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On Zygmund-type inequalities involving polar derivative of a lacunary-type polynomial

by
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Abstract

In this paper, we prove results on inequalities involving polar derivative of a polynomial with restricted zeros, which generalize and sharpen some of the known results.

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1 Introduction and statement of results

The inequalities for polynomials and related classes of functions are the important and crucial tools in obtaining inverse theorems in Approximation Theory. The progress in this domain mostly depends upon obtaining the corresponding analogue or generalization of Markov's and Bernstein's inequalities. These inequalities have been the starting point of a considerable literature in polynomial approximations, and many mathematicians have been working for last hundred years. Over a period, Bernstein's inequality and its variants concerning the growth of polynomials have been generalized in different domains, in different norms, and for different classes of functions. These type of inequalities have wide applications in Numerical Analysis, Polynomial Approximation Theory and their applications in allied sciences and engineering. Here we study some of the new inequalities centered around Bernstein-type inequalities for polar derivatives of polynomials. Let us begin with the famous Bernstein's inequality.

If $P(z)$ is a polynomial of degree n then from a well-known inequality due to Bernstein [4], we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is sharp and equality holds, if $P(z)$ has all its zeros at the origin. If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then Erdös conjectured and later Lax [11] proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible and equality holds for $P(z) = a + bz^n$, where $|a| = |b|$.

For polynomials having no zeros in $|z| > 1$ the corresponding inequality was proved by Turán [18].

Zygmund [19] extended the Bernstein's inequality (1.1) to L^p norm as

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq n\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{1.3}$$

for any polynomial $P(z)$ of degree n and for any $p \geq 1$. The result (1.3) is sharp and equality holds if $P(z)$ has all its zeros at the origin.

The above inequality of Zygmund was further extended by Arestov [1] for $0 < p < 1$. De Bruijn [5] proved an analogue of Zygmund's result for the class of polynomials having no zeros in the disc $|z| < 1$. Rahman and Schmeisser [15] showed that de Bruijn's result is true for all $p > 0$. Govil and Rahman[10] generalized and sharpened the above inequality due to de Bruijn for polynomials of degree n having all its zeros in $|z| \geq K \geq 1$, for any $p \geq 1$. Gardner and Weems [7] proved that the above result of Govil and Rahman[10] is true for $0 < p < 1$ by proving a theorem in more generalized form, which is presented below.

Theorem 1. *If $P(z) = a_0 + \sum_{\nu=m}^{n} a_\nu z^\nu$ $1 \leq m \leq n$, is a polynomial of degree n having all its zeros in $|z| \geq K \geq 1$, then for any $p > 0$,*

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq nG_p\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{1.4}$$

where $G_p = \left\{\dfrac{2\pi}{\int_0^{2\pi}|t_{n,m}+e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$, *and* $t_{n,m} = \dfrac{n|a_0|K^{m+1}+m|a_m|K^{2m}}{n|a_0|+m|a_m|K^{m+1}}$.

As mentioned earlier, different versions of Bernstein's inequality have appeared in the literature in more generalized forms in which the underlying polynomials are replaced by more general class of functions. The one such generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative. The results presented here on polar derivatives are the natural extensions of the Bernstein's inequality appeared in the literature for the ordinary derivative of a complex polynomial. These inequalities have their own significance and beauty.

Let us introduce the concept of polar derivative now. If $P(z)$ is a polynomial of degree n, then the *polar derivative of $P(z)$ with respect to a complex number α* is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha\{P(z)\}$ is a polynomial of degree atmost $n-1$ and one could get the sense of 'generalization' from the fact that

$$\lim_{\alpha\to\infty}\frac{D_\alpha\{P(z)\}}{\alpha} = P'(z),$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.

Bernstein-type inequalities have been extended from 'ordinary derivative' to 'polar derivative' of complex polynomials, and a detailed chronological evolution of the results is presented in a recently published book chapter by Govil and Kumar [8]. In this paper, we extend the Theorem 1 involving ordinary derivative of a polynomial having all its zeros in $|z| \geq K \geq 1$, to the one in more generalized form, involving polar derivative of a polynomial having all its zeros in $|z| \geq K \geq 1$.

