The lattice structure of all lattice preradicals on modular complete lattices, and applications (I)

by

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Abstract

Based on the concept of a linear morphism of lattices, recently introduced in the literature, we present and investigate in this paper the latticial counterpart of the big lattice $R$-pr of all preradicals on the category $\text{Mod-}R$ of all unital right $R$-modules over an associative ring $R$ with identity.

Key Words: modular lattice, upper continuous lattice, linear morphism of lattices, fully invariant submodule, fully invariant element, lattice preradical, big lattice, Grothendieck category, hereditary torsion theory.

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Introduction

In this paper we shall again illustrate a general strategy which consists on putting a module-theoretical definition/result into a latticial frame, in order to translate that definition/result to Grothendieck categories and to module categories equipped with a hereditary torsion theory. Thus, we provide the latticial counterpart of the big lattice $R$-pr of all preradicals on the category $\text{Mod-}R$ of all unital right $R$-modules over an associative ring $R$ with identity.

In Section 0 we list some definitions and results about lattices, especially from [2]. We also present from [3] and [5] the concepts of a linear morphism of lattices and of a lattice preradical, respectively, and list some of their basic properties.

Section 1 is devoted to the investigation of the main properties of the big lattice $L$-pr of all lattice preradicals on all modular complete lattices.

In Section 2 we present the latticial counterparts of the basic operations on $R$-pr, as well as the relationship between them.

Applications of our latticial results to Grothendieck categories and module categories equipped with a hereditary torsion theory will be given in a subsequent paper.

0 Preliminaries

All lattices considered in this paper are assumed to be bounded, i.e., to have a least element denoted by 0 and a last element denoted by 1, and $L$ will always denote such a lattice. If the lattices $L$ and $L'$ are isomorphic, we denote this by $L \simeq L'$. The opposite lattice of $L$
will be denoted by $L^0$. By a big lattice we mean any class, not necessarily a set, satisfying the usual axioms of a lattice.

We denote by $\mathcal{L}$ (respectively, $\mathcal{M}$, $\mathcal{C}$) the class of all bounded (respectively, bounded modular, complete) lattices.

For a lattice $L$ and elements $a \leq b$ in $L$ we write

$$b/a := [a, b] = \{ x \in L \mid a \leq x \leq b \}.$$ 

A subfactor of $L$ is any interval $b/a$ of $L$ with $a \leq b$, and an initial interval of $b/a$ is any interval $c/a$ with $a \leq c \leq b$.

A lattice $L$ is said to be simple in case it has exactly two elements, so, $L$ is simple if $L = \{0, 1\}$ and $0 \neq 1$. An element $a \in L$ is said to be an atom if $a \neq 0$ and $a/0 = \{0, a\}$, i.e., $a/0$ is a simple lattice. We denote by $A(L)$ the set, possibly empty, of all atoms of $L$. The socle $\text{Soc}(L)$ of a complete lattice $L$ is the join of all atoms of $L$, i.e., $\text{Soc}(L) := \bigvee A(L)$; if $L$ has no atoms, then $\text{Soc}(L) = 0$. Notice that the atoms of the lattice $\mathcal{L}(M_R)$ of all submodules of a right $R$-module $M_R$ are precisely the simple submodules of $M_R$. In general, a Grothendieck category may have no simple objects, see [10]). The reader is referred to [16] and/or [8] and for basic notions and facts on Grothendieck categories. As in [12], a lattice $L$ is said to be atomic if for every $0 \neq x \in L$ there exists an atom $a \in L$ such that $a \leq x$.

An element $m \in L$ is said to be a coatom if it is a maximal element of $L \setminus \{1\}$, i.e., $m$ is an atom of the opposite lattice $L^\circ$, and $M(L)$ will denote the set, possibly empty, of all coatoms of $L$, so $M(L) = A(L^\circ)$. The lattice $L$ is called coatomic if its opposite lattice $L^\circ$ is atomic, i.e., for every element $x \in L \setminus \{1\}$ there exists $m \in M(L)$ such that $x \leq m$.

