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Some bounds for the largest eigenvalue of a signed graph

by

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Abstract

For a vertex i of a signed graph, let d_i, m_i and T_i^- denote its degree, average 2-degree and the number of unbalanced triangles passing through i, respectively. We prove that

$$\rho \leq \max\left\{\frac{-d_i+\sqrt{5d_i^2+4(d_im_i-4T_i^-)}}{2}\ :\ 1\leq i\leq n\right\},$$

where ρ stands for the largest eigenvalue. This bound is tight at least for regular signed graphs that are switching equivalent to their underlying graphs. We also derive a general lower bound for ρ and certain practical consequences. A discussion, including the cases of equality in inequalities obtained and some examples, is given.

Key Words: Signed graph, switching equivalence, index, vertex degree, net-balance.

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1 Introduction

A *signed graph* $\dot{G}$ is obtained from a (simple) graph G by accompanying each edge e by the sign $\sigma(e)\in\{1,-1\}$ (chosen in any way for any edge). We say that G is the underlying graph of $\dot{G}$. The set of vertices of $\dot{G}$ is denoted by $V(\dot{G})$ and the number of vertices is denoted by n.

The $n\times n$ adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0,1)$-adjacency matrix of G by reversing the sign of all 1's which correspond to negative edges. The corresponding eigenvalues are real and form the spectrum of $\dot{G}$. The largest eigenvalue is called the index and denoted by ρ. For U subset of the vertex set, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of each edge between a vertex in U and a vertex in $V(\dot{G})\setminus U$. The signed graph $\dot{G}^U$ is said to be *switching equivalent* to $\dot{G}$. The switching equivalence is an equivalence relation and switching equivalent signed graphs share the same spectrum.

In this paper we focus on presentation of our contribution, and so avoid the details about signed graphs and a list of known results. For those who need a more detailed approach, we refer to [4, 6, 7].

Continuing our research on the index of a signed graph reported in [2, 4, 6], we establish a sharp upper bound for this eigenvalue, and also a general lower bound expressed in terms of walks of an arbitrary length. Using the latter bound we derive some particular results. After introducing the notation, we present our results.

2 Notation

We denote by d_i the degree of a vertex $i \in V(\dot{G})$; in particular, Δ stands for the maximum vertex degree in $\dot{G}$. We also write d_i^+ and d_i^- for the positive and negative vertex degree (i.e., the number of positive and negative edges incident with i). If the vertices i and j are adjacent, then we write $i \sim j$. The existence of a positive (resp. negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ (resp. $i \overset{-}{\sim} j$). For $d_i > 0$, we denote

$$m_i^+ = \frac{1}{d_i}\left(\sum_{j \overset{+}{\sim} i} d_j^+ + \sum_{j \overset{-}{\sim} i} d_j^-\right),\ m_i^- = \frac{1}{d_i}\left(\sum_{j \overset{+}{\sim} i} d_j^- + \sum_{j \overset{-}{\sim} i} d_j^+\right)$$

and $m_i = m_i^+ + m_i^-$ (*the average* 2*-degree*, that is the average degree of the neighbours of i). For $d_i = 0$, we define all these parameters to be zero.

For (not necessary distinct) vertices i and j, we use c_{ij} to denote the number of their common neighbours (so, joined to both of them by any edge), c_{ij}^+ for the number of common neighbours joined to i by a positive edge and to j by any edge, and c_{ij}^{++} for the number common neighbours that are joined to both of them by a positive edge. We also use the similar notation for all the remaining possibilities.

A *walk* in a (signed) graph is a sequence of alternate vertices and edges such that the consecutive vertices are the endpoints of the corresponding edge. A walk in a signed graph is *balanced* if the number of negative edges contained is not odd. Otherwise, it is unbalanced. We use $w_k^+(i,j)$ and $w_k^+(i)$ to denote the number of balanced walks of length k starting at i and terminating at j and the number of balanced walks of length k starting at i, respectively, and similarly for the numbers of unbalanced ones. Since every cycle can be considered as a walk, we may talk about balanced or unbalanced cycles. In particular, the number of unbalanced cycles with 3 vertices (triangles) passing through a vertex i is denoted by T_i^-.

The *net-balance* ϱ_i of a vertex i is defined by $\varrho_i = d_i^+ - d_i^-$. We say that a signed graph is *net-balanced* if the net-balance is a constant on the vertex set. Following this concept, we also denote $m_i^+ - m_i^-$ by ϱ_{m_i} and $w_k(i)^+ - w_k(i)^-$ by $\varrho_{w_k(i)}$.

