Abstract

Fraenkel-Mostowski set theory represents an alternative set theory with multiple applications in mathematics and computer science. It deals with ‘finitely supported sets’ instead of ‘sets’. The notions of ‘finite’ and ‘Dedekind finite’ are different in this framework. Our main purpose is to provide a complete characterization of Dedekind finite and Dedekind infinite sets in Fraenkel-Mostowski set theory, and to study each class of Dedekind finite and Dedekind infinite sets in an hierarchically defined model of the Fraenkel-Mostowski set theory.

Key Words: Fraenkel-Mostowski set theory, Dedekind finiteness, invariant set

2010 Mathematics Subject Classification: Primary 03E10. Secondary 03E25, 03E30.

1 Set theoretical Fraenkel-Mostowski approach

The Fraenkel-Mostowski permutation models of Zermelo-Fraenkel set theory with atoms (ZFA) were developed in 1930s by Fraenkel, Lindenbaum and Mostowski in order to prove the independence of the axiom of choice from the other axioms of ZFA, where ZFA is Zermelo-Fraenkel (ZF) set theory with the Axiom of Extensionality weakened to allow the existence of atoms. However, these models have been recently rediscovered by Gabbay and Pitts, and presented as a new axiomatic set theory called the Fraenkel-Mostowski set theory (FM); this new approach is used to explain formally concepts such as renaming, binding and fresh name in computer science. The axioms of Fraenkel-Mostowski set theory are precisely the ZFA axioms over an infinite set of atoms, together with the special axiom of finite support which claims that for each element \( x \) in an arbitrary set we can find a finite set supporting \( x \) according to an hierarchically constructed group action of the group of all permutation of atoms. Inductively defined finitely supported sets involving the name-abstraction together with Cartesian product and disjoint union can encode syntax modulo renaming of bound variables. More exactly, because they have no internal structure, atoms can be used to represent names. The finite support axiom is motivated by the fact that syntax can only involve finitely many distinct (free) names. The action of a permutation on an element actually represents the renaming of the ‘bound names’ of that element. Fresh names for the bound variables of a term can always be chosen from the atoms that are outside of the support (outside the set of the free names) of the related term. Binding is modelled by a certain concept of FM abstraction generalizing the notion of abstraction in the \( \lambda \)-calculus; actually FM set theory provides a formal framework for dealing with \( \lambda \)-terms modulo \( \alpha \)-conversion. The construction of the universe of all FM-sets [5] is inspired
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by the construction of the universe of all admissible sets over an arbitrary collection of atoms [3]. The FM-sets represent a generalization of hereditary finite sets (which are particular admissible sets used to describe ‘Gandy machines’); actually, any FM-set is an hereditary finitely supported set.

The theory of nominal sets represents an alternative approach to FM set theory. These sets can be defined both in the ZF framework [8] and in the FM framework [5]. A nominal set is defined as a usual ZF-set endowed with a particular group action of the group of permutations of a fixed countably infinite ZF-set \( A \) (generally called the set of atoms by analogy with the FM framework) that satisfies a certain finite support requirement. There exists also an alternative definition for nominal sets in the FM framework (when the fixed ZF-set \( A \) is replaced by the set of atoms in FM set theory; this is possible because we did not require a certain internal structure for the elements of \( A \)). They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations of atoms) elements in an hierarchical construction. This is because nominal sets need to be closed under the group actions with who they are equipped. This definition corresponds to Tarski’s concept of logicality because the nominal sets developed in a model of FM set theory (namely the FM von-Neumann cumulative hierarchy \( FM(A) \)) are logical notions in Tarski’s sense [1]. The two ways of defining nominal sets finally lead to similar properties. Motivated by Tarski’s approach regarding logicality we convene to use ‘invariant’ instead of ‘nominal’.

Since the theory of invariant sets makes sense over both ZF and ZFA, the FM-sets can be expressed as finitely supported subsets of invariant sets defined over ZFA. The theory of nominal sets over a fixed set \( A \) whose elements can be checked only for equality (i.e. whose elements do not have an internal structure) is extended in [4] to the theory of generalized nominal sets by using new data symmetries over arbitrary (unfixed) sets of data which may have a certain internal structure. Nominal sets and the related extension are used in various areas such as logic, proof theory, game theory, topology, domain theory, automata theory [1]. The theory of nominal sets has also been developed for possible non-countable families of atoms and used in order to describe a theory of algebraic structures with finite support. This approach allow us tho characterize infinite algebraic structures, hierarchically constructed from a set of basic elements, in a finitary manner, by employing only finitely many characteristics (i.e. by analyzing only their finite supports) [1].

The translation of classical results from the framework of ZF (or ZFA) sets to the one of invariant sets is not simple because the family of invariant sets is not closed under subsets constructions, meaning that there exist subsets of invariant sets that fail to be finitely supported. Various translating procedures are developed in [1] and several examples of non-transferable statements are presented including the ZF (or ZFA) choice principles or results regarding cardinalities and maximality in ZFA (or in ZFA). Since choice principles (including countable choice axiom) fail in FM, we ascertain that the concepts of finiteness and Dedekind finiteness are different in this framework. The notion of ‘finite set’ means ‘a set having a bijection with a finite ordinal (such a bijection would be obviously finitely supported)’, while the notion of ‘FM Dedekind finite set’ means ‘a set having no finitely supported injection onto a proper subset’. In this paper, our purpose is to describe the properties of Dedekind (in)finite sets in the FM axiomatic framework, and to present various examples of such sets.
2 Invariant sets

Let \( A \) be a fixed finite infinite ZF-set. Alternatively, the theory of invariant sets can be adequately reformulated if \( A \) is the set of atoms from ZFA set theory. A \textit{transposition} is a function \((a\, b) : A \to A\) defined by \((a\, b)(a) = b, (a\, b)(b) = a, \) and \((a\, b)(c) = c\) for \( c \neq a, b \). A \textit{(finite) permutation} of \( A \) is a bijection of \( A \) generated by composing finitely many transpositions, i.e. a bijection of \( A \) that leaves unchanged all but finitely many atoms of \( A \). From the proof of Corollary 10, any finitely supported bijection of \( A \) would be necessarily a finite permutation of \( A \). Thus, the notions ‘permutation (bijection) of \( A \)’ and ‘finite permutation of \( A \)’ coincide in the framework of FM-sets where only finitely supported objects are allowed. We denote by \( S_A \) the set of all (finite) permutations of \( A \) which will be called simply ‘permutations’.

\textbf{Definition 1.} Let \( X \) be a ZF-set.

1. An \( S_A \)-action on \( X \) is a function \( \cdot : S_A \times X \to X \) having the properties that \( \text{Id} \cdot x = x \) and \( \pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x \) for all \( \pi, \pi' \in S_A \) and \( x \in X \).

An \( S_A \)-set is a pair \((X, \cdot)\) where \( X \) is a ZF-set, and \( \cdot : S_A \times X \to X \) is an \( S_A \)-action on \( X \).

2. Let \((X, \cdot)\) be an \( S_A \)-set. We say that \( S \subseteq A \) supports \( x \) (or \( x \) is \( S \)-supported) whenever for each \( \pi \in \text{Fix}(S) \) we have \( \pi \cdot x = x \), where \( \text{Fix}(S) = \{ \pi | \pi(a) = a \text{ for all } a \in S \} \).