Theorem 2. *If $P(z)=a_0+\sum_{\nu=m}^{n}a_\nu z^\nu$ $1\le m\le n$, is a polynomial of degree n having all its zeros in $|z|\ge K\ge 1$, then for any $p>0$, and for every complex number α, with $|\alpha|\ge 1$,*

$$\left\{\int_0^{2\pi}|D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}}\le n(|\alpha|+t_{n,m})G_p\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{1.5}$$

where $G_p=\left\{\dfrac{2\pi}{\int_0^{2\pi}|t_{n,m}+e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$, and $t_{n,m}$ is as given in Theorem 1.

In the limiting case, when $p\to\infty$, the result is best possible and equality holds for the polynomial $(z+K)^n$ with real $\alpha\ge 1$, and $K\ge 1$.

It is equally interesting to get the analogous result involving polar derivative of a polynomial whose zeros all lie in $|z|\le K\le 1$, which is presented below.

Theorem 3. *If $P(z)=a_n z^n+\sum_{\nu=m}^{n}a_{n-\nu}z^{n-\nu}$ $0\le m\le n-1$, is a polynomial of degree n having all its zeros in $|z|\le K\le 1$, then for any complex α with $|\alpha|\le 1$, and $p>0$,*

$$\left\{\int_0^{2\pi}|D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}}\le n(1+|\alpha|s_{n,m})H_p\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{1.6}$$

where $H_p=\left\{\dfrac{2\pi}{\int_0^{2\pi}|s_{n,m}+e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$, and $s_{n,m}=\dfrac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}$. Again in the limiting case $p\to\infty$, the above inequality is sharp in the case $K\le 1, 1\le m\le n$ and equality holds for the polynomial $(z+K)^n$ with non-negative real $\alpha\le 1$ and $K\le 1$.

Corollary 1. *It is quite natural to see Theorem 1 is a special case of the result we presented in Theorem 2. If we divide inequality (1.5) by $|\alpha|$ and make $|\alpha|\to\infty$, Theorem 2 reduces to Theorem 1. In this way, our result is the extended version of the Theorem 1 due to Gardner and Weems.*

Corollary 2. *If we make $p\to\infty$, in Theorem 2, we get the following important result that, if $P(z)=a_0+\sum_{\nu=m}^{n}a_\nu z^\nu$ $1\le m\le n$, is a polynomial of degree n having all its zeros in $|z|\ge K\ge 1$, then for any $p>0$, and for every complex number α, with $|\alpha|\ge 1$,*

$$\max_{|z|=1}|D_\alpha\{P(z)\}|\le n\frac{(|\alpha|+t_{n,m})}{t_{n,m}+1}\max_{|z|=1}|P(z)|, \tag{1.7}$$

where $t_{n,m}$ is as given in Theorem 1.

The above Corollary 1.6 was independently proved by Dewan et al. [6] as well.

Corollary 3. *Note that the constant $t_{n,m}$ in Theorem 2 is always greater than or equal to K. Hence by taking $a=|\alpha|,\ b=t_{n,m},\ c=K,\ d=1$ in Lemma 5, it follows easily that*

$$(|\alpha|+t_{n,m})G_p\le(|\alpha|+K)F_p.$$

Using this in Theorem 2 we get a result that, if $P(z)$ is a polynomial of degree n having no zeros in $|z|<K$, $K\ge 1$ then for any $p>0$ and for every complex number α, with $|\alpha|\ge 1$,

$$\left\{\int_0^{2\pi}|D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}}\le n(|\alpha|+K)F_p\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{1.8}$$

where $F_p = \left\{ \frac{2\pi}{\int_0^{2\pi} |K + e^{i\theta}|^p d\theta} \right\}^{\frac{1}{p}}$. *The result is best possible and equality holds for the polynomial* $(z+K)^n$ *with real* $\alpha \geq 1$ *and* $K \geq 1$, *in the limiting case* $p \to \infty$.