3 Results

Theorem 1. *For the index ρ of a signed graph,*

$$\rho \leq \max\left\{\frac{-d_i + \sqrt{5d_i^2 + 4(d_i m_i - 4T_i^-)}}{2}\ :\ 1 \leq i \leq n\right\}. \tag{3.1}$$

Proof. Let $\mathbf{x}$ be an eigenvector associated with ρ and x_i a coordinate that is largest in modulus. Without loss of generality, we may assume that $x_i = 1$. Considering the eigenvalue equation for x_i, we obtain

$$\begin{aligned}\rho &= \sum_{k \overset{+}{\sim} i} x_k - \sum_{k \overset{-}{\sim} i} x_k \\ &= \sum_{\substack{k \overset{+}{\sim} i \\ k \sim j}} x_k - \sum_{\substack{k \overset{-}{\sim} i \\ k \sim j}} x_k + \sum_{\substack{k \overset{+}{\sim} i \\ k \nsim j}} x_k - \sum_{\substack{k \overset{-}{\sim} i \\ k \nsim j}} x_k,\end{aligned} \tag{3.2}$$

where j is a neighbour of i. And similarly for x_j:

$$\rho x_j = \sum_{\substack{k \overset{+}{\sim} j \\ k \sim i}} x_k - \sum_{\substack{k \overset{-}{\sim} j \\ k \sim i}} x_k + \sum_{\substack{k \overset{+}{\sim} j \\ k \nsim i}} x_k - \sum_{\substack{k \overset{-}{\sim} j \\ k \nsim i}} x_k. \tag{3.3}$$

Summing (3.2) and (3.3), we arrive at

$$\begin{aligned}\rho(1+x_j) &= 2\sum_{\substack{k \overset{+}{\sim} i \\ k \overset{+}{\sim} j}} x_k - 2\sum_{\substack{k \overset{-}{\sim} i \\ k \overset{-}{\sim} j}} x_k + \sum_{\substack{k \overset{+}{\sim} i \\ k \nsim j}} x_k - \sum_{\substack{k \overset{-}{\sim} i \\ k \nsim j}} x_k + \sum_{\substack{k \overset{+}{\sim} j \\ k \nsim i}} x_k - \sum_{\substack{k \overset{-}{\sim} j \\ k \nsim i}} x_k, \\ &\leq 2c_{ij}^{++} + 2c_{ij}^{--} + d_i^+ - 2c_{ij}^+ + d_i^- - c_{ij}^- + d_j^+ - c_{ji}^+ + d_j^- - c_{ji}^- \\ &= 2(c_{ij}^{++} + c_{ij}^{--}) + d_i + d_j - 2c_{ij} \\ &= d_i + d_j - 2(c_{ij}^{+-} + c_{ij}^{-+}).\end{aligned} \tag{3.4}$$

Taking the summation over all j such that $j \overset{+}{\sim} i$, we get

$$\rho\left(d_i^+ + \sum_{j \overset{+}{\sim} i} x_j\right) \leq d_i^+ d_i + \sum_{j \overset{+}{\sim} i}\big(d_j - 2(c_{ij}^{+-} + c_{ij}^{-+})\big). \tag{3.5}$$

Acting in the same manner, by subtracting (3.3) from (3.2), we get

$$\begin{aligned}\rho(1-x_j) &\leq 2c_{ij}^{+-} + 2c_{ij}^{-+} + d_i^+ - c_{ij}^+ + d_i^- - c_{ij}^- + d_j^+ - c_{ji}^+ + d_j^- - c_{ji}^- \\ &= d_i + d_j - 2(c_{ij}^{++} + c_{ij}^{--}),\end{aligned} \tag{3.6}$$

while by taking the summation over all j such that $j \overset{-}{\sim} i$, we get

$$\rho\left(d_i^- - \sum_{j \overset{-}{\sim} i} x_j\right) \leq d_i^- d_i + \sum_{j \overset{-}{\sim} i}\big(d_j - 2(c_{ij}^{++} + c_{ij}^{--})\big). \tag{3.7}$$