For basic notation and terminology on lattices the reader is referred to [1], [2], [12], and/or [16], but especially to [2]. In particular, for any $L \in \mathcal{L}$, one denotes by $D(L)$ the set of all complement elements of $L$ (D for “Direct summand”).

As in [3], a mapping $f : L \to L'$ between a lattice $L$ with least element 0 and greatest element 1 and a lattice $L'$ with least element 0' and greatest element 1' is called a linear morphism if there exist $k \in L$, called a kernel of $f$, and $a' \in L'$ such that the following two conditions are satisfied.

- $f(x) = f(x \lor k)$, $\forall x \in L$.
- $f$ induces a lattice isomorphism

$$\tilde{f} : 1/k \longrightarrow a'/0', \tilde{f}(x) = f(x), \forall x \in 1/k.$$

If $f : L \to L'$ is a linear morphism of lattices, then $f$ is an increasing mapping, commutes with arbitrary joins (i.e., $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$ for any family $(x_i)_{i \in I}$ of elements of $L$, provided both joins exist), preserves intervals (i.e., for any $u \leq v$ in $L$, one has $f(v/u) = f(v)/f(u)$), and its kernel $k$ is uniquely determined; moreover, for any $a \in L$, the restriction $f_a : a/0 \to L'$, $f_a(x) = f(x)$, $\forall x \in a/0$, of $f$ to $a/0$ is a linear morphism with kernel $a \land k$.

As in [3], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, (for “Linear Modular”) if for any $L, L' \in \mathcal{M}$ one takes as morphisms from $L$ to $L'$ all the linear morphisms from $L$ to $L'$.

The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices. The monomorphisms (respectively,
epimorphisms) in the category $\mathcal{LM}$ are exactly the injective (respectively, surjective) linear morphisms. Moreover, the subobjects of $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$ for any $a \in L$. For all these properties of linear morphisms of lattices the reader is referred to [3].

For any $L, L' \in \mathcal{LM}$ we denote, as in any category, by $\text{Hom}_{\mathcal{LM}}(L, L')$ the set of all linear morphisms of lattices between $L$ and $L'$, and by $\text{End}_{\mathcal{LM}}(L)$ the set $\text{Hom}_{\mathcal{LM}}(L, L)$ of all endomorphisms of $L$ in the category $\mathcal{LM}$.

We present now after [5] the concept of a lattice preradical as a functor $r : \mathcal{LM} \to \mathcal{LM}$ satisfying the following two conditions.

- For any $L \in \mathcal{LM}$, $r(L)$ is a subobject of $L$, i.e., an interval $[0, a]$, $a \in L$.
- For any morphism $f : L \to L'$ in $\mathcal{LM}$, $r(f) : r(L) \to r(L')$ is the restriction of $f$ to $r(L)$ and $r(L')$, i.e., $f(r(L)) \subseteq r(L')$.

In other words, a lattice preradical is nothing else than a subfunctor of the identity functor $1_{\mathcal{LM}}$ of the category $\mathcal{LM}$.

We denote by $0 : \mathcal{LM} \to \mathcal{LM}$ the functor defined by $0(L) = 0$, $\forall L \in \mathcal{LM}$, and by $1$ the identity functor $1_{\mathcal{LM}}$ of $\mathcal{LM}$, which are clearly lattice preradicals.

If $r : \mathcal{LM} \to \mathcal{LM}$ is a lattice preradical, then for any $L \in \mathcal{LM}$ and $a \in L$, the subobject $r(a/0)$ of $L$ in $\mathcal{LM}$ is necessarily an initial interval of $a/0$. We denote

$$r(a/0) := a^r/0.$$ 

In particular $r(L) = 1^L/0$.