Corollary 4. *Again, as is easy to observe that the constant* $s_{n,m}$ *in our Theorem 2 is always greater than or equal to* $\frac{1}{K}$. *Then taking* $a = \frac{1}{|\alpha|},\ b = s_{n,m},\ c = \frac{1}{K},\ d = 1$ *in Lemma 5, one easily arrives at*

$$(1 + |\alpha| s_{n,m}) H_p \leq (|\alpha| + K) F_p,$$

which establishes a result that if $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq K,\ \ K \leq 1$ *then for any* $p > 0$ *and for every complex number* α, *with* $|\alpha| \leq 1$,

$$\left\{ \int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta \right\}^{\frac{1}{p}} \leq n(|\alpha| + K) F_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}, \tag{1.9}$$

where F_p *is same as in Corollary 3. Again in the limiting case, when* $p \to \infty$, *the above inequality is sharp and equality holds for* $P(z) = (z+K)^n, K \leq 1$ *with non-negative real* $\alpha \leq 1$.

The results derived in Corollaries 3 and 4 were independently proved by Rather [17] (see also [16]).

Corollary 5. *If we make* $p \to \infty$ *in Theorem 3 we get the following interesting result. If* $P(z) = a_n z^n + \sum_{\nu=0}^{m} a_\nu z^\nu\ \ 0 \leq m \leq n-1$, *is a polynomial of degree* n *having all its zeros in* $0 < |z| \leq K \leq 1$, *then*

$$\max_{|z|=1} |D_\alpha\{P(z)\}| \leq n \frac{(1 + |\alpha| s_{n,m})}{1 + s_{n,m}} \max_{|z|=1} |P(z)|, \tag{1.10}$$

where $s_{n,m}$ *is as given in Theorem 3. the above inequality is sharp in the case* $K \leq 1, 1 \leq m \leq n$ *and equality holds for the polynomial* $(z+K)^n$ *with non-negative real* $\alpha \leq 1$ *and* $K \leq 1$.

2 Lemmas

We need following lemmas for proving our theorems. The following lemma is due to Govil and Kumar [9].

Lemma 1. *Let* $z_1,\ z_2$ *be two complex numbers independent of* α, *where* α *being real. Then for* $p > 0$

$$\int_0^{2\pi} |z_1 + z_2 e^{i\alpha}|^p d\alpha = \int_0^{2\pi} ||z_1| + |z_2| e^{i\alpha}|^p d\alpha. \tag{2.1}$$

The next Lemma is also given in the paper due to Govil and Kumar [9], but we present a different proof for it.

Lemma 2. *Let p, q be any two positive real numbers such that $p \geq qx$ where $x \geq 1$. If γ is any real such that $0 \leq \gamma \leq 2\pi$, then*

$$(p+qy)|x+e^{i\gamma}| \leq (x+y)|p+qe^{i\gamma}|, \tag{2.2}$$

for any $y \geq 1$.

Proof. We have

$$\left|\frac{x+e^{i\gamma}}{p+qe^{i\gamma}}\right|^2 = \frac{x^2+1+2x\cos\gamma}{p^2+q^2+2pq\cos\gamma}.$$

It is easily seen that the hypotheses $p \geq qx$, and $x \geq 1$, imply $\frac{x^2+1+2x\cos\gamma}{p^2+q^2 2pq\cos\gamma}$ is a non-decreasing function of $\cos\gamma$ and hence

$$\left|\frac{x+e^{i\gamma}}{p+qe^{i\gamma}}\right|^2 \leq \frac{x^2+1+2x}{p^2+q^2+2pq} = \left(\frac{x+1}{p+q}\right)^2,$$

which implies

$$\left|\frac{x+e^{i\gamma}}{p+qe^{i\gamma}}\right| \leq \left(\frac{x+1}{p+q}\right).$$

One can also easily check with the hypothesis that,

$$\left(\frac{x+1}{p+q}\right) \leq \left(\frac{x+y}{p+qy}\right).$$

Thus we have

$$\left|\frac{x+e^{i\gamma}}{p+qe^{i\gamma}}\right| \leq \left(\frac{x+y}{p+qy}\right),$$

which proves the lemma.

∎

Next lemma is due to Quazi [14, p. 338] (see also [3]).

Lemma 3. *If $P(z) = a_0 + \sum_{\nu=m}^{n} a_\nu z^\nu$ $1 \leq m \leq n$, is a polynomial of degree n having all its zeros in $|z| \geq K \geq 1$, then*

$$t_{n,m}|P'(z)| \leq |Q'(z)|,$$

on $|z| = 1$, where $t_{n,m} = \dfrac{n|a_0|K^{m+1} + m|a_m|K^{2m}}{n|a_0| + m|a_m|K^{m+1}}$, and $Q(z) = z^n\overline{P(1/\overline{z})}$.

