A hybrid power mean of the two-term exponential sums 
and the reciprocal of the quartic Gauss sums 
by 
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Abstract

The main purpose of this paper is using the elementary methods and the properties 
of the classical Gauss sums to study the computational problems of two kinds hybrid 
power mean of the two-term exponential sums and the reciprocal of quartic Gauss 
sums, and give two interesting computational formulae for them.

Key Words: The reciprocal of the quartic Gauss sums, two-term exponential 
sums, hybrid power mean, computational formula.

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1 Introduction

The study of the properties of exponential sums is a very important content in analytic 
number theory, many important number theory problems are closely related to it. Therefore, 
the exponential sums has aroused the attention and interest of many number theorists. 
In this paper, we also work on these contents. First we give the definitions of two kinds 
exponential sums. Let \( q \geq 3 \) be a positive integer. For any positive integer \( k \geq 2 \) and 
integer \( m \) with \( (m, q) = 1 \), the \( k \)-th Gauss sums \( G(m, k; q) \) and the two-term exponential 
sums \( E(m, k; q) \) are defined as

\[
G(m, k; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k}{q}\right)
\]

and

\[
H(m, k; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right),
\]

where \( e(y) = e^{2\pi iy} \).

About the upper bound estimates of \( G(m, k; q) \) and \( H(m, k; q) \), some authors had studied 
them, and obtained a series of interesting results, see [2]-[8]. For example, from the A. Weil’s 
important work [2], one can get the best upper bound estimates

\[
\left| \sum_{a=0}^{p-1} e\left(\frac{ma^k}{p}\right) \right| \ll_k \sqrt{p} \quad \text{and} \quad \left| \sum_{a=0}^{p-1} e\left(\frac{ma^k + a}{p}\right) \right| \ll_k \sqrt{p},
\]

where \( p \) be a prime, and \( \ll_k \) denotes the big-\( O \) constant depend only on \( k \).
Shen Shimeng and Zhang Wenpeng [3] studied the recursive properties of the quartic Gauss sums, and proved a fourth-order linear recursive formula for it. Zhang Wenpeng and Han Di [4] studied the sixth power mean of the two-term exponential sums, and obtained an exact computational formula.

But for the lower bound estimates of $G(m, k; q)$ and $H(m, k; q)$, we still do not know how large they are?

About this aspect of the work, it seems that none had studied it yet, at least we have not seen any related papers before. In this paper, we will consider the following two kind hybrid power means:

$$
\sum_{m=1}^{p-1} \left| \frac{H(m, 3; p)}{G(m, 4; p)} \right|^k \quad \text{and} \quad \sum_{m=1}^{p-1} \frac{1}{\left| G(m, 4; p) \right|^k}.
$$

It is clear that if one can get an exact computational formula for (1.1), then we can also get some information about (1.1) more or less. Therefore, the research on the hybrid power mean (1.1) is a meaningful topic.

This paper, we will use the elementary methods and the properties of the classical Gauss sums to study these problems, and prove two interesting identities for mean value (1.1). That is, we will prove the following two results:

**Theorem 1.** Let $p$ be a prime with $p \equiv 5 \mod 8$, then we have the identity

$$
\sum_{m=1}^{p-1} \left| \frac{H(m, 3; p)}{G(m, 4; p)} \right|^2 = \begin{cases} 
\frac{3p(p - 2) - 2\sqrt{p}\alpha}{9p - 4\alpha^2} & \text{if } 3 \mid (p - 1), \\
\frac{3p^2 + 2\sqrt{p}\alpha}{9p - 4\alpha^2} & \text{if } 3 \nmid (p - 1),
\end{cases}
$$

where $\alpha = \alpha(p) = \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right)$ is an integer with $\alpha \neq 0$, $\left( \frac{a}{p} \right)$ denotes the Legendre symbol mod $p$, and $\overline{a}$ denotes the solution of the equation $ax \equiv 1 \mod p$.