If $a \leq b$ in $L$, the inclusion mapping $\iota : a/0 \hookrightarrow b/0$ is clearly a linear morphism. Applying now $r$ we obtain $r(\iota) : a^r/0 \to b^r/0$ as a restriction of $\iota$, and so $a^r \leq b^r$.

Notice that $a^r = b^r$ for two elements $a, b$ in $L$ does not necessarily imply $a = b$, as one may see with the following simple example: $L$ is the lattice of all subgroups of the Abelian group $\mathbb{Z}[i]$ of Gaussian integers, $a = \mathbb{Z}[i]$, $b = \mathbb{Z}$, and $r$ is the Jacobson radical $\text{Jac}$ on the class of all Abelian groups. Then $a^r = b^r = 0$, but $a \neq b$.

As in [6], a non-empty class $\mathcal{H}$ of lattices is said to be weakly hereditary if $a/0 \in \mathcal{H}$ for any $L \in \mathcal{H}$ and $a \in L$. According to [9], an abstract class of lattices is a subclass $\mathcal{H} \subseteq \mathcal{L}$ which is closed under lattice isomorphisms, i.e., if $L, K \in \mathcal{L}$, $K \cong L$, and $L \in \mathcal{H}$, then $K \in \mathcal{H}$. Thus, a hereditary class of lattices as defined in [9] is nothing else than a weakly hereditary class which additionally is an abstract class.

For any non-empty subclass $\mathcal{D}$ of $\mathcal{M}$ we shall denote by $\mathcal{LD}$ the full subcategory of $\mathcal{LM}$ having $\mathcal{D}$ as the class of its objects. We shall also use the notation $\mathcal{M}_\mathcal{D}$ for the class $\mathcal{M} \cap \mathcal{C}$ of all modular complete lattices.

Let $\mathcal{H}$ be a weakly hereditary subclass of $\mathcal{M}$. As in [6], a weakly lattice preradical on $\mathcal{H}$ is any functor $r : \mathcal{LH} \to \mathcal{LH}$ satisfying the following two conditions.

- $r(L)$ is an initial interval of $L$ for any $L \in \mathcal{LH}$.
- For any morphism $f : L \to L'$ in $\mathcal{LH}$, $r(f) : r(L) \to r(L')$ is the restriction and corestriction of $f$ to $r(L)$ and $r(L')$, respectively.

The lattice preradicals defined in [5] are precisely the weakly lattice preradicals on hereditary classes $\mathcal{H} \subseteq \mathcal{M}$. As in the case of “true” lattice preradicals, for a weakly lattice preradical $r$ on the weakly hereditary class $\mathcal{H} \subseteq \mathcal{M}$, we set $r(a/0) := a^r/0$ for any $a \in L$ and $L \in \mathcal{H}$. 

If \( a \leq b \) in \( L \) then \( a/0, b/0 \) are both in \( \mathcal{H} \) because \( \mathcal{H} \) is weakly hereditary. The inclusion mapping \( \iota: a/0 \to b/0 \) is clearly a linear morphism, so it is a morphism in \( \mathcal{L}\mathcal{H} \).

Applying now \( \mathcal{L} \) we obtain \( r(\iota): a^r/0 \to b^r/0 \) as a restriction of \( \iota \), and so \( a^r \leq b^r \).

The latticial counterpart of the concept of a fully invariant submodule of a module is that of a fully invariant element introduced in [7] as follows. Let \( L \in \mathcal{M} \). An element \( a \in L \) is said to be fully invariant, abbreviated \( FI \), if \( f(a) \leq a \) for any \( f \in \text{End}_{\mathcal{L}\mathcal{M}}(L) \), and the set of all fully invariant elements of \( L \) will be denoted by \( \text{FI}(L) \).

Throughout this paper \( R \) will denote an associative ring with non-zero identity element, and \( \text{Mod-}R \) the category of all unital right \( R \)-modules. The notation \( M_R \) will be used to designate a unital right \( R \)-module \( M \), and \( N \leq M \) will mean that \( N \) is a submodule of \( M \). The lattice of all submodules of a module \( M_R \) will be denoted by \( \mathcal{L}(M_R) \). The reader is referred to [17] for basic notation, notions, and facts on rings, modules and categories.

