subseteq V(I)$ and K is finitely generated it follows that $I^n K = 0$ for some positive integer n. Moreover, from the hypothesis $M = IM$ it follows that $I^n M = M$. So, we have

$$\begin{aligned} \mathrm{Hom}_R(M, K) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(R/I^n, K)) \\ &\cong \mathrm{Hom}_R(M \otimes_R R/I^n, K) \\ &\cong \mathrm{Hom}_R(M/I^n M, K) \\ &\cong \mathrm{Hom}_R(0, K) \\ &\cong 0. \end{aligned}$$

□

The following lemma is a generalization of [23, Theorem 2.1] in the sense of Serre subcategory of the category of R-modules.

Lemma 2.2. *Let R be a Noetherian ring and $I = (x_1, ..., x_n)$ be an ideal of R and let M be an R-module. Let $\mathscr{S}$ be a Serre subcategory of the category of R-modules. Then the following statements are equivalent:*

(i) *The R-module* $\mathrm{Ext}^i_R(R/I, M)$ *belongs to $\mathscr{S}$, for all integers $i \geq 0$,*

(ii) *The R-module* $\mathrm{Tor}^R_i(R/I, M)$ *belongs to $\mathscr{S}$, for all integers $i \geq 0$,*

(iii) *The Koszul cohomology module* $\mathrm{H}^i(x_1, ..., x_n; M)$ *belongs to* $\mathscr{S}$*, for all integers* $i = 0, ..., n$.

Proof. Follows from the method of the proof [25, Theorem 2]. ∎

The equivalent conditions in the following lemma is quite useful in the proof of Lemma 2.4.

Lemma 2.3. *Let* R *be a Noetherian ring and* I *a proper non-zero ideal of* R*. Then for an* R*-module* M*, the following statements are equivalent:*
(i) $\mathrm{H}^n_I(M) = 0$*, for all integers* $n \geq 0$*,*
(ii) $\mathrm{Ext}^n_R(R/I, M) = 0$*, for all integers* $n \geq 0$*,*
(iii) $\mathrm{Tor}^R_n(R/I, M) = 0$*, for all integers* $n \geq 0$*.*

Proof. (i)⇔(ii) Follows applying [5, Theorem 2.9 (i)⇔(ii)] to the zero Serre category.
(ii)⇔(iii) Follows applying Lemma 2.2, getting $\mathscr{S}$ equal to the zero Serre category. ∎

The next result is of assistance in the proof of the main theorems in this paper.

Lemma 2.4. *Let* R *be a Noetherian ring,* I *a proper non-zero ideal of* R *and* K *be a finitely generated* R*-module with* $\mathrm{Supp}_R(K) \subseteq V(I)$*. Also, let* M *be an* R*-module satisfying in the equivalent conditions of Lemma 2.3. Then we have* $\mathrm{Ext}^n_R(M, K) = 0$*, for all integers* $n \geq 0$*.*

Proof. We argue using induction on n. For $n = 0$, the assertion follows from Lemma 2.1. Now, let $n > 0$, and assume that inductively the assertion holds for all R-modules satisfying the equivalent conditions of Lemma 2.3, and for all integers smaller than n. Then we must prove the assertion for n. Since $\mathrm{Supp}_R(K) \subseteq V(I)$ and K is finitely generated it follows that $I^s K = 0$, for some positive integer s. Since by hypothesis we have $\mathrm{H}^n_I(M) = 0$, for all integers $n \geq 0$, it follows that $\mathrm{H}^n_{I^s}(M) = 0$, for all integers $n \geq 0$. So, M satisfies the equivalent conditions of Lemma 2.3, for the ideal I^s instead of I. So, replacing I by I^s, without lose of generality we may assume that $IK = 0$. Now, let $I = (a_1, ..., a_t)$. We define the R-homomorphism $f : \oplus^t_{i=1} M \longrightarrow M$ as follows:

$$f(x_1, x_2, ..., x_t) = \Sigma^t_{i=1} a_i x_i.$$

Then we have $\mathrm{Im}(f) = IM = M$. So, f is an epimorphism. Let $N = \mathrm{Ker}(f)$. Then from the exact sequence

$$0 \to N \to \oplus^t_{i=1} M \xrightarrow{f} M \to 0, \quad (*)$$

it follows that R-module N, satisfies the equivalent conditions of Lemma 2.3 and so by inductive hypothesis we have $\mathrm{Ext}_R^{n-1}(N,K)=0$. For each $1\leq j\leq t$, let $\iota_j : M \to \oplus_{i=1}^t M$ and $\pi_j : \oplus_{i=1}^t M \to M$ be the natural monomorphism and natural epimorphism, respectively. Then, for each $1\leq j\leq t$, the R-homomorphism $f\circ\iota_j : M\to M$ is the R-homomorphism $M\overset{a_j}{\to} M$. In particular, since $a_j\in \mathrm{Ann}_R(K)$ and the functor $\mathrm{Ext}_R^n(-,K)$ is R-linear, it follows that

$$\mathrm{Ext}_R^n(\iota_j,K)\circ \mathrm{Ext}_R^n(f,K)=\mathrm{Ext}_R^n(f\circ\iota_j,K)=0.$$

On the other hand since $\Sigma_{j=1}^t \iota_j\circ\pi_j = 1_{\oplus_{i=1}^t M}$ and the functor $\mathrm{Ext}_R^n(-,K)$ is additive, it follows that

$$\begin{aligned}
\mathrm{Ext}_R^n(f,K) &= 1_{\mathrm{Ext}_R^n(\oplus_{i=1}^t M,K)}\circ \mathrm{Ext}_R^n(f,K)\\
&= \mathrm{Ext}_R^n(1_{\oplus_{i=1}^t M},K)\circ \mathrm{Ext}_R^n(f,K)\\
&= \mathrm{Ext}_R^n(\Sigma_{j=1}^t \iota_j\circ\pi_j,K)\circ \mathrm{Ext}_R^n(f,K)\\
&= (\Sigma_{j=1}^t \mathrm{Ext}_R^n(\iota_j\circ\pi_j,K))\circ \mathrm{Ext}_R^n(f,K)\\
&= (\Sigma_{j=1}^t \mathrm{Ext}_R^n(\pi_j,K)\circ \mathrm{Ext}_R^n(\iota_j,K))\circ \mathrm{Ext}_R^n(f,K)\\
&= \Sigma_{j=1}^t \mathrm{Ext}_R^n(\pi_j,K)\circ \mathrm{Ext}_R^n(\iota_j,K)\circ \mathrm{Ext}_R^n(f,K)\\
&= \Sigma_{j=1}^t \mathrm{Ext}_R^n(\pi_j,K)\circ 0\\
&= 0.
\end{aligned}$$

Now, the exact sequence $(*)$ yields an exact sequence

$$\mathrm{Ext}_R^{n-1}(N,K)\longrightarrow \mathrm{Ext}_R^n(M,K)\overset{\mathrm{Ext}_R^n(f,K)}{\longrightarrow}\mathrm{Ext}_R^n(\oplus_{i=1}^t M,K),$$

which implies that $\mathrm{Ext}_R^n(M,K)=0$, as required. This completes the proof of inductive step. □

We are now ready to state and prove the main theorem of this paper. The following theorem is a generalization of [20, Theorem 2.8]. In fact, we remove I-torsion condition from R-module M in this theorem.