**Theorem 2.** Let $p$ be a prime with $p \equiv 5 \mod 8$, then for any real number $k \geq 0$, we have the identity

$$
\sum_{m=1}^{p-1} \frac{1}{\left| G(m, 4; p) \right|^2} = \frac{p - 1}{2} \cdot \frac{(3\sqrt{p} + 2\alpha)^k + (3\sqrt{p} - 2\alpha)^k}{p^2 \cdot (9p - 4\alpha^2)^k}.
$$

Taking $k = 1$, from Theorem 2 we may immediately deduce the following two interesting corollaries:

**Corollary 1.** Let $p$ be a prime with $p \equiv 5 \mod 8$, then we have the identity

$$
\sum_{m=1}^{p-1} \frac{1}{\left| G(m, 4; p) \right|^2} = \frac{3(p - 1)}{9p - 4\alpha^2}.
$$

**Corollary 2.** Let $p$ be a prime with $p \equiv 5 \mod 8$, then we have the estimate

$$
\max_{1 \leq m \leq p-1} \left| \sum_{a=0}^{p-1} e \left( \frac{ma^3}{p} \right) \right| \geq \sqrt{\frac{3p - 4}{3} \cdot \alpha^2}.
$$
Some notes: First, if \( p \) is a prime with \( p \equiv 3 \mod 4 \), then we have the identity

\[
|G(m, 4; p)|^2 = \left| \sum_{a=0}^{p-1} \left( 1 + \left( \frac{a}{p} \right) \right) e \left( \frac{ma^2}{p} \right) \right|^2 = \left| \sum_{a=0}^{p-1} e \left( \frac{ma^2}{p} \right) \right|^2 = p.
\]

So in this case, the problems we are studying is trivial.

Second, the constant \( \alpha = \alpha(p) \) is an integer, and it closely related to prime \( p \). In fact, we have a very important square-sum formula

\[
p = \left( \sum_{a=1}^{\frac{p-1}{2}} \left( \frac{a + \pi}{p} \right) \right)^2 + \left( \sum_{a=1}^{\frac{p-1}{2}} \left( \frac{ra + \pi}{p} \right) \right)^2 \equiv \alpha^2 + \beta^2,
\]

where \( r \) be any integer with \( \left( \frac{r}{p} \right) = -1 \) (see Theorem 4-11 in [9]).

Third, for prime \( p \) with \( p \equiv 5 \mod 8 \), it is clear that our Corollary 2 obtained a nontrivial lower bound estimate for \( \max_{1 \leq m \leq p-1} |G(m, 3; p)| \).

Finally, if \( p \) is a prime with \( p \equiv 1 \mod 8 \), then the situation is more complicated, we can not give an exact computational formula that looks like Theorem 1 or Theorem 2. These are two open problems. We need further study.

2 Several Lemmas

This section, we will give several simple lemmas, which is necessary in the proofs of our theorems. Hereinafter, we shall use some elementary number theory knowledge and the properties of the classical Gauss sums, all of them can be found in reference [1], so they will not be repeated here.

**Lemma 1.** Let \( p \) be an odd prime, then we have the identity

\[
\sum_{m=1}^{p-1} |H(m, 3; p)|^2 = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right|^2 = \begin{cases} p(p-2) & \text{if } 3 \mid (p-1), \\ p^2 & \text{if } 3 \nmid (p-1), \end{cases}
\]

**Proof.** First if \( p \equiv 1 \mod 3 \), then the congruence equation \( x^3 \equiv 1 \mod p \) has three distinct integer solutions in a reduced residue system mod \( p \). So from the properties of the trigonometric sums and reduced residue system mod \( p \) we have

\[
\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right|^2 = \sum_{m=1}^{p-1} \left| 1 + \sum_{a=1}^{p-1} e \left( \frac{ma^3 + a}{p} \right) \right|^2
\]

\[
= \sum_{m=1}^{p-1} \left( 1 + 2 \sum_{a=1}^{p-1} e \left( \frac{ma^3 + a}{p} \right) + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e \left( \frac{mb^3(a^3-1) + b(a-1)}{p} \right) \right)
\]

\[
= (p-1) + 2 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e \left( \frac{ma^3 - 1 + b(a-1)}{p} \right)
\]
Now Lemma 1 follows from (2.1) and (2.2).