As in [11] or [16], a preradical on \( \text{Mod-}R \) is a subfunctor \( q \) of the identity functor \( 1_{\text{Mod-}R} \) of \( \text{Mod-}R \). This means that \( q \) assigns to each right \( R \)-module \( M \) a submodule \( q(M) \) of \( M \) such that each morphism \( f : M \to N \) in \( \text{Mod-}R \) induces by restriction a morphism \( q(f) : q(M) \to q(N) \), i.e., \( f(q(M)) \leq q(N) \). We call these preradicals module preradicals.

Notice that any lattice preradical naturally induces a module preradical, or more generally a preradical on any locally small Abelian category, but not conversely (see [4]).

\section{The big lattice \( \mathcal{L}\text{-pr} \)}

In this section we show that the class of all lattice preradicals on all modular complete lattices is a big complete lattice, we shall denote \( \mathcal{L}\text{-pr} \).

\begin{lemma}
Let \( f : L \to L' \) be a linear morphism of lattices, and let \( a < b \) in \( L \) and \( c' \in L' \) such that \( f(b) \leq c' \). Then the restriction \( f_\mid : b/a \to c'/f(a) \) of \( f \) is a linear morphism of lattices.
\end{lemma}

\begin{proof}
If \( k \) is the kernel of \( f \), then \( \overline{f} = a \lor (b \land k) \) is the kernel of \( f_\mid \) because

\[ f_\mid(x \lor \overline{k}) = f((x \lor a) \lor (b \land k)) = f(x \lor a) \lor f(b \land k) = f(x) = f_\mid(x), \quad \forall x \in b/a. \]

As \( \overline{k} = a \lor (b \land k) = b \land (a \lor k) \) by modularity, and \( f(b) \leq c' \), we claim that \( f_\mid \) induces the lattice isomorphism

\[ f_\mid : b/(b \land (a \lor k)) \to f(b)/f(a), \quad f_\mid(x) = f(x), \quad \forall x \in b/(b \land (a \lor k)), \]

which means exactly that \( f_\mid \) is a linear morphism of lattices.

In order to prove our claim, observe first that, by modularity, we have the canonical lattice isomorphism

\[ \varphi : b/(b \land (a \lor k)) \cong (b \lor (a \lor k))/(a \lor k), \quad \varphi(x) := x \lor (a \lor k), \quad \forall x \in b/(b \land (a \lor k)). \]

Now, observe that \( \varphi(x) = x \lor k, \quad \forall x \in b/(b \land (a \lor k)) \). Indeed, by modularity, we have \( b \land (a \lor k) = a \lor (b \land k) \), so \( b/(b \land (a \lor k)) = b/(a \lor (b \land k)) \). Thus, \( x \in b/(a \lor (b \land k)) \) for any \( x \in b/(b \land (a \lor k)) \), and then \( a \lor (b \land k) \leq x \leq b \). It follows that \( a \leq x \). Consequently

\[ \varphi(x) = x \lor (a \lor k) = (x \lor a) \lor k = x \lor k, \]

\end{proof}
as desired.

On the other hand, since $f$ is a linear morphism of lattices, $f$ induces the lattice isomorphism

$$
\overline{f} : 1/k \longrightarrow f(1)/0', \overline{f}(x) = f(x), \forall x \in 1/k,
$$

where $0'$ is the least element of $L'$. Using now the properties, presented in Preliminaries, of linear morphisms of lattices, we deduce that the restriction of the linear isomorphism $\overline{f}$ to the subinterval $(b \lor k)/(a \lor k)$ of $1/k$ produces a lattice isomorphism

$$
\psi : (b \lor k)/(a \lor k) \longrightarrow f(b)/f(a), \psi(x) = f(x), \forall x \in (b \lor k)/(a \lor k),
$$