The following result is due to Aziz (see [2]).

Lemma 4. *If $P(z)$ is a polynomial of degree n then for every complex α with $|\alpha| \neq 0$*

$$|D_\alpha\{Q(e^{i\theta})\}| = |\alpha||D_{\frac{1}{\overline{\alpha}}}\{P(e^{i\theta})\}|$$

where $Q(z)$ is as given in Lemma 3.

Lemma 5. *If $a \geq 1$, $b \geq c \geq 1$ and $p > 0$ then for any $1 \leq d \leq a$, we have*

$$(a+b)\left\{\int_0^{2\pi} |de^{i\theta}+c|^p d\theta|\right\}^{1/p} \leq (a+c)\left\{\int_0^{2\pi} |de^{i\theta}+b|^p d\theta\right\}^{1/p}. \tag{2.3}$$

Proof. If $b = c$, the result is trivial. It is a simple exercise that for any $a \geq 1$, $b > c \geq 1$, $1 \leq d \leq a$,

$$\left|\frac{de^{i\theta}+c}{de^{i\theta}+b}\right| \leq \frac{d+c}{d+b} \leq \frac{a+c}{a+b},$$

by which it follows that for any $p > 0$,

$$(a+b)^p|de^{i\theta}+c|^p \leq (a+c)^p|de^{i\theta}+b|^p.$$

Now the result follows by the well-known property of the integrals. □

Definition 1. *For $\gamma = (\gamma_0, \ldots, \gamma_n) \in C^{n+1}$ and $P(z) = \sum_{k=0}^{n} c_k z^k$, define*

$$\Lambda_\gamma P(z) = \sum_{k=0}^{n} \gamma_k c_k z^k.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:
(a) $P(z)$ has all its zeros in $\{z \in C : |z| \leq 1\}$.
(b) $P(z)$ has all its zeros in $\{z \in C : |z| \geq 1\}$.

Lemma 6. *Let $\phi(x) = \psi(\log x)$ where ψ is a convex non-decreasing function on $\mathbf{R}$. Then for all polynomials $P(z)$ of degree n and each admissible operator Λ_γ*

$$\int_0^{2\pi} \phi(|\Lambda_\gamma P(e^{i\theta})|)d\theta \leq \int_0^{2\pi} \phi(c(\gamma,n)|P(e^{i\theta})|)d\theta,$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

The proof of the above lemma was given by Arestov [1].

Next we state a famous theorem due to Laguerre [12].

Lemma 7. *If $P(z)$ is a polynomial of degree n having no zeros in the circular domain C and if $\eta \in C$ then*

$$(\eta - z)P'(z) + nP(z) \neq 0$$

for $z \in C$.

3 Proofs of the theorems

Proof of the Theorem 2. We have for any $p > 0$

$$\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta \int_0^{2\pi} |t_{n,m} + e^{i\gamma}|^p d\gamma$$
$$= \int_0^{2\pi}\int_0^{2\pi} |(t_{n,m} + e^{i\gamma})|^p |D_\alpha\{P(e^{i\theta})\}|^p d\theta d\gamma$$

$$= \int_0^{2\pi}\int_0^{2\pi} |(t_{n,m}+e^{i\gamma})|^p |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) + \alpha P'(e^{i\theta})|^p d\theta d\gamma$$
$$\leq \int_0^{2\pi}\int_0^{2\pi} |(t_{n,m}+e^{i\gamma})|^p \left[|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + |\alpha P'(e^{i\theta})|\right]^p d\theta d\gamma$$
$$\leq \int_0^{2\pi}\int_0^{2\pi} |(t_{n,m}+e^{i\gamma})|^p \left[|Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|\right]^p d\theta d\gamma,$$

which implies

$$\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta \int_0^{2\pi} |t_{n,m}+e^{i\gamma}|^p d\gamma$$
$$\leq \int_0^{2\pi}\int_0^{2\pi} \left|(t_{n,m}+e^{i\gamma})\left[|Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|\right]\right|^p d\theta d\gamma. \tag{3.1}$$