Theorem 2.5. *Let R be a Noetherian ring, I a proper non-zero ideal of R and K be a finitely generated R-module with* $\mathrm{Supp}_R(K)\subseteq V(I)$. *Also, let M be an I-ETH-cofinite R-module (e.g.* $\mathrm{Ext}_R^i(R/I,M)$ *is finitely generated for all integers $i\geq 0$) and* $\dim(M)\leq 1$. *Then the R-module* $\mathrm{Ext}_R^n(M,K)$ *is finitely generated, for all integers $n\geq 0$.*

Proof. Using the exact sequence

$$0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0, \quad (*)$$

it is easy to see that the R-modules $\mathrm{Hom}_R(R/I, \Gamma_I(M))$ and $\mathrm{Ext}^1_R(R/I, \Gamma_I(M))$ are finitely generated and so by [8, Proposition 2.6], it follows that $\Gamma_I(M)$ is I-cofinite with dimension at most one. So, by [20, Theorem 2.8], it follows that the R-module $\mathrm{Ext}^n_R(\Gamma_I(M), K)$ is finitely generated, for all integers $n \geq 0$. So, considering the exact sequence, without lose of generality we may assume that $\Gamma_I(M) = 0$, then we have $\Gamma_I(\mathrm{E}_R(M)) = 0$. In fact since $\mathrm{E}_R(M)$ is injective it follows that $\mathrm{H}^i_I(\mathrm{E}_R(M)) = 0$, for all integers $i \geq 0$ and hence by Lemma 2.4, it follows that $\mathrm{Ext}^n_R(\mathrm{E}_R(M), K) = 0$, for all integers $n \geq 0$. Next, consider the exact sequence

$$0 \to M \to \mathrm{E}_R(M) \to N \to 0, \quad (**)$$

Then $\mathrm{H}^1_I(M) \cong \Gamma_I(N)$. If $\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{H}^1_I(M)) \subseteq \mathrm{Supp}_R(M)$, then $\mathrm{H}^1_{IR_\mathfrak{p}}(M_\mathfrak{p}) \cong \mathrm{H}^1_I(M)_\mathfrak{p} \neq 0$. Since $\dim M \leq 1$, it is easy to see that $\dim R/\mathfrak{p} = 0$ or $\dim R/\mathfrak{p} = 1$. If $\dim R/\mathfrak{p} = 1$ then $M_\mathfrak{p}$ is a zero dimensional $R_\mathfrak{p}$-module that implies $\mathrm{H}^1_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by using Grothendieck vanishing theorem [11, Theorem 6.1.2] which is a contradiction. Thus $\dim R/\mathfrak{p} = 0$ and so $\mathfrak{p}$ is a maximal ideal. So we have the following inclusion

$$\mathrm{Supp}_R(\mathrm{H}^1_I(M)) \subseteq \mathrm{Max}\, R.$$

Moreover, since $\Gamma_I(M) = 0$ so by [23, Lemma 7.9] or [5, Corollary 4.3], we have

$$\mathrm{Hom}_R(R/I, \mathrm{H}^1_I(M)) \cong \mathrm{Ext}^1_R(R/I, M).$$

Thus $\mathrm{Hom}_R(R/I, \mathrm{H}^1_I(M))$ is finitely generated with support in $\mathrm{Max}(R)$. So, the R-module $\mathrm{Hom}_R(R/I, \mathrm{H}^1_I(M))$ is of finite length. Now it follows from Melkersson result ([23, Proposition 4.1]) that $\mathrm{H}^1_I(M) \cong \Gamma_I(N)$ is I-cofinite and so by Irani-Bahmanpour result [20, Theorem 2.8], it follows that the R-module $\mathrm{Ext}^n_R(\Gamma_I(N), K)$ is finitely generated, for all integers $n \geq 0$. From the exact sequence $(**)$ we can deduce that $\mathrm{H}^i_I(N/\Gamma_I(N)) = 0$, for all integers $i \geq 0$ and so by Lemma 2.4 it follows that $\mathrm{Ext}^n_R(N/\Gamma_I(N), K) = 0$, for all integers $n \geq 0$. Now it follows from the exact sequence

$$0 \to \Gamma_I(N) \to N \to N/\Gamma_I(N) \to 0,$$

that the R-module $\mathrm{Ext}^n_R(N, K)$ is finitely generated, for all integers $n \geq 0$. Now, it follows from the exact sequence $(**)$ that the R-module $\mathrm{Ext}^n_R(M, K)$ is finitely generated, for all integers $n \geq 0$. This completes the proof. □

The following corollary is a generalization of [3, Corollary 3.11] which also generalizes Theorem 2.5 to a more larger class of modules.