If $3 \nmid (p - 1)$, then the congruence equation $x^3 \equiv 1 \pmod{p}$ has only one solution in a reduced residue system mod $p$. So this time we have

$$
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) = \sum_{m=1}^{p-1} \frac{ma^3 + a}{p} = (p - 1) + 2 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} e\left(\frac{m(a^3 - 1) + b(a - 1)}{p}\right)
$$

$$
= p + 1 + (p - 1)^2 + (p - 2) = p^2. \tag{2.2}
$$

Now Lemma 1 follows from (2.1) and (2.2). \hfill \Box

**Lemma 2.** Let $p$ be prime with $p \equiv 5 \pmod{8}$, then we have the identity

$$
\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \cdot |H(m, 3; p)|^2 = \sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) = -\left(\frac{3}{p}\right) \cdot p,
$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol mod $p$.

**Proof.** Note that $H(m, 3; p)$ is a real number and $\chi^3_2 = \chi_2$, so from the definition and properties of the classical Gauss sums mod $p$ we have

$$
\sum_{m=1}^{p-1} \chi_2(m) \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) = \sum_{m=1}^{p-1} \chi_2(m) \left(1 + 2 \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{mb^3(a^3 - 1) + b(a - 1)}{p}\right)\right)
$$

$$
= \sum_{m=1}^{p-1} \chi_2(m) + 2\tau(\chi_2) \sum_{a=1}^{p-1} \chi_2(a^3) e\left(\frac{a}{p}\right) + \tau(\chi_2) \sum_{a=1}^{p-1} \chi_2(a^3 - 1) \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b(a - 1)}{p}\right)
$$

$$
= 2\tau^2(\chi_2) + \tau^2(\chi_2) \sum_{a=1}^{p-1} \chi_2(a^3-1) \chi_2(a-1)
$$

$$
= 2\tau^2(\chi_2) + \tau^2(\chi_2) \sum_{a=2}^{p-1} \chi_2(a^2 + a + 1). \tag{2.3}
$$

Since $p \equiv 1 \pmod{4}$, so we have $\tau(\chi_2) = \sqrt{p}$ and

$$
\sum_{a=2}^{p-1} \frac{a^2 + a + 1}{p} = \sum_{a=0}^{p-1} \left(\frac{4a^2 + 4a + 4}{p}\right) - 1 - \left(\frac{3}{p}\right)
$$
Zhang Wenpeng, Chen Zhuoyu

\[ \begin{align*}
= & \sum_{a=0}^{p-1} \left( \frac{(2a+1)^2 + 3}{p} \right) - 1 - \left( \frac{3}{p} \right) = \sum_{a=0}^{p-1} \left( \frac{a^2 + 3}{p} \right) - 1 - \left( \frac{3}{p} \right) \\
= & -2 - \left( \frac{3}{p} \right). 
\end{align*} \]

(2.4)

From (2.3) and (2.4) we may immediately deduce the identity

\[ \sum_{m=1}^{p-1} \left( \frac{a}{m} \right)^2 \sum_{a=0}^{p-1} e\left( \frac{ma^3 + a}{p} \right) = - \left( \frac{3}{p} \right) \cdot p. \]

This proves Lemma 2.

Lemma 3. Let \( p \) be an odd prime with \( p \equiv 1 \mod 4 \), \( \psi \) be any fourth-order character mod \( p \). Then we have the identity

\[ \tau^2(\psi) + \tau^2(\bar{\psi}) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left( \frac{a + \pi}{p} \right) = 2 \sqrt{p} \cdot \alpha. \]

Proof. See Lemma 2.2 in [8]. \( \text{\textendash} \) qed

Lemma 4. Let \( p \) be an odd prime, then for any real number \( k \geq 0 \) and \( x \) with \( |x| < 1 \), we have

\[ \sum_{n=0}^{p-1} \left( \frac{1}{1 - \chi_2(m)x} \right)^k = \frac{p - 1}{2} \cdot \frac{(1 - x)^k + (1 + x)^k}{(1 - x^2)^k}, \]

where \( \chi_2 \) be the Legendre symbol mod \( p \).