3. Let \((X, \cdot)\) be an \( S_A \)-set. We say that \( X \) is an invariant set if for each \( x \in X \) there exists a finite set \( S_x \subseteq A \) which supports \( x \).

4. Let \( X \) be an \( S_A \)-set and let \( x \in X \). If there exists a finite set supporting \( x \) (particularly, if \( X \) is an invariant set), then there exists at least a finite set \( \text{supp}(x) \) supporting \( x \) which is called the support of \( x \) and is constructed as the intersection of all finite sets supporting \( x \) [5]. An empty supported element is called equivariant; this means that \( x \in X \) is equivariant if and only if \( \pi \cdot x = x \), for all \( \pi \in S_A \).

\textbf{Proposition 1.} [1] Let \((X, \cdot)\) and \((Y, \odot)\) be \( S_A \)-sets.

1. The set \( A \) of atoms is an \( S_A \)-set with the \( S_A \)-action \( \cdot : S_A \times A \to A \) defined by \( \pi \cdot a := \pi(a) \) for all \( \pi \in S_A \) and \( a \in A \). \((A, \cdot)\) is an invariant set with \( \text{supp}(a) = \{a\} \) for each \( a \in A \).

2. If \( x \in X \) is finitely supported, then \( \pi \cdot x \) is finitely supported and \( \text{supp}(\pi \cdot x) = \pi(\text{supp}(x)) \) for any \( \pi \in S_A \).

3. Any ordinary ZF-set \( X \) defined without involving any atoms is an invariant with the single possible \( S_A \)-action \( \cdot : S_A \times X \to X \) defined by \( \pi \cdot x := x \) for all \( \pi \in S_A \) and \( x \in X \).

4. The Cartesian product \( X \times Y \) is also an \( S_A \)-set with the \( S_A \)-action \( \odot : S_A \times (X \times Y) \to (X \times Y) \) defined by \( \pi \odot (x, y) = (\pi \cdot x, \pi \cdot y) \) for all \( \pi \in S_A \) and all \( x \in X, y \in Y \). If \((X, \cdot)\) and \((Y, \odot)\) are invariant sets, then \((X \times Y, \odot)\) is also an invariant set.

5. The powerset \( \wp(X) = \{ Z | Z \subseteq X \} \) is also an \( S_A \)-set with the \( S_A \)-action \( \ast : S_A \times \wp(X) \to \wp(X) \) defined by \( \pi \ast Z := \{ \pi \cdot z | z \in Z \} \) for all \( \pi \in S_A \), and all \( Z \subseteq X \).
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For each invariant set \((X, \cdot)\), we denote by \(\varphi_{fs}(X)\) the set formed from those subsets of \(X\) which are finitely supported according to the action \(\cdot\). \((\varphi_{fs}(X), \cdot|_{\varphi_{fs}(X)})\) is an invariant set, where \(\cdot|_{\varphi_{fs}(X)}\) represents the action \(\cdot\) restricted to \(\varphi_{fs}(X)\).

6. We define the disjoint union of \(X\) and \(Y\) by \(X + Y = \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}\). \(X + Y\) is an \(S_A\)-set with the \(S_A\)-action \(\cdot : S_A \times (X + Y) \to (X + Y)\) defined by \(\pi \cdot z = (0, \pi \cdot x)\) if \(z = (0, x)\) and \(\pi \cdot z = (1, \pi \circ y)\) if \(z = (1, y)\). If \((X, \cdot)\) and \((Y, \circ)\) are invariant sets, then \((X + Y, \cdot)\) is also an invariant set.

**Definition 2.** 1. Let \((X, \cdot)\) be an invariant set. A subset \(Z\) of \(X\) is called finitely supported if and only if \(Z \in \varphi_{fs}(X)\) with the notations from Proposition 1. A subset \(Z\) of \(X\) with the property that all of its elements are supported by the same set of atoms is called uniformly supported.

2. Let \((X, \cdot)\) be a finitely supported subset of an invariant set \((Y, \cdot)\). A subset \(Z\) of \(Y\) is called finitely supported subset of \(X\) (and we denote this by \(Z \in \varphi_{fs}(X)\)) if and only if \(Z \in \varphi_{fs}(Y)\) and \(Z \subseteq X\). Similarly, we say that a uniformly supported subset of \(Y\) contained in \(X\) is a uniformly supported subset of \(X\).

From Definition 1, a subset \(Z\) of an invariant set \((X, \cdot)\) is finitely supported by a set \(S \subseteq A\) if and only if \(\pi \cdot z \in Z\) for all \(z \in Z\) and all \(\pi \in Fix(S)\). It is worth noting that not any subset of an invariant set is finitely supported. For example, if \(B \subseteq A\) and \(B\) is finite, then \(supp(B) = B\). If \(C \subseteq A\) and \(C\) is cofinite (i.e. its complementary is finite), then \(supp(C) = A \setminus C\). However, if \(D \subseteq A\) is neither finite nor cofinite, then \(D\) is not finitely supported.

Since functions are particular relations (i.e. particular subsets of a Cartesian product of two sets), we have the following results.

**Definition 3.** Let \(X\) and \(Y\) be two invariant sets, \(Z\) a finitely supported subset of \(X\), and \(T\) a finitely supported subset of \(Y\). A function \(f : Z \to T\) is finitely supported if \(f \in \varphi_{fs}(X \times Y)\).

**Proposition 2.** [1] Let \((X, \cdot)\) and \((Y, \circ)\) be two invariant sets, \(Z\) a finitely supported subset of \(X\), and \(T\) a finitely supported subset of \(Y\). A function \(f : Z \to T\) is supported by a finite set \(S \subseteq A\) if and only if for all \(x \in Z\) and all \(\pi \in Fix(S)\) we have \(\pi \cdot x \in Z, \pi \circ f(x) \in T\) and \(f(\pi \cdot x) = \pi \circ f(x)\).

The previous results remain consistent if \(A\) is considered to be the set of atoms in the ZFA framework (characterized by the axiom “\(y \in x \Rightarrow x \notin A\)” ) and if ‘ZF’ is replaced by ‘ZFA’ in their statements. This is possible because we did not require a certain internal structure for the elements of \(A\).

3 Fraenkel-Mostowski axiomatic set theory

Let us denote the set of atoms in ZFA by \(A\). The axioms of FM set theory are the ZFA axioms [7] together with the additional axiom of finite support (axiom 11).

**Definition 4.** The following axioms characterize FM set theory:
1. \( \forall x. (\exists y. y \in x) \Rightarrow x \notin A \)
2. \( \forall x, y. (x \notin A \text{ and } y \notin A \text{ and } \forall z. (z \in x \Leftrightarrow z \in y)) \Rightarrow x = y \)
3. \( \forall x, y. \exists z. z = \{x, y\} \)
4. \( \forall x. \exists y. y = \{z \mid z \subseteq x\} \)
5. \( \forall x. \exists y. y \notin A \text{ and } y = \{z \mid \exists w. (z \in w \text{ and } w \in x)\} \)
6. \( \forall x. \exists y. y \notin A \text{ and } y = \{f(z) \mid z \in x\}, \text{ for each functional formula } f(z) \)
7. \( \forall x. \exists y. y \notin A \text{ and } y = \{z \mid z \in x \text{ and } p(z)\}, \text{ for each formula } p(z) \)
8. \( (\forall x. (\forall y \in x. p(y)) \Rightarrow p(x)) \Rightarrow \forall x. p(x) \)
9. \( \exists x. (\emptyset \in x \text{ and } (\forall y, y \in x \Rightarrow y \cup \{y\} \in x)) \)
10. \( A \text{ is not finite.} \)
11. \( \forall x. \exists S \subset A. S \text{ is finite and } S \text{ supports } x. \)

We describe a model \( \nu(A) \) of ZFA by generalizing the classical von-Neumann hierarchy on ordinals.