Since the zeros of $P(z)$ satisfy $|z| \geq K \geq 1$, it follows by Lemma 3, that on $|z| = 1$,

$$t_{n,m}|P'(e^{i\theta})| \leq |Q'(e^{i\theta})|.$$

The above by using Lemma 2 and the fact that $\left||Q'(e^{i\theta})| + e^{i\gamma}|P'(e^{i\theta})|\right| = \left||P'(e^{i\theta})| + e^{i\gamma}|Q'(e^{i\theta})|\right|$, clearly gives that for every α with $|\alpha| \geq 1$. we have

$$\left[|Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|\right] |t_{n,m}+e^{i\gamma}| \leq (t_{n,m}+|\alpha|)\left||P'(e^{i\theta})| + e^{i\gamma}|Q'(e^{i\theta})|\right|. \tag{3.2}$$

Now, if we use the above inequality (3.2) in (3.1), we get

$$\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta \int_0^{2\pi} |t_{n,m}+e^{i\gamma}|^p d\gamma$$
$$\leq (|\alpha| + t_{n,m})^p \int_0^{2\pi}\int_0^{2\pi} \left||P'(e^{i\theta})| + e^{i\gamma}|Q'(e^{i\theta})|\right|^p d\theta d\gamma. \tag{3.3}$$

Since $P(z)$ has no zeros in $|z| < 1$, by Lemma 7, it follows that for any complex number η, with $|\eta| < 1$, $nP(z) - (z-\eta)P'(z) \neq 0$ for $|z| < 1$.

Therefore setting $\eta = -ze^{-i\gamma}$, $\gamma \in \mathbf{R}$, the operator Λ defined by

$$\Lambda P(z) = (e^{i\gamma}+1)zP'(z) - ne^{i\gamma}P(z)$$

is admissible and so by Lemma 6 with $\psi(x) = e^{px}$, and for $p > 0$,

$$\int_0^{2\pi} |(e^{i\gamma}+1)e^{i\theta}P'(e^{i\theta}) - ne^{i\gamma}P(e^{i\theta})|^p d\theta \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

Therefore

$$\int_0^{2\pi} |e^{i\theta}P'(e^{i\theta}) + e^{i\gamma}[e^{i\theta}P'(e^{i\theta}) - nP(e^{i\theta})]|^p d\theta \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

Hence

$$\int_0^{2\pi}\int_0^{2\pi} |e^{i\theta}P'(e^{i\theta}) + e^{i\gamma}[e^{i\theta}P'(e^{i\theta}) - nP(e^{i\theta})]|^p d\theta d\gamma \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

By applying Lemma 1 to the left hand side of the above inequality, we will have

$$\int_0^{2\pi}\int_0^{2\pi} ||P'(e^{i\theta})| + e^{i\gamma}|e^{i\theta}P'(e^{i\theta} - nP(e^{i\theta})||^p d\theta d\gamma \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta$$

which is equivalent to

$$\int_0^{2\pi}\int_0^{2\pi} ||P'(e^{i\theta})| + e^{i\gamma}|Q'(e^{i\theta})||^p d\theta d\gamma \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta d\gamma. \tag{3.4}$$

Combining (3.3) and (3.4), it follows that

$$\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta \int_0^{2\pi} |t_{n,m} + e^{i\gamma}|^p d\gamma \leq (|\alpha| + t_{n,m})^p 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta, \tag{3.5}$$

and the inequality (1.5) can now be obtained by raising the power $\frac{1}{p}$ on both the sides of (3.5), and then doing some rearrangement of terms. □

Proof of the Theorem 3. Since $P(z)$ has all its zeros in $0 < |z| \leq K, \quad K \leq 1$ the polynomial $Q(z) = z^n\overline{P(1/\overline{z})}$ has zeros in $|\frac{1}{z}| \geq \frac{1}{K} \geq 1$. Therefore applying Theorem 2 to the polynomial $Q(z)$, we get for $|\alpha| \geq 1$, and $p > 0$

$$\left\{\int_0^{2\pi} |D_\alpha\{Q(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \leq n(|\alpha| + s_{n,m})H_p \left\{\int_0^{2\pi} |Q(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}.$$