and clearly

$$
\overline{f}| = \psi \circ \varphi,
$$

so

$$
\overline{f}|(x) = (\psi \circ \varphi)(x) = (\psi(\varphi(x)) = \psi(x \lor k) = f(x \lor k) = f(x), \forall x \in b/(b \land (a \lor k)),
$$

and we are done. \qed

We denote by $\mathcal{L}$-pr the class of all lattice preradicals on all modular complete lattices. For any family $(r_i)_{i \in I}$ in $\mathcal{L}$-pr, $I$ any set, we set

$$
\bigvee_{i \in I} r_i : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c \quad \text{and} \quad \bigwedge_{i \in I} r_i : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c
$$

by

$$
(\bigvee_{i \in I} r_i)(L) := (\bigvee_{i \in I} 1^{r_i})/0 \quad \text{and} \quad (\bigwedge_{i \in I} r_i)(L) := (\bigwedge_{i \in I} 1^{r_i})/0, \forall L \in \mathcal{LM}_c.
$$

Instead of considering families $(r_i)_{i \in I}$ of lattice preradicals we may consider classes $S$ of lattice preradicals, and denote

$$
\bigvee S : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c \quad \text{and} \quad \bigwedge S : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c
$$

by

$$
(\bigvee S)(L) := (\bigvee_{r \in S} 1^r)/0 \quad \text{and} \quad (\bigwedge S)(L) := (\bigwedge_{r \in S} 1^r)/0, \forall L \in \mathcal{LM}_c.
$$

Notice that though $S$ is a class and not necessarily a set, for each $L \in \mathcal{M}_c$, $\{r(L) \mid r \in S\}$ is a set.

The class $\mathcal{L}$-pr becomes clearly a big poset with respect to the following order relation:

$$
r \leq s \quad \overset{\text{def}}{\iff} \quad r(L) \subseteq s(L), \forall L \in \mathcal{L}.
$$

Clearly $0 \leq r \leq 1, \forall r \in \mathcal{L}$-pr, i.e., $0$ is the least element of the big poset $\mathcal{L}$-pr, and $1$ is the last element of $\mathcal{L}$-pr.

**Lemma 1.2.** For any $\emptyset \neq S \subseteq \mathcal{L}$-pr we have $\bigvee S \in \mathcal{L}$-pr and $\bigwedge S \in \mathcal{L}$-pr.

**Proof.** For every $r \in S$ we have $r(L) = 1^r/0 \subseteq L$, and so, because $L$ is a set,

$$
(\bigvee S)(L) := (\bigvee_{r \in S} 1^r)/0
$$
is an initial interval of $L$.

Now, if $f : L \rightarrow L'$ is a linear morphism of complete modular lattices, then, by definition, $f(r(L)) \subseteq r'(L')$. Since $r(L) = r(1/0) = 1'/0$ and $r(L') = r(1/0') = 1'/0'$, we have $f(1') \leq 1'$, $\forall r \in S$. Using the fact that $f$ commutes with arbitrary joins, we deduce that

$$f(\bigvee_{r \in S} 1') = \bigvee_{r \in S} f(1') \leq \bigvee_{r \in S} 1'^r,$$

so, by Lemma 1.1, the restriction

$$(\bigvee S)(f) : (\bigvee_{r \in S} 1')/0 \rightarrow (\bigvee_{r \in S} 1'^r)/0'$$

of $f$ is a linear morphism of lattices, hence $\bigvee S \in L\text{-pr}$. As above,

$$(\bigwedge S)(L) := (\bigwedge_{r \in S} 1')/0$$

is an initial interval of $L$.