- \( \nu_0(A) = \emptyset; \)
- \( \nu_{\alpha+1}(A) = A + \nu_\alpha(A) \) for every non-limit ordinal \( \alpha; \)
- \( \nu_\lambda(A) = \bigcup_{\alpha < \lambda} \nu_\alpha(A) \) (\( \lambda \) a limit ordinal);
- \( \nu(A) = \bigcup_{\alpha} \nu_\alpha(A), \)

where + is the disjoint union of sets. Using the names \( \text{atm} \) and \( \text{set} \) for the functions \( x \mapsto (0, x) \) and \( x \mapsto (1, x) \) we have that every element \( x \) of \( \nu(A) \) is either of the form \( \text{atm}(a) \) with \( a \in A \), or of the form \( \text{set}(X) \) where \( X \) is a set formed at an earlier ordinal stage than \( x \). We call ZFA-sets the elements of the form \( \text{set}(X) \), and atoms the elements of the form \( \text{atm}(a) \).

On \( \nu(A) \) we can recursively define an \( S_A \)-action \( \cdot \) as follows:
\[
\pi \cdot \text{atm}(a) = \text{atm}(\pi(a)), \quad \pi \cdot \text{set}(X) = \text{set}(\{\pi \cdot x \mid x \in X\}).
\]

A model of axiomatic FM set theory is represented by the von-Neumann cumulative hierarchy \( FM(A) \) which is a subset of \( \nu(A) \) defined as follows:

- \( FM_0(A) = \emptyset; \)
- \( FM_{\alpha+1}(A) = A + \text{fs}(FM_\alpha(A)) \) for every non-limit ordinal \( \alpha; \)
- \( FM_\lambda(A) = \bigcup_{\alpha < \lambda} FM_\alpha(A) \) (\( \lambda \) a limit ordinal);
- \( FM(A) = \bigcup_{\alpha} FM_\alpha(A). \)

From Proposition 1, each \( FM_\alpha(A) \) and \( FM(A) \) are invariant sets. A ZFA-set \( X \) (i.e. an element \( X \in \nu(A) \) that is not an atom) is an FM-set (i.e. an element in \( FM(A) \setminus A \)) if and only if it is finitely supported as an element of \( \nu(A) \) under the action \( \cdot \) and \( Y \) is a FM-set or an atom for all \( Y \in X \). \( FM(A) \) is a subset of \( \nu(A) \) which is itself an invariant set with the action \( \cdot \) defined as above. Any FM-set (i.e. any set belonging to the model \( FM(A) \)) is an hereditary finitely supported subset of the invariant set \( FM(A) \). An FM set \( X \) is not itself
closed under the $S_A$-action · defined on $FM(A)$, unless $supp(X) = \emptyset$. Thus, the restriction on a certain FM set $X$ of the $S_A$-action · on $FM(A)$ does not necessarily lead to a new group action of $S_A$ on $X$. Since invariant sets need to be closed under the actions they are equipped (meaning that ‘being an invariant set’ means ‘being an equivariant element at the following order stage in the hierarchical construction’), the invariant (or nominal) sets in the FM cumulative hierarchy are defined as those equivariant (i.e. empty supported) elements of $FM(A)$.

4 Dedekind finite sets in FM set theory

We start the original part of our paper by studying the concepts of Dedekind (in)finiteness in the axiomatic framework of FM-sets. In ZF, a set is called Dedekind infinite if there exists an injection into one of its proper subsets. Otherwise, it is called a Dedekind finite set. Within invariant sets, the previous definition can be reformulated as below.

**Definition 5.** Let $X$ be a finitely supported subset of an invariant set $Y$ (particularly $X$ is an FM-set). The set $X$ is FM Dedekind finite if and only if any finitely supported injection $f : X \rightarrow X$ is also a surjection. Otherwise, it is an FM Dedekind infinite set.

**Theorem 1.** Let $(X, \cdot)$ be a finitely supported subset of an invariant set $(Y, \cdot)$. Then $X$ is FM Dedekind infinite (i.e. there exists a finitely supported injection of $X$ into one of its finitely supported proper subsets) if and only if there exists a finitely supported injective mapping $f : \mathbb{N} \rightarrow X$.

**Proof.** Let us suppose that $X$ is FM Dedekind infinite, and $g : X \rightarrow X$ is an injection supported by the finite set $S \subseteq A$ with the property that $Im(g) \subseteq X$. This means that there exists $supp(g) \subseteq S$ and there exists $x_0 \in X$ such that $x_0 \notin Im(g)$. We can form a sequence of elements from $X$ which has the first term $x_0$ and the general term $x_{n+1} = g(x_n)$ for all $n \in \mathbb{N}$. Since $x_0 \notin Im(g)$ it follows that $x_0 \neq g(x_0)$. Since $g$ is injective and $x_0 \notin Im(g)$, by induction we obtain that $g^n(x_0) \neq g^m(x_0)$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Furthermore, $x_{n+1}$ is supported by $supp(g) \cup supp(x_n)$ for all $n \in \mathbb{N}$. Indeed, let $\pi \in Fix(supp(g) \cup supp(x_n))$. According to Proposition 2, $\pi \cdot x_{n+1} = \pi \cdot g(x_n) = g(\pi \cdot x_n) = g(x_n) = x_{n+1}$. Since $supp(x_{n+1})$ is the least set supporting $x_{n+1}$, we obtain $supp(x_{n+1}) \subseteq supp(g) \cup supp(x_n)$ for all $n \in \mathbb{N}$. By finite recursion, we have $supp(x_n) \subseteq supp(g) \cup supp(x_0)$ for all $n \in \mathbb{N}$. Since all $x_n$ are supported by the same set of atoms $supp(g) \cup supp(x_0)$, we have that the function $f : \mathbb{N} \rightarrow X$, defined by $f(n) = x_n$, is also finitely supported (by the set $supp(g) \cup supp(x_0) \cup supp(X)$ not depending on $n$). Indeed, for any $\pi \in Fix(supp(g) \cup supp(x_0) \cup supp(X))$ we have $f(\pi \circ n) = f(n) = x_n = \pi \cdot x_n = \pi \cdot f(n)$, $\forall n \in \mathbb{N}$, where by $\circ$ we denoted the trivial $S_A$-action on $\mathbb{N}$. Furthermore, because $\pi$ fixes $supp(X)$ pointwise we have $\pi \cdot f(n) \in X$ for all $n \in \mathbb{N}$. From Proposition 2 we have that $f$ is finitely supported. Obviously, $f$ is also injective.