Corollary 2.6. *Let R be a Noetherian ring and I be an ideal of R. Let M be an* $\mathrm{FD}_{\leq 1}$ (*or weakly Laskerian*) *and I-ETH-cofinite R-module. Then, the R-modules* $\mathrm{Ext}^n_R(M,K)$ *are finitely generated, for all finitely generated R-modules K with* $\mathrm{Supp}_R(K) \subseteq V(I)$ *and all integers $n \geq 0$.*

Proof. The assertion follows from the definition of $\mathrm{FD}_{\leq 1}$ modules using Theorem 2.5. ∎

As applications to local cohomology we prove the following corollaries which generalize [20, Theorem 2.9].

Corollary 2.7. *Let $I \in \Phi$ be an ideal of a Noetherian ring R, M a non-zero finite R-module such that* $\mathrm{H}^i_\Phi(M)$ *are* $\mathrm{FD}_{\leq 1}$(*or weakly Laskerian*) *R-modules for all $i \geq 0$. Then for each finite R-module K with* $\mathrm{Supp}_R(K) \subseteq V(I)$, *the R-modules* $\mathrm{Ext}^j_R(\mathrm{H}^i_\Phi(M), K)$ *are finitely generated for all $i \geq 0$ and $j \geq 0$.*

Proof. By [2, Theorem 2.7 (i)], it follows that $\mathrm{H}^i_\Phi(M)$ is I-ETH-cofinite for all $i \geq 0$. Now the assertion follows by Corollary 2.6. ∎

Corollary 2.8. *Let Φ be a system of ideals of R and $I \in \Phi$. If* $\dim M/\mathfrak{a}M \leq 1$ (*e.g.,* $\dim R/\mathfrak{a} \leq 1$) *for all $\mathfrak{a} \in \Phi$, then for each finite R-module K with* $\mathrm{Supp}_R(K) \subseteq V(I)$, *the R-modules* $\mathrm{Ext}^j_R(\mathrm{H}^i_\Phi(M), K)$ *are finitely generated for all $i \geq 0$ and $j \geq 0$.*

Proof. Since by [9, Lemma 2.1],

$$\mathrm{H}^i_\Phi(M) \cong \varinjlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_{\mathfrak{a}}(M),$$

it is easy to see that $\mathrm{Supp}_R(\mathrm{H}^i_\Phi(M)) \subseteq \bigcup_{\mathfrak{a}\in\Phi} \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ and therefore

$$\mathrm{dimSupp}\,\mathrm{H}^i_\Phi(M) \leq \sup\{\mathrm{dimSupp}\,\mathrm{H}^i_{\mathfrak{a}}(M) | \mathfrak{a} \in \Phi\} \leq 1,$$

thus $\mathrm{H}^i_\Phi(M)$ is $\mathrm{FD}_{\leq 1}$ R-module and the assertion follows by Corollary 2.7. ∎

Remark 2.9. *Let I and J be two ideals of R. Replacing Φ by $\tilde{W}(I,J)$ and $\mathrm{H}^i_{\Phi}(M)$ by $\mathrm{H}^i_{I,J}(M)$, the Corollaries 2.7 and 2.8 are true for local cohomology modules defined by a pair of ideals. Because by [24, Definition 3.1 and Theorem 3.2], it is easy to see that the local cohomology modules defined by a pair of ideals is a special case of local cohomology modules with respect to a system of ideals.*

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