Summing (3.5) and (3.7), we arrive at

$$\begin{aligned}\rho\left(d_i^+ + d_i^- + \sum_{j \overset{+}{\sim} i} x_j - \sum_{j \overset{-}{\sim} i} x_j\right) &\leq (d_i^+ + d_i^-)d_i + \sum_{j \sim i} d_j \\ &\quad + 2\left(\sum_{j \overset{+}{\sim} i}(c_{ij}^{+-} + c_{ij}^{-+}) + \sum_{j \overset{-}{\sim} i}(c_{ij}^{++} + c_{ij}^{--})\right),\end{aligned}$$

i.e.,

$$d_i\rho+\rho^2\leq d_i^2+d_im_i-4T_i^-,$$

which leads to

$$\rho\leq\frac{-d_i+\sqrt{5d_i^2+4(d_im_i-4T_i^-)}}{2}.$$

Taking the maximum over all vertices, we arrive at (3.1). ∎

In our approach, a signed graph is regular if and only if its underlying graph is regular. If $\dot{G}$ is a regular signed graph switching equivalent to its underlying graph, then the equality in (3.1) is attained; this follows directly. Conversely, if the equality holds, then the equality in (3.4) must hold for all $j \overset{+}{\sim} i$ and the equality in (3.6) must hold for all $j \overset{-}{\sim} i$. If, in addition, $\dot{G}$ does not contain unbalanced triangles, then we conclude that

$$x_k=\begin{cases} x_i & \text{if } k\overset{+}{\sim} i,\\ -x_i & \text{if } k\overset{-}{\sim} i,\end{cases}$$

which means that all the coordinates of $\mathbf{x}$ are equal in modulus. If $A_{\dot{G}}$ is the adjacency matrix of $\dot{G}$ and D is the $n\times n$ diagonal matrix with diagonal entries

$$d_{kk}=\begin{cases} 1 & \text{if } x_k=1,\\ -1 & \text{if } x_k=-1,\end{cases}$$

then $D^{-1}A_{\dot{G}}D$ is the adjacency matrix of a switching equivalent graph (so, all the edges being positive) and the all-1 eigenvector $D\mathbf{x}$ corresponds to the index. Thus, this graph is regular, and consequently $\dot{G}$ is regular and switching equivalent to its underlying graph. The case when our signed graph has unbalanced triangles seems to be much harder. (Observe that the imposed equalities do not give any condition for x_k when i and k belong to such a triangle.)

Since $T_i^-\geq 0$ and $d_im_i\leq\Delta^2$, we conclude that the bound (3.1) does not exceed the value obtained for $T_i^-=0$ and $d_im_i=\Delta^2$, i.e., does not exceed the maximum vertex degree Δ, which makes it always non-trivial.

In particular case of graphs, the bound (3.1) reduces to the bound that can be expressed in the same way with the term concerning unbalanced triangles removed. This result is less important, since we already know a finer bound $\rho\leq\max\{\sqrt{d_im_i}\ :\ 1\leq i\leq n\}$ [1, 5].

Proceed now with lower bounds.

Theorem 2. *For the index of a signed graph $\dot{G}$,*

$$\rho^k\sum_{i=1}^{n}\varrho_{w_r(i)}^2\geq\sum_{1\leq i,j\leq n}\varrho_{w_r(i)}\varrho_{w_r(j)}\varrho_{w_k(i,j)}. \tag{3.8}$$

Proof. Applying the Rayleigh principle with respect to the vector $\mathbf{y}$ whose ith coordinate is equal to $\varrho_{w_r(i)}$ $(=w_r^+(i)-w_r^-(i))$, we get

$$\rho^k\mathbf{y}^T\mathbf{y}\geq\mathbf{y}^TA_{\dot{G}}^k\mathbf{y}, \tag{3.9}$$

which, since the (i,j)-entry of $A_{\dot{G}}^k$ is equal to $\varrho_{w_k(i,j)}$, leads to (3.8). ∎

If $\varrho_{w_r(i)} \neq 0$ holds for at least one vertex i, then the inequality (3.8) can be written as

$$\rho^k \geq \frac{1}{\sum_{i=1}^{n} \varrho^2_{w_r(i)}} \sum_{1 \leq i,j \leq n} \varrho_{w_r(i)} \varrho_{w_r(j)} \varrho_{w_k(i,j)}.$$

A similar result for graphs can be found in [3]. Considering the block matrix of the form

$$\begin{pmatrix} I-J & J-I \\ J-I & I-J \end{pmatrix}$$

(where I and J stand for the unit and the all-1 matrix of equal size, respectively), we obtain an example of the signed graph with $\varrho_{w_r(i)} = 0$, for all i.

Here is a corollary.