Conversely, suppose there exists a finitely supported injective mapping $f : \mathbb{N} \rightarrow X$. According to Proposition 2, it follows that for any $\pi \in Fix(supp(f))$ we have $\pi \cdot f(n) = f(\pi \circ n) = f(n)$ and $\pi \cdot f(n) \in X$ for all $n \in \mathbb{N}$. Let us define $g : X \rightarrow X$ by

$$g(x) = \begin{cases} f(n+1), & \text{if there exists } n \in \mathbb{N} \text{ such that } x = f(n); \\ x, & \text{if } x \notin Im(f). \end{cases}$$
We claim that \( g \) is supported by \( \text{supp}(f) \cup \text{supp}(X) \). Indeed, let \( \pi \in \text{Fix}(\text{supp}(f) \cup \text{supp}(X)) \) and \( x \in X \). If there is some \( n \) such that \( x = f(n) \), we have that \( \pi \cdot x = \pi \cdot f(n) = f(n) \), and so \( g(\pi \cdot x) = g(f(n)) = f(n + 1) = \pi \cdot f(n + 1) = \pi \cdot g(x) \). If \( x \notin \text{Im}(f) \), we prove by contradiction that \( \pi \cdot x \notin \text{Im}(f) \). Indeed, suppose that \( \pi \cdot x \in \text{Im}(f) \). Then there is \( y \in X \) such that \( \pi \cdot x = f(y) \) or, equivalently, \( x = \pi^{-1} \cdot f(y) \). However, since \( \pi \in \text{Fix}(\text{supp}(f)) \), from Proposition 2 we have \( \pi^{-1} \cdot f(y) = f(\pi^{-1} \cdot y) \), and so we get \( x = f(\pi^{-1} \cdot y) \in \text{Im}(f) \) which contradicts the assumption that \( x \notin \text{Im}(f) \). Thus, \( \pi \cdot x \notin \text{Im}(f) \), and so \( g(\pi \cdot x) = \pi \cdot x = \pi \cdot g(x) \). We obtained that \( g(\pi \cdot x) = \pi \cdot x = \pi \cdot g(x) \) for all \( x \in X \) and all \( \pi \in \text{Fix}(\text{supp}(f) \cup \text{supp}(X)) \). Furthermore, \( \pi \cdot g(x) \in \pi \cdot X = X \) (where by \( \pi \cdot \cdot \cdot \) we denoted the \( S_A \)-action on \( \mathcal{V}_{f_A}(Y) \)), and so \( g \) is finitely supported. Since \( f \) is injective, it follows immediately that \( g \) is injective. Furthermore, \( \text{Im}(g) = X \setminus \{f(0)\} \) which is a proper subset of \( X \), finitely supported by \( \text{supp}(f(0)) \cup \text{supp}(X) = \text{supp}(f) \cup \text{supp}(X) \). \( \square \)

**Corollary 1.** Let \((X, \cdot)\) be a finitely supported subset of an invariant set \((Y, \cdot)\). Then \(X\) is FM Dedekind infinite if and only if there exists an infinite, uniformly supported, injective countable subset of the invariant set \(Y\) that is contained in \(X\).

**Proof.** Let us assume that \(X\) is FM Dedekind infinite. According to Theorem 1, there exists a finitely supported injective mapping \(f: \mathbb{N} \rightarrow X\). Thus, according to Proposition 2, for any \(\pi \in \text{Fix}(\text{supp}(f))\) we have \(\pi \cdot f(n) = f(\pi \circ n) = f(n)\) for all \(n \in \mathbb{N}\), where \(\circ\) represents the trivial \(S_A\)-action on \(\mathbb{N}\). Therefore, \(f(n)\) is supported by \(\text{supp}(f)\) for all \(n \in \mathbb{N}\), and so \(f(\mathbb{N})\) is an infinite, injective, uniformly supported countable subset of \(Y\) contained in \(X\).

Conversely, let us assume there is an infinite, injective, uniformly supported countable subset \((x_n)_{n \in \mathbb{N}}\) of the invariant set \(Y\) that is contained in \(X\). Thus, there exists a finite set \(S \subseteq A\) such that \(\text{supp}(x_n) \subseteq S\) for all \(n \in \mathbb{N}\). Let us define the injective function \(f: \mathbb{N} \rightarrow X\) by \(f(n) = x_n\) for all \(n \in \mathbb{N}\). Let \(\pi \in \text{Fix}(S)\). Since \(\pi\) fixes \(S\) pointwise and \(S\) supports \(x_n\) for all \(n \in \mathbb{N}\), we have \(\pi \cdot f(n) = \pi \cdot x_n = x_n = f(n)\) for all \(n \in \mathbb{N}\). According to Proposition 2, because we have \(f(\pi \circ n) = f(n) = \pi \cdot f(n)\) and \(\pi \cdot f(n) = f(n) \in X\) for all \(n \in \mathbb{N}\), we obtain that \(f\) is finitely supported by \(S\). Therefore, from Theorem 1, we obtain that \(X\) is FM Dedekind infinite. \( \square \)

**Corollary 2.** Let \((X, \cdot)\) be a finitely supported subset of an invariant set \((Y, \cdot)\) and \((Z, \cdot)\) another finitely supported subset of \(Y\) with \(X \subseteq Z\). If \(X\) is FM Dedekind infinite, then \(Z\) is FM Dedekind infinite.

**Proof.** Suppose \(X\) is FM Dedekind infinite. From Theorem 1, there exists a finitely supported injective mapping \(f: \mathbb{N} \rightarrow X\). Obviously, because the codomain of \(f\) is contained in \(Z\), we remark that \(f\) is a finitely supported injection from \(\mathbb{N}\) to \(Z\). \( \square \)

**Corollary 3.** Let \((X, \cdot)\) be a finitely supported subset of an invariant set \((Y, \cdot)\) and \((Z, \cdot)\) another finitely supported subset of \(Y\) with \(X \subseteq Z\). If \(Z\) is FM Dedekind finite, then \(X\) is FM Dedekind finite.

**Proof.** The result is equivalent with Corollary 2. **\( \square \)**
Corollary 4. Let \((X, \cdot)\) be an FM Dedekind infinite finitely supported subset of an invariant set. Then \(\varphi_{\text{fin}}(X) = \{Z \subseteq X \mid Z \text{ finite}\}\) is FM Dedekind infinite.

Proof. Since \(X\) is FM Dedekind infinite, according to Theorem 1 we can define finitely supported injection \(f : \mathbb{N} \to X\). However, there exists a mapping \(i : X \to \varphi_{\text{fin}}(X)\) defined by \(i(x) = \{x\}\) for all \(x \in X\), which is supported by \(\text{supp}(X)\) according to Proposition 2. According to Proposition 2, we remark that \(i \circ f : \mathbb{N} \to \varphi_{\text{fin}}(X)\) is a finitely supported injection from \(\mathbb{N}\) to \(\varphi_{\text{fin}}(X)\) (supported by \(\text{supp}(f) \cup \text{supp}(X)\)). \(\square\)

Corollary 5. Let \((X, \cdot)\) be an FM Dedekind infinite finitely supported subset of an invariant set. Then \(\varphi_{\text{fs}}(X)\) is FM Dedekind infinite.