Now, if $f : L \rightarrow L'$ is a linear morphism of complete modular lattices, then, by definition $f(r(L)) \subseteq r'(L')$, and, as above $f(1') \leq 1'$, $\forall r \in S$. It follows that

$$\bigwedge_{r \in S} f(1') \leq \bigwedge_{r \in S} 1'^r.$$ 

On the other hand, we have $\bigwedge_{r \in S} 1^r \leq 1'$, $\forall r \in S$. Therefore

$$f(\bigwedge_{r \in S} 1') = f(\bigwedge_{r \in S} 1^r) \leq \bigwedge_{r \in S} f(1').$$

We deduce that $f(\bigwedge_{r \in S} 1') \leq \bigwedge_{r \in S} 1'^r$.

Again by Lemma 1.1, the restriction

$$(\bigwedge S)(f) : (\bigwedge_{r \in S} 1')/0 \rightarrow (\bigwedge_{r \in S} 1'^r)/0'$$

of $f$ is a linear morphism of lattices, which shows that $\bigwedge S \in L\text{-pr}$, as desired. \qed

The next result shows even more, that $L\text{-pr}$ is a complete modular big lattice.

**Proposition 1.3.** The class $L\text{-pr}$ is a complete modular big lattice with respect to the operations $\bigvee$ and $\bigwedge$ defined above, having $0$ as least element and $1$ as last element.

**Proof.** Let $\emptyset \neq S \subseteq L\text{-pr}$ and $t \in L\text{-pr}$ be such that $r \leq t$, $\forall r \in S$. For $L \in \mathcal{M}_c$ we have $$(\bigvee S)(L) = (\bigvee_{r \in S} 1^r)/0.$$ As $r \leq t$, we have $r(L) \leq t(L)$, so $1'/0 \subseteq 1'/0$, $\forall r \in S$, and then $1' \leq 1'$, $\forall r \in S$. Therefore $\bigvee_{r \in S} 1' \leq 1'$. It follows that $$(\bigvee_{r \in S} 1')/0 \subseteq 1'/0.$$ This shows that $(\bigvee S)(L) \subseteq t(L)$, $\forall L \in \mathcal{M}_c$, i.e., $\bigvee S \leq t$. Since $r \subseteq \bigvee S$, $\forall r \in S$, we deduce that $\bigvee S$ is the supremum of $S$. In a similar way one shows that $\bigwedge S$ is the infimum of $S$.

We are now going to prove that the lattice $L\text{-pr}$ is modular. To do that, let $r, s, t \in L\text{-pr}$ with $r \leq s$. We have to show the equality $r \vee (s \wedge t) = s \wedge (r \vee t)$. For any $L \in \mathcal{M}_c$ we have

$$(r \vee (s \wedge t))(L) = (1' \vee (1^s \wedge 1^t))/0.$$ 

Because $1' \leq 1^s$ and $L$ is a modular lattice, it follows that

$$(r \vee (s \wedge t))(L) = (1' \vee (1^s \wedge 1^t))/0 = (1^s \wedge (1' \vee 1^t))/0 = (s \wedge (r \vee t))(L).$$
This shows that \( r \lor (s \land t) = s \land (r \lor t) \). We conclude that \( \mathcal{L}\text{-pr} \) is a complete modular big lattice. \( \square \)

**Remarks 1.4.** (1) The big poset structure of \( \mathcal{L}\text{-pr} \) is exactly the one induced by its big lattice structure above.

(2) If we consider now the subclass \( \mathcal{U}\text{-pr} \) of \( \mathcal{L}\text{-pr} \) consisting of all lattice preradicals on all upper continuous modular lattices, then it is clear that \( \mathcal{U}\text{-pr} \) is a big sublattice of \( \mathcal{L}\text{-pr} \) that is modular and upper continuous. \( \square \)

## 2 Operations on \( \mathcal{L}\text{-pr} \)

In this section we present the latticial counterparts of the basic operations on \( R\text{-pr} \) (see [13], [14], [15]) as well as the relationship between them.