If $|\alpha| \leq 1$ then $\frac{1}{|\alpha|} \geq 1$ and hence by replacing α by $\frac{1}{\overline{\alpha}}$ in the above equation, we get for $|\alpha| \leq 1$,

$$\left\{\int_0^{2\pi} |D_{\frac{1}{\overline{\alpha}}}\{Q(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \leq n\left(\frac{1}{|\alpha|} + s_{n,m}\right) H_p \left\{\int_0^{2\pi} |Q(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}},$$

which by Lemma 4 is equivalent to

$$\left\{\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \leq n(1 + |\alpha|s_{n,m})H_p \left\{\int_0^{2\pi} |Q(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}.$$

From the fact that $|Q(e^{i\theta})| = |P(e^{i\theta})|$, the above inequality reduces to (1.6), and the proof of Theorem 3 is thus complete. □

Remark 1. *A result due to Aziz and Rather [3, Lemma 2, page 19] played a significant role and used many times in proving such integral inequalities by several researchers. But the proof of this result given by Aziz and Rather [3] heavily depends on a result due to Melas [13], whose proof is quite indirect in the present context. In the proof of Theorem 2 of this paper, we established a straight forward proof (see equation (3.4)) to [3, Lemma 2, page 19] by using Arestov's result (see Lemma 6). The proof of main theorem in the paper due to Govil and Kumar [9] can also be re-presented using Arestov's Lemma (Lemma 6) without using [3, Lemma 2, page 19] explicitly.*

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References

[1] V. V. Arestov, On integral inequalities for trigonometric polynomials and their derivatives, *Izv. Akad. Nauk SSSR Ser. Mat.*, **45**, 3–22 (1981).

[2] A. Aziz, Inequalities for the polar derivative of a polynomial, *J. Approx. Th.*, **55**, 183–193 (1988).

[3] A. Aziz, N. A. Rather, Some Zygmund type L^q inequalities for polynomials, *J. Math. Anal. Appl.*, **289**, 14–29 (2004).

[4] S. N. Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle. *Gauthier-Villars*, Paris (1926).

[5] N. G. De Bruijn, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensch. Proc. Ser. A*, **50**, 1265–1272 (1947).

[6] K. K. Dewan, N. Singh, A. Mir, Extensions of some polynomial inequalities to the polar derivative, *J. Math. Anal. Appl.*, **352**, 807–815 (2009).

[7] R. B. Gardner, A. Weems, A Bernstein-type L^p inequality for a certain class of polynomials, *J. Math. Anal. Appl.,* **219**, 472–478 (1998).

[8] N. K. Govil, P. Kumar, On Bernstein-type Inequalities for the Polar Derivative of a Polynomial, *Progress in Approximation Theory and Applicable Complex Analysis*, Springer Optimization and Its Applications, **117**, 41-74 (2017).

[9] N. K. Govil, P. Kumar, On L^p inequalities involving polar derivative of a polynomial, *Acta. Math. Hung.*, **152**, 130–139 (2017).

[10] N. K. Govil, Q. I. Rahman, Functions of exponential type not vanishing in a half-plane and related polynomials, *Trans. Amer. Math. Soc.*, **137**, 501-517 (1969).

[11] P. D. Lax, Proof of a conjecture due to Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc.*, **50**, 509–513 (1944).

[12] E. Laguerre, *Oeuvres Vol.1; Nouvelles Annales de Mathematiques*, **17**, No 2, (1878).

[13] A. D. Melas, Polynomial with zeros in a disk, problems and solutions, *Amer. Month. Monthly*, **103**, 177 181 (1996).

[14] M. A. Quazi, On the maximum modulus of polynomials, *Proc. Amer. Math. Soc.*, **115**, 237–243 (1992).

[15] Q. I. Rahman, G. Schmeisser, L^p inequalities for polynomials, *J. Appox. Th.*, **53**, 26–32 (1988).

[16] N. A. Rather, L^p Inequalities for the polar derivative of a polynomial, *J. Ineq. Pure Appl. Math.*, **9** Art 103, 1–10 (2008).

[17] N. A. Rather, Some integral inequalities for the polar derivative of a polynomial, *Mat. Balk*, **22**, 207–216 (2008).

[18] P. Turán, Über die Ableitung von Polynomen, *Compositio Mathematica*, **7**, 89-95 (1939).

[19] A. Zygmund, A remark on conjugate series, *Proc. London Math. Soc.*, **34**, 392–400 (1932).

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