Proof. Let \(X\) be an FM Dedekind finite set. The result follows from Corollary 4 and Corollary 2 because \(\varphi_{\text{in}}(X)\) is FM Dedekind infinite and \(\varphi_{\text{fin}}(X) \subseteq \varphi_{\text{fs}}(X)\). \(\square\)

Corollary 6. Let \((X, \cdot)\) be an FM-set. Then \(X\) is FM Dedekind infinite (i.e. there exists a finitely supported injection of \(X\) into one of its finitely supported proper subsets) if and only if there exists a finitely supported injective mapping \(f : \mathbb{N} \to X\).

Proof. The result follows from Theorem 1 because FM-sets are finitely supported subsets of invariant sets. Indeed, assume that \(X\) is a non-empty FM-set. It follows that there is an ordinal \(\alpha\) such that \(X \in FM_{\alpha}(A) = A + \varphi_{\text{fs}}(FM_{\alpha-1}(A))\). Since \(X\) is not an atom, we have that \(X \in \varphi_{\text{fs}}(FM_{\alpha-1}(A))\), and so \(X \subseteq FM_{\alpha-1}(A)\). Thus, \(X\) is a finitely supported subset of the invariant set \(FM_{\alpha-1}(A)\). \(\square\)

Corollary 7. Let \((X, \cdot)\) be an invariant set. Then \(X\) is FM Dedekind infinite if and only if there exists a finitely supported injective mapping \(f : \mathbb{N} \to X\).

Corollary 8. The set \(A\) of atoms in FM Dedekind finite.

Proof. If \(A\) was FM Dedekind infinite then there would be a finitely supported injection \(f : \mathbb{N} \to A\). Thus, \(f(2\mathbb{N})\) would be a simultaneously infinite and cofinite subset of \(A\), finitely supported by \(\text{supp}(f)\). However, any finitely supported subset of \(A\) has to be either finite or cofinite. \(\square\)

Corollary 9. The set \(\varphi_{\text{fs}}(A)\) is FM Dedekind finite.

Proof. If \(\varphi_{\text{fs}}(A)\) was FM Dedekind infinite, then there would be finitely supported injection \(f : \mathbb{N} \to \varphi_{\text{fs}}(A)\), i.e. there would be an infinite injective sequence of subsets of \(A\) whose terms are all supported by the same finite set \(\text{supp}(f)\). However, there could exist only finitely many subsets of \(A\) supported by \(\text{supp}(f)\), namely the subsets of \(\text{supp}(f)\) and the supersets of \(A \setminus \text{supp}(f)\) (where a superset of \(A \setminus \text{supp}(f)\) is of form \(A \setminus X\) with \(X \subseteq \text{supp}(f)\)). \(\square\)

Corollary 10. The set \(P_A\) of all finitely supported one-to-one mappings from \(A\) onto \(A\) is FM Dedekind finite.
Proof. We prove first that a function \( f : A \to A \) is a bijection on \( A \) in FM if and only if it is a bijection that leaves unchanged all but finitely many elements of \( A \), i.e. in FM any bijection of \( A \) should be a (finite) permutation of \( A \). Indeed, we claim that for each \( a \in A \) we have that \( a \notin \text{supp}(f) \) implies \( f(a) = a \). Let \( a \notin \text{supp}(f) \). Assume, by contradiction, that \( f(a) \neq a \). Let us consider two atoms \( b, c \notin \text{supp}(f) \) such that \( a, b, c \) are all different (such atoms exist because \( \text{supp}(f) \) is finite, while \( A \) is infinite). Since \( \text{supp}(f) \) supports \( f \) and \( (a b) \notin \text{Fix}(\text{supp}(f)) \), we have \( f(b) = f((a b)(a)) = (a b)(f(a)) \). However, \( f(a) \neq a \). Since \( f \) is a bijection, it follows that \( f(a) = b \) (otherwise, we would have \( f(b) = f(a) \) with \( b \neq a \)). However, from \( f((a c)(a)) = (a c)(f(a)) \), it follows that \( f(c) = f((a c)(a)) = (a c)(b) = b = f(a) \), which contradicts the bijectivity of \( f \). Thus, \( f(a) = a \). This means \( S_f \stackrel{\text{def}}{=} \{ a \in A \mid f(a) \neq a \} \subseteq \text{supp}(f) \). Since \( \text{supp}(f) \) is finite, we have that \( S_f \) is finite.

Suppose, by contradiction, that \( P_A \) is FM Dedekind infinite, that is, there exists a finitely supported injective mapping \( g \) from \( \mathbb{N} \) to \( P_A \). Thus, \( g(\mathbb{N}) \) is an infinite sequence of distinct bijections of \( A \), all of these bijections being supported by the same finite set \( \text{supp}(g) \). However, we proved above that \( P_A \) coincides with the set \( S_A \) of all (finite) permutations of \( A \). By (1), for each finitely supported bijection \( \sigma : A \to A \) we have that the set of atoms changed by \( \sigma \) is contained in \( \text{supp}(\sigma) \). For a finite set \( S \) of atoms there are at most \( |S|! \) bijections of \( A \) onto \( A \) supported by \( S \). This assertion follows because whenever a bijection \( \sigma : A \to A \) is finitely supported by \( S \) we have \( \{ a \in A \mid \sigma(a) \neq a \} \subseteq \text{supp}(\sigma) \subseteq S \), and so the highest possible number of bijections of \( A \) onto \( A \) supported by \( S \) is the number of permutations of \( S \). Thus, there could be at most \( |\text{supp}(g)|! \) finitely supported bijections of \( A \) onto \( A \) supported by \( \text{supp}(g) \), and so there does not exist an infinite sequence of distinct bijections of \( A \) onto \( A \) uniformly supported by \( \text{supp}(g) \).}

**Corollary 11.** The set \( \wp_{fs}(A + \{0\}) \) is FM Dedekind finite.

**Proof.** If \( \wp_{fs}(A + \{0\}) \) was FM Dedekind infinite, as in the proof of Corollary 9, there would be an infinite injective sequence of subsets of \( A + \{0\} \) all whose terms are supported by the same finite set \( S \). However, there could exist only finitely many subsets of \( A + \{0\} \) supported by \( S \), namely the subsets of \( S + \{0\} \) and the supersets of \( (A + \{0\}) \setminus \{(0, a) \mid a \in S\} \).}

**Corollary 12.** The invariant sets \( \text{FM}_0(A) \), \( \text{FM}_1(A) \) and \( \text{FM}_2(A) \) are FM Dedekind finite.