Proof. Without loss of generality we can assume that \( k > 0 \). Let the power series expansion of \( 1/(1 - x)^k \) and \( 1/(1 + x)^k \) are

\[ \frac{1}{(1 - x)^k} = \sum_{n=0}^{\infty} a_n x^n, \quad \frac{1}{(1 + x)^k} = \sum_{n=0}^{\infty} a_n (-1)^n x^n. \]

(2.5)

For any integer \( m \) with \( (m, p) = 1 \), note that \( \chi_2^2(m) = 1 \), from (6) we have

\[ \frac{1}{(1 - \chi_2(m)x)^k} = \sum_{n=0}^{\infty} a_n \chi_2^2(m) x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \chi_2(m) \sum_{n=1}^{\infty} a_{2n-1} x^{2n-1} \]

(2.6)

and

\[ \frac{1}{(1 + \chi_2(m)x)^k} = \sum_{n=0}^{\infty} a_n (-1)^n \chi_2^2(m) x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} - \chi_2(m) \sum_{n=1}^{\infty} a_{2n-1} x^{2n-1}. \]

(2.7)

Combining (2.5), (2.6) and (2.7) we have

\[ \frac{1}{(1 - \chi_2(m)x)^k} + \frac{1}{(1 + \chi_2(m)x)^k} = 2 \sum_{n=0}^{\infty} a_{2n} x^{2n} = \frac{1}{(1 + x)^k} + \frac{1}{(1 - x)^k}. \]

(2.8)
On the other hand, we also have

\[
\sum_{m=1}^{p-1} \frac{1}{(1 + \chi_2(m)x)^k} = (p - 1) \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{m=1}^{p-1} \chi_2(m) \sum_{n=1}^{\infty} a_{2n-1} x^{2n-1}
\]

\[
= (p - 1) \sum_{n=0}^{\infty} a_{2n} x^{2n}.
\]

(2.9)

Applying (2.8) and (2.9) we can deduce the identity

\[
\sum_{m=1}^{p-1} \frac{1}{(1 - \chi_2(m)x)^k} = p \frac{1}{2} \left( \frac{1}{(1 - x)^k} + \frac{1}{(1 + x)^k} \right) = \frac{p - 1}{2} \frac{(1 - x)^k + (1 + x)^k}{(1 - x^2)^k}.
\]

This proves Lemma 4. \qed

3 Proofs of the theorems

Now we will complete the proofs of our main results. First we prove Theorem 1. For any prime $p$ with $p \equiv 5 \mod 8$, let $\psi$ be a fourth-order character mod $p$, then $\psi(-1) = -1$. For any integer $m$ with $(m, p) = 1$, from the properties of the classical Gauss sums we have

\[
\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = \sum_{a=0}^{p-1} \left(1 + \psi(a) + \psi(a^2) + \psi(a^3)\right) = \chi_2(m) \sqrt{p} + \bar{\psi}(m) \tau(\psi) + \psi(m) \tau(\bar{\psi}).
\]

(3.1)

Since $\overline{\psi}(m) \tau(\psi) + \psi(m) \tau(\overline{\psi}) = - \left(\overline{\psi}(m) \tau(\psi) + \psi(m) \tau(\overline{\psi})\right)$ is a pure imaginary number, and $\chi_2(m) \sqrt{p}$ is a real number. So from (3.1), Lemma 3 and note that $\psi^2 = \chi_2 = \overline{\psi}^2$ we have

\[
|G(m, 4; p)|^2 = \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 = p + |\overline{\psi}(m) \tau(\psi) + \psi(m) \tau(\overline{\psi})|^2
\]

\[
= 3p - \chi_2(m) \left(\tau^2(\psi) + \tau^2(\overline{\psi})\right) = 3p - 2\chi_2(m) \sqrt{p} \alpha.
\]