Corollary 1. *For the index ρ of a signed graph,*

$$\rho \sum_{i=1}^{n} \varrho_i^2 \geq 2 \sum_{i \sim j} \varrho_i \varrho_j \sigma(ij),$$

$$\rho^2 \sum_{i=1}^{n} \varrho_i^2 \geq \sum_{1 \leq i,j \leq j} \varrho_i \varrho_j (c_{ij}^{++} + c_{ij}^{--} - c_{ij}^{+-} - c_{ij}^{-+})$$

and

$$\rho^2 \sum_{i=1}^{n} d_i^2 \varrho_{m_i}^2 \geq \sum_{1 \leq i,j \leq j} d_i d_j \varrho_{m_i} \varrho_{m_j} (c_{ij}^{++} + c_{ij}^{--} - c_{ij}^{+-} - c_{ij}^{-+}).$$

Proof. Since $\varrho_{w_1(i)} = d_i^+ - d_i^- = \varrho_i$,

$$\varrho_{w_1(i,j)} = \begin{cases} \sigma(ij) & \text{if } i \sim j, \\ 0 & \text{otherwise,} \end{cases}$$

$\varrho_{w_2(i)} = d_i(m_i^+ - m_i^-) = d_i \varrho_{m_i}$ and $\varrho_{w_2(i,j)} = c_{ij}^{++} + c_{ij}^{--} - c_{ij}^{+-} - c_{ij}^{-+}$, the desired inequalities are obtained by setting (in (3.8)) $(r,k) = (1,1)$, $(r,k) = (1,2)$ and $(r,k) = (2,2)$, respectively. □

Lower bounds from the previous corollary are attained for the first graph illustrated in Fig. 1. This is unsurprising because the bound (3.8) is attained whenever $\mathbf{y}$ is an eigenvector associated with ρ. In particular, if $\mathbf{y}$ is a constant vector, then it is associated with ρ if and only if $\dot{G}$ is net-balanced and its index is equal to the net-balance [4]. This occurs in the case of our signed graph. The second signed graph illustrated is also net-balanced, its net-balance occurs in its spectrum, but not as the largest eigenvalue. Consequently, none of mentioned bounds is attained.

We conclude by a practical range for ρ.

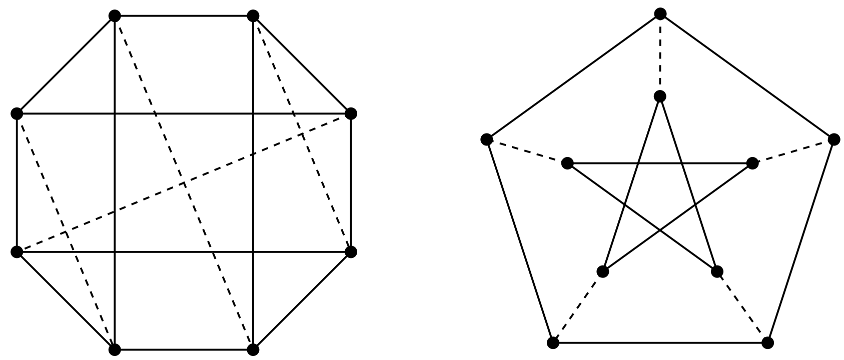

Figure 1: Two net-balanced signed graphs. (Negative edges are dashed.)

Theorem 3. *If $\dot{H}$ is a spanning subgraph induced by all the positive edges of a sigend graph $\dot{G}$, then*

$$\rho(\dot{H}) - \Delta^- \leq \rho(\dot{G}) \leq \rho(\dot{H}) + \Delta^-,$$

where Δ^- stands for the maximum negative vertex degree in $\dot{G}$.

Proof. Since $A_{\dot{G}} = P + N$, where P and N are the submatrices of the same size induced by positive and negative edges, we have $\rho(P) - \rho(-N) \leq \rho(\dot{G}) \leq \rho(P) + \rho(N)$ (as follows by Courant-Weyl inequalities, see also [6]). Therefore, we have

$$\begin{aligned} &\rho(\dot{G}) \geq \rho(P) - \rho(-N) \geq \rho(P) - \Delta^-, \\ &\rho(\dot{G}) \leq \rho(P) + \rho(N) = \rho(P) - \lambda_n(-N) \leq \rho(P) + \rho(-N) \leq \rho(P) + \Delta^-, \end{aligned}$$

giving the result. ∎

A bound for the index of a signed graph (in particular, the range obtained in the previous theorem) sometimes might not give a good estimate, but in such situations one should always bear in mind that an appropriate switching equivalent signed graph can be found.

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