**Proof.** The set \( \text{FM}_0(A) \) coincide with the empty set, and so it is FM Dedekind finite. \( \text{FM}_1(A) = A + \{0\} \). If there was an infinite finitely supported injective countable sequence in \( \text{FM}_1(A) \), then there would be an infinite injective sequence of atoms all supported by the same finite set, which is a contradiction. Now, suppose that there is a countable injective sequence of elements from \( \text{FM}_2(A) = A + \wp_{fs}(A + \{0\}) \), all whose terms are supported by the same set \( S \). Thus, at least one of the sets \( A \) and \( \wp_{fs}(A + \{0\}) \) contains an infinite uniformly supported family of elements. However, this is a contradiction according to the proofs of Corollary 8 and Corollary 11.}

**Theorem 2.** Let \((X, \cdot)\) be an infinite finitely supported subset of an invariant set \((Y, \cdot)\). Then the set \( \wp_{fs}(\wp_{fin}(X)) \) is FM Dedekind infinite.
The family \( \varphi_{\text{fin}}(X) \) represents the family of those finite subsets of \( X \) (they are finitely supported as subsets of the invariant set \( Y \) in the sense of Definition 2). Obviously, \( \varphi_{\text{fin}}(X) \) is a finitely supported subset of the invariant set \( \varphi_{fs}(Y) \), supported by \( \text{supp}(X) \). This is because whenever \( Z \) is an element of \( \varphi_{\text{fin}}(X) \) (i.e. whenever \( Z \) is a finite subset of \( X \)) and \( \pi \) fixes \( \text{supp}(X) \) pointwise, we have that \( \pi \ast Z \) is also a finite subset of \( X \). The family \( \varphi_{fs}(\varphi_{\text{fin}}(X)) \) represents the family of those subsets of \( \varphi_{\text{fin}}(X) \) which are finitely supported as subsets of the invariant set \( \varphi_{fs}(Y) \) in the sense of Definition 2. As above, according to Proposition 1(2), we have that \( \varphi_{fs}(\varphi_{\text{fin}}(X)) \) is a finitely supported subset of the invariant set \( \varphi_{fs}(\varphi_{fs}(Y)) \), supported by \( \text{supp}(\varphi_{\text{fin}}(X)) \subseteq \text{supp}(X) \).

Let \( X_i \) be the set of all \( i \)-size subsets from \( X \), i.e. \( X_i = \{ Z \subseteq X | |Z| = i \} \). Since \( X \) is infinite, it follows that each \( X_i, i \geq 1 \) is non-empty. Obviously, we have that any \( i \)-size subset \( \{x_1, \ldots, x_i\} \) of \( X \) is finitely supported (as a subset of \( Y \)) by \( \text{supp}(x_1) \cup \ldots \cup \text{supp}(x_i) \). Therefore, \( X_i \subseteq \varphi_{\text{fin}}(X) \) and \( X_i \subseteq \varphi_{fs}(Y) \) for all \( i \in \mathbb{N} \). Since \( \cdot \) is a group action, the image of an \( i \)-size subset of \( X \) under an arbitrary (finite) permutation is an \( i \)-size subset of \( Y \). However, any (finite) permutation of atoms that fixes \( \text{supp}(X) \) pointwise also leaves \( X \) invariant, and so for any (finite) permutation \( \pi \in \text{Fix}(\text{supp}(X)) \) we have that \( \pi \ast Z \) is an \( i \)-size subset of \( X \) whenever \( Z \) is an \( i \)-size subset of \( X \). Thus, each \( X_i \) is a subset of \( \varphi_{\text{fin}}(X) \) finitely supported by \( \text{supp}(X) \), and so \( X_i \in \varphi_{fs}(\varphi_{\text{fin}}(X)) \).

We define \( f : \mathbb{N} \rightarrow \varphi_{fs}(\varphi_{\text{fin}}(X)) \) by \( f(n) = X_n \). We claim that \( \text{supp}(X) \) supports \( f \). Indeed, let \( \pi \in \text{Fix}(\text{supp}(X)) \). Since \( \text{supp}(X) \) supports \( X_n \) for all \( n \in \mathbb{N} \), we have \( \pi \ast f(n) = \pi \ast X_n = X_n = f(n) = f(\pi \circ n) \) (where \( \circ \) is the trivial \( S_A \)-action on \( \mathbb{N} \)) and \( \pi \ast f(n) = \pi \ast X_n = X_n \in \varphi_{fs}(\varphi_{\text{fin}}(X)) \) for all \( n \in \mathbb{N} \). According to Proposition 2, we have that \( f \) is finitely supported. Furthermore, \( f \) is injective and, by Theorem 1, we have that \( \varphi_{fs}(\varphi_{\text{fin}}(X)) \) is FM Dedekind infinite. \( \Box \)

**Corollary 13.** Let \( (X, \cdot) \) be an infinite finitely supported subset of an invariant set \( (Y, \cdot) \). Then the set \( \varphi_{fs}(\varphi_{fs}(X)) \) is FM Dedekind infinite.

**Proof.** The result follows from Theorem 2 and Corollary 2 because \( \varphi_{fs}(\varphi_{\text{fin}}(X)) \) is FM Dedekind infinite, \( \varphi_{fs}(\varphi_{\text{fin}}(X)) \subseteq \varphi_{fs}(\varphi_{fs}(X)) \), and both \( \varphi_{fs}(\varphi_{\text{fin}}(X)) \) and \( \varphi_{fs}(\varphi_{fs}(X)) \) are finitely supported subsets of the invariant set \( \varphi_{fs}(\varphi_{fs}(Y)) \) (supported by \( \text{supp}(X) \)). \( \Box \)

**Corollary 14.** The invariant sets \( \varphi_{fs}(\varphi_{fs}(A)) \) and \( \varphi_{fs}(\varphi_{\text{fin}}(A)) \) are FM Dedekind infinite.

**Corollary 15.** The set \( FM_\alpha(A) \) is FM Dedekind infinite whenever \( \alpha > 2 \) and \( \alpha \) is not as limit ordinal.

**Proof.** Denote by \( \alpha - 1 \) the predecessor of \( \alpha \). We have that \( FM_\alpha(A) = A + \varphi_{fs}(A + \varphi_{fs}(FM_{\alpha-2}(A))) \). Since \( FM_{\alpha-2}(A) \) is an infinite invariant set for \( \alpha > 2 \), from Theorem 2, we have that \( \varphi_{fs}(\varphi_{fs}(FM_{\alpha-2}(A))) \) is FM Dedekind infinite. Since \( \varphi_{fs}(\varphi_{fs}(FM_{\alpha-2}(A))) \subseteq A + \varphi_{fs}(A + \varphi_{fs}(FM_{\alpha-2}(A))) \), from Corollary 2, it follows that \( FM_\alpha(A) \) is FM Dedekind infinite. \( \Box \)

**Corollary 16.** The set \( FM_\lambda(A) \) is FM Dedekind infinite whenever \( \lambda \) is a limit ordinal.
Proof. This follows from Corollary 2 and Corollary 15 because $FM_\alpha(A)$ is a superset of some FM Dedekind infinite set $FM_\alpha(A)$ with $\alpha > 2$.

Corollary 17. The FM universe $FM(A)$ is FM Dedekind infinite.

Proposition 3. If $X$ and $Y$ are FM Dedekind finite, finitely supported subsets of an invariant set $(Z, \cdot)$, then $X \times Y$ is FM Dedekind finite.