(3.2)

Note that the estimate $|\alpha| < \sqrt{p}$, so $\left|\frac{2\chi_2(m) \alpha}{3\sqrt{p}}\right| < 1$, from (3.2) and the properties of the geometric series we have

\[
\frac{1}{|G(m, 4; p)|^2} = \frac{1}{3p \left(1 - \frac{2\chi_2(m) \alpha}{3\sqrt{p}}\right)} = \frac{1}{3p} \sum_{i=0}^{\infty} \left(\frac{2\chi_2(m) \alpha}{3\sqrt{p}}\right)^i
\]

\[
= \frac{1}{3p} \sum_{i=0}^{\infty} \left(\frac{4\alpha^2}{9p}\right)^i + \frac{\chi_2(m)}{3p} \sum_{i=1}^{\infty} \left(\frac{2\alpha}{3\sqrt{p}}\right)^{2i-1}
\]

\[
= \frac{3}{9p - 4\alpha^2} + \chi_2(m) \cdot \frac{2\alpha}{\sqrt{p}} \cdot \frac{1}{9p - 4\alpha^2}.
\]

(3.3)
From (3.3), Lemma 1 and Lemma 2 we may immediately deduce the identity

\[
\sum_{m=1}^{p-1} |H(m, 3; p)|^2 = \sum_{m=1}^{p-1} \left( \frac{3|H(m, 3; p)|^2}{9p - 4\alpha^2} + \frac{2\alpha}{\sqrt{p}} \cdot \frac{\chi_2(m)}{9p - 4\alpha^2} \right)
\]

\[
= \frac{3}{9p - 4\alpha^2} \sum_{m=1}^{p-1} |H(m, 3; p)|^2 + \frac{2\alpha}{\sqrt{p}} \cdot \frac{1}{9p - 4\alpha^2} \sum_{m=1}^{p-1} \chi_2(m)|H(m, 3; p)|^2
\]

\[
= \begin{cases} 
\frac{3(p-2)}{9p - 4\alpha^2} - \left( \frac{3}{p} \right) \cdot \frac{2\sqrt{\alpha}}{9p - 4\alpha^2} & \text{if } 3 \mid (p - 1), \\
\frac{3p^2}{9p - 4\alpha^2} - \left( \frac{3}{p} \right) \cdot \frac{2\sqrt{\alpha}}{9p - 4\alpha^2} & \text{if } 3 \nmid (p - 1).
\end{cases}
\] (3.4)

It is clear that if \(3 \mid (p - 1)\) and \(8 \mid (p - 5)\), then \((\frac{3}{p}) = (\frac{9}{3}) = 1\). If \(3 \nmid (p - 1)\) and \(8 \mid (p - 5)\), then \((\frac{3}{p}) = (\frac{9}{3}) = -1\). From these properties and (3.4) we may immediately deduce Theorem 1.

Now we prove Theorem 2. If \(k = 0\), then the conclusion is obvious. For any real number \(k > 0\), from (3.2) and Lemma 4 we have

\[
\sum_{m=1}^{p-1} \frac{1}{|G(m, 4; p)|^{2k}} = \sum_{m=1}^{p-1} \frac{1}{(3p - \chi_2(m)2\sqrt{p})^k} = \frac{1}{(3p)^k} \sum_{m=1}^{p-1} \frac{1}{(1 - \chi_2(m)\frac{2\alpha}{3\sqrt{p}})^k}
\]

\[
= \frac{1}{(3p)^k} \cdot \frac{p-1}{2} \left( \frac{1}{(1 - 2\alpha\frac{2\alpha}{3\sqrt{p}})^k} + \frac{1}{(1 + 2\alpha\frac{2\alpha}{3\sqrt{p}})^k} \right)
\]

\[
= \frac{p-1}{2} \cdot \frac{(3\sqrt{p} + 2\alpha)^k + (3\sqrt{p} - 2\alpha)^k}{p \cdot (9p - 4\alpha^2)^k}.
\]

This completes the proof of Theorem 2.

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References

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