Proof. Let us assume, by contradiction, that $X \times Y$ is FM Dedekind infinite. According to Theorem 1, there is a finitely supported injective mapping $f : \mathbb{N} \to X \times Y$. Thus, according to Proposition 2, there exists an infinite injective sequence $f(\mathbb{N}) = \{(x_i, y_i)\}_{i \in \mathbb{N}} \subseteq X \times Y$ such that $supp((x_i, y_i)) \subseteq supp(f)$ for all $i \in \mathbb{N}$ (1). Fix some $j \in \mathbb{N}$; we claim that $supp((x_j, y_j)) = supp(x_j) \cup supp(y_j)$. Let $U = (x_j, y_j)$, and $S = supp(x_j) \cup supp(y_j)$. Obviously, $S$ supports $U$. Indeed, let us consider $\pi \in Fix(S)$. We have that $\pi \in Fix(supp(x_j))$ and $\pi \in Fix(supp(y_j))$. Therefore, $\pi \cdot x_j = x_j$ and $\pi \cdot y_j = y_j$, and so $\pi \otimes (x_j, y_j) = (\pi \cdot x_j, \pi \cdot y_j) = (x_j, y_j)$, where $\otimes$ represent the $S_A$ action on $X \times Y$ described in Proposition 1(4). Thus, $supp(U) \subseteq S$. It remains to prove that $S \subseteq supp(U)$. Fix $\pi \in Fix(supp(U))$. Since $supp(U)$ supports $U$, we have $\pi \otimes (x_j, y_j) = (x_j, y_j)$, and so $(\pi \cdot x_j, \pi \cdot y_j) = (x_j, y_j)$, from which we get $\pi \cdot x_j = x_j$ and $\pi \cdot y_j = y_j$. Thus, $supp(x_j) \subseteq supp(U)$ and $supp(y_j) \subseteq supp(U)$. Hence $S = supp(x_j) \cup supp(y_j) \subseteq supp(U)$.

According to relation (1) we obtain, $supp(x_i) \cup supp(y_i) \subseteq supp(f)$ for all $i \in \mathbb{N}$. Thus, $supp(x_i) \subseteq supp(f)$ for all $i \in \mathbb{N}$ and $supp(y_i) \subseteq supp(f)$ for all $i \in \mathbb{N}$ (2). Since the sequence $\{(x_i, y_i)\}_{i \in \mathbb{N}}$ is infinite and injective, then at least one of the sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ is infinite. Assume that $(x_i)_{i \in \mathbb{N}}$ is infinite. Then there exists an infinite subset $B$ of $\mathbb{N}$ such that $(x_i)_{i \in B}$ is injective, and so there exists an injection $u : B \to X$ defined by $u(i) = x_i$ for all $i \in B$ which is supported by $supp(f)$ (according to relation (2) and Proposition 2). However, since $B$ is an infinite subset of $\mathbb{N}$, there exists a ZF bijection $h : \mathbb{N} \to B$. The construction of $h$ requires only the fact that $\mathbb{N}$ is well-ordered which is obtained from the Peano construction of $\mathbb{N}$ and does not involve a form of the axiom of choice. Since both $B$ and $\mathbb{N}$ are trivial invariant sets (see Proposition 1(3)), it follows that $h$ is equivariant. Thus, $u \circ h$ is an injection from $\mathbb{N}$ to $X$ which is finitely supported by $supp(u) \subseteq supp(f)$. This contradicts the assumption that $X$ is FM Dedekind finite.

Corollary 18. If $X$ and $Y$ are two Dedekind finite FM-sets, then $X \times Y$ is FM Dedekind finite.

Corollary 19. The invariant sets $A \times A$, $\varphi_{f_A}(A) \times \varphi_{f_A}(A)$, $A \times \varphi_{f_A}(A)$ and every other finite Cartesian combination between $A$ and $\varphi_{f_A}(A)$, are FM Dedekind finite.

Proposition 4. If $X$ and $Y$ are FM Dedekind finite finitely supported subsets of an invariant set $(Z, \cdot)$, then $X + Y$ is FM Dedekind finite.

Proof. Let us assume, by contradiction, that $X + Y$ is FM Dedekind infinite. According to Theorem 1, there is a finitely supported injective mapping $f : \mathbb{N} \to X + Y$. Thus, there exists an infinite injective sequence $(z_i)_{i \in \mathbb{N}} \subseteq X + Y$ such that $supp(z_i) \subseteq supp(f)$ for all $i \in \mathbb{N}$. According to the construction of the disjoint union of two $S_A$-sets (see Proposition 1(6)), as in the proof of Proposition 3, there should exist an infinite subsequence of $(z_i)$ of form
Corollary 20. If $X$ and $Y$ are two Dedekind finite FM-sets, then $X + Y$ is FM Dedekind finite.

From the proof of Theorem 1, we have that every FM Dedekind infinite set contains an infinite uniformly supported subset. Thus, a set containing no infinite uniformly supported subset will be necessarily FM Dedekind finite. In Corollary 8, Corollary 9 and Corollary 10 we proved that the sets $A, \varphi_f(A)$ and $P_A$ contain no infinite uniformly supported subsets, and so these sets are FM Dedekind finite. Furthermore, those FM-sets that do not contain infinite uniformly supported subsets have the following property.

Proposition 5. Let $X$ be a finitely supported subset of an invariant set $Y$ such that $X$ does not contain an infinite uniformly supported subset. Then $\varphi_{fin}(X) = \{Z \subseteq X \mid Z$ finite $\}$ does not contain an infinite uniformly supported subset. 

Proof. Assume, by contradiction, that $\varphi_{fin}(X)$ contains an infinite subset $\mathcal{F}$ such that all the elements of $\mathcal{F}$ are different and supported by the same finite set $S$. Therefore, we can express $\mathcal{F} = (X_i)_{i \in I} \subseteq \varphi_{fin}(X)$ with the properties that $X_i \neq X_j$ whenever $i \neq j$ and $supp(X_i) \subseteq S$ for all $i \in I$. Fix an arbitrary $j \in I$. We claim that $X_j$ has the property that $supp(x) \subseteq S$ for all $x \in X_j$. First we remind that $X_j \in \varphi_{fin}(X)$, i.e. $X_j$ is a finite subset of $X$. Thus, $X_j = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$, with $x_1, \ldots, x_n \in X$. Let $S_0 = supp(x_1) \cup \ldots \cup supp(x_n)$. We prove that $S_0 = supp(X_j)$. Obviously, $S_0$ supports $X_j$. Indeed, let us consider $x \in Fix(S_0)$. We have that $\pi \in Fix(supp(x_i))$ for each $i \in \{1, \ldots, n\}$. Therefore, $\pi \cdot x_i = x_i$ for each $i \in \{1, \ldots, n\}$ because $supp(x_i)$ supports $x_i$ for each $i \in \{1, \ldots, n\}$, and so $supp(X_j) \subseteq S_0$. It remains to prove that $S_0 \subseteq supp(X_j)$. Consider $a \in S_0$. This means there exists $k \in \{1, \ldots, n\}$ such that $a \in supp(x_k)$. Due to the infiniteness of $A$ we can consider an atom $b$ such that $b \notin supp(X_j)$ and $b \notin supp(x_i)$ for all $i \in \{1, \ldots, n\}$. We prove by contradiction that $(b \cdot a) \cdot x_k \notin X_j$. Indeed, suppose that $(b \cdot a) \cdot x_k \in X_j$. In this case there is $y \in X_j$ with $(b \cdot a) \cdot x_k = y$. Since $a \notin supp(x_k)$, we have $b = (b \cdot a)(a) \in (b \cdot a)(supp(x_k))$. However, according to Proposition 1(2), we have $supp(y) = (b \cdot a)(supp(x_k))$. We obtain that $b \in supp(y)$ for some $y \in X_j$, which is a contradiction with the choice of $b$. Therefore, $(b \cdot a) \star X_j \neq X_j$, where $\star$ is the standard $S_A$-action on $\varphi(Y)$. Since $b \notin supp(X_j)$, we prove by contradiction that $a \in supp(X_j)$. Indeed, suppose that $a \notin supp(X_j)$. It follows that the transposition $(b \cdot a)$ fixes each element from $supp(X_j)$, i.e. $(b \cdot a) \in Fix(supp(X_j))$. Since $supp(X_j)$ supports $X_j$, it follows that $(b \cdot a) \star X_j = X_j$, which is a contradiction. Thus, $a \in supp(X_j)$, and so $S_0 \subseteq supp(X_j)$. Thus $S_0 = supp(X_j)$. Now, since $supp(X_j) \subseteq S$, we have $\cup_{x \in X_j} supp(x) = S_0 = supp(X_j) \subseteq S$, and so $supp(x) \subseteq S$ for all $x \in X_j$. Since $j$ has been arbitrarily chosen from $I$, it follows that every element from every set of form $X_i$ is supported by $S$, and so $\cup_{x \in X_i} x$ is an uniformly supported subset of $X$ (all its elements being supported by $S$). Furthermore, $\cup_{i \in I} X_i$ is infinite because the sequence $(X_i)_{i \in I}$ is infinite and $X_i \neq X_j$ whenever $i \neq j$. Otherwise,
if $\bigcup_i X_i$ was finite, the family $(X_i)_{i \in I}$ would be contained in the finite set $\mathcal{V} (\bigcup_i X_i)$, and so it couldn’t be infinite with the property that $X_i \neq X_j$ whenever $i \neq j$. We were able to construct an infinite uniformly supported subset of $X$, namely $\bigcup_i X_i$, and this contradicts the hypothesis that $X$ does not contain an infinite uniformly supported subset.

**Corollary 21.** Let $X$ be a finitely supported subset of an invariant set $Y$ such that $X$ does not contain an infinite uniformly supported subset. Then $\mathcal{V}_{fin}(X) = \{Z \subseteq X \mid Z \text{ finite}\}$ is FM Dedekind finite.

**Proof.** According to Proposition 5, if $X$ does not contain an infinite uniformly supported subset, then $\mathcal{V}_{fin}(X)$ does not contain an infinite uniformly supported subset. Suppose, by contradiction, that $\mathcal{V}_{fin}(X)$ is FM Dedekind infinite. According to Theorem 1, there exists a finitely supported injective mapping $f : \mathbb{N} \to \mathcal{V}_{fin}(X)$. Thus, because $\mathbb{N}$ is a trivial invariant set, according to Proposition 2, there exists an infinite injective sequence $f(\mathbb{N}) = (X_i)_{i \in \mathbb{N}} \subseteq \mathcal{V}_{fin}(X)$ such that $\text{supp}(X_i) \subseteq \text{supp}(f)$ for all $i \in \mathbb{N}$. We obtained that $\mathcal{V}_{fin}(X)$ contains an infinite uniformly supported subset $(X_i)_{i \in \mathbb{N}}$, which is a contradiction. \qed

**Corollary 22.** The invariant sets $\mathcal{V}_{fin}(A)$, $\mathcal{V}_{fin}(\mathcal{V}_{fs}(A))$ and $\mathcal{V}_{fin}(P_A)$ are FM Dedekind finite.

**Proof.** According to (the proofs of) Corollary 8, Corollary 9 and Corollary 10, none of the invariant sets $A$, $\mathcal{V}_{fs}(A)$ and $P_A$ contains an infinite uniformly supported subset. For $A$ this assertion is obvious. For $\mathcal{V}_{fs}(A)$, there are only finitely many elements in $\mathcal{V}_{fs}(A)$ (i.e. only finitely many subsets of $A$) supported by a certain set $S$, namely the subsets of $S$ and the supersets of $A \setminus S$, and so $\mathcal{V}_{fs}(A)$ cannot contain an infinite uniformly supported subset. For $P_A$, there are at most $|S|!$ elements in $P_A$ (i.e. at most $|S|!$ bijections of $A$ onto $A$) supported by a certain set $S$, and so $P_A$ cannot contain an infinite uniformly supported subset; this follows because whenever a bijection $f : A \to A$ is supported by $S$ we should have $\{a \in A \mid f(a) \neq a\} \subseteq S$. The result follows from Corollary 21. \qed

## 5 Conclusion

FM set theory deals with an alternative definition of a set, that is, any FM-set has associated a finite support with respect to the canonical hierarchical action of the group of all permutations of atoms. An alternative ZF approach to FM set theory is represented by the theory of invariant sets which are ZF-sets equipped with permutation group actions satisfying a finite support requirement. The construction of invariant sets also makes sense under the FM axioms. Analyzed in the FM universe, the invariant sets are those empty supported elements from the FM cumulative von-Neumann universe $FM(A)$. The FM approach was successfully used in modelling syntax involving binding operations, in studying the languages over infinite alphabets or automata theory. FM translations of algebraic structures have already triggered significant applications in the classical algebra, in the theory of abstract interpretation, in domain theory, or in topology (see [1]). The theory of
fuzzy sets has also been recently developed within finitely supported structures, by considering an association of FM type between a certain invariant set and a finitely supported membership function characterizing the related invariant set [2].

In the framework of FM-sets, the notions of finite and Dedekind finite sets are different. The goal of this paper was to provide a characterization of Dedekind (in)finiteness in FM axiomatic set theory, and to present various examples of FM Dedekind finite and FM Dedekind infinite sets. We proved that a certain finitely supported subset of an invariant set (particularly a certain FM-set) is FM Dedekind infinite if and only if it contains a countably infinite uniformly supported subset. Examples of FM Dedekind (in)finite sets are presented below.

1. The following sets and also their subsets are FM Dedekind finite.
   - Any finite subset of an invariant set (particularly, any finite FM-set).
   - The set $A$ of atoms.
   - The FM powerset $\varphi_{fs}(A)$ of the set of atoms.
   - The set $P_A$ of all finitely supported bijections of $A$ onto $A$.
   - The invariant sets $FM_0(A), FM_1(A)$ and $FM_2(A)$.
   - The invariant sets $\varphi_{fin}(A), \varphi_{fin}(\varphi_{fs}(A))$ and $\varphi_{fin}(P_A)$.
   - Any construction of finite powersets having the form $\varphi_{fin}(\varphi_{fin}(\varphi_{fin}(\ldots \varphi_{fin}(A))))$, $\varphi_{fin}(\varphi_{fin}(\varphi_{fin}(\varphi_{fs}(A))))$ or $\varphi_{fin}(\varphi_{fin}(\varphi_{fin}(P_A)))$.
   - Every finite Cartesian combination between $A, \varphi_{fs}(A)$ and $P_A$.
   - The disjoint unions $A + P_A, A + \varphi_{fs}(A), \varphi_{fs}(A) + P_A, A + \varphi_{fs}(A) + P_A$.

2. The following sets and also their supersets, their powersets and the families of their finite subsets, are FM Dedekind infinite.
   - The invariant sets $\varphi_{fs}(\varphi_{fs}(A))$ and $\varphi_{fs}(\varphi_{fin}(A))$.
   - The invariant sets $FM_\alpha(A)$ whenever $\alpha > 2$ and $\alpha$ is not a limit ordinal.
   - The invariant set $FM_\lambda(A)$ whenever $\lambda$ is a limit ordinal.
   - The invariant FM universe $FM(A)$.

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Received: 07.03.2018
Accepted: 26.06.2018

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