For any \( r, s \in \mathcal{L}\text{-pr} \) we define \( r \cdot s \) and \( r : s \) as follows:

\[
(r \cdot s)(L) := r(s(L))
\]

and

\[
(r : s)(L) = 1^{(r \cdot s)}/0, \quad \text{where } 1^{(r \cdot s)} \text{ is defined by } 1^{(r \cdot s)}/1^r = s(1/1^r), \forall L = 1/0 \in \mathcal{M}_c.
\]

Recall that for any \( L, K \in \mathcal{LM} \) and \( f \in \text{Hom}_{\mathcal{LM}}(K,L) \) we denote, as usually, by \( f^{-1}(A) \), \( A \subseteq L \), the inverse image of the subset \( A \) of \( L \) under the mapping \( f \), i.e.,

\[
f^{-1}(A) := \{ b \in K \mid f(b) \in A \};
\]

in particular, for any \( a \in L \),

\[
f^{-1}(a/0) = \{ b \in K \mid f(b) \in a/0 \} = \{ b \in K \mid f(b) \leq a \}.
\]

**Proposition 2.1.** For any set \( r, s \in \mathcal{L}\text{-pr} \) we have \( r \cdot s \in \mathcal{L}\text{-pr} \) and \( r : s \in \mathcal{L}\text{-pr} \).

**Proof.** Since \( s(L) = 1^s/0 \), by the definition of \( r \cdot s \), we deduce that

\[
(r \cdot s)(L) = r(s(L)) = r(1^s)/0 = (1^s)/0
\]

is an initial interval of \( 1^s/0 \subseteq L \), so \( r \cdot s)(L) \) is a subobject of \( L \). Now, if \( f : L \rightarrow L' \) is a linear morphism of lattices, then \( s(f) : s(L) \rightarrow s(L') \) is a linear morphism of lattices which shows that \( r(s(f)) : r(s(L)) \rightarrow r(s(L')) \), and so, \( r \cdot s \in \mathcal{L}\text{-pr} \).

We are now going to prove that \( r : s \in \mathcal{L}\text{-pr} \). Remember that \( r(L) = r(1/0) = 1^r/0 \). By [5, Example 0.2(2)], the mapping

\[
p : 1/0 \rightarrow 1/1^r, \quad p(x) = x \lor 1^r, \forall x \in L,
\]

is a surjective linear morphism with kernel \( 1^r \). Notice that, by definition, \( 1^r \leq 1^{(r \cdot s)} \), hence \( p(1^{(r \cdot s)}/0) = 1^{(r \cdot s)}/1^r = s(1/1^r) \). Therefore \( 1^{(r \cdot s)}/0 \leq p^{-1}(s(1/1^r)) \).

Denote \( z_p := \bigvee_{x \in p^{-1}(1^{(r \cdot s)}/1^r)} x \). We claim that
Indeed, let \( y \in p^{-1}(1^{(rs)}/1^r) \). Then \( p(y) \in 1^{(rs)}/1^r \), so \( y \leq \bigvee_{x \in p^{-1}(1^{(rs)}/1^r)} x = z_p \), i.e., \( y \in z_p/0 \). It follows that \( p^{-1}(1^{(rs)}/1^r) \subseteq z_p/0 \).

For the opposite inclusion, let \( y \in z_p/0 \). Then, \( y \leq z_p = \bigvee_{x \in p^{-1}(1^{(rs)}/1^r)} x \). As \( p \) is a linear morphism, then

\[
p(y) \leq p\left(\bigvee_{x \in p^{-1}(1^{(rs)}/1^r)} x\right) = \bigvee_{x \in p^{-1}(1^{(rs)}/1^r)} p(x) \leq 1^{(rs)},
\]

so \( y \in p^{-1}(1^{(rs)}/1^r) \) because clearly \( p(y) \geq 1^r \). This shows that \( z_p/0 \subseteq p^{-1}(1^{(rs)}/1^r) \) and proves our claim.

We have \( p^{-1}(s(1/1^r)) = z_p/0 \). Indeed, \( s(1/1^r) = 1^{(rs)}/1^r \), and so

\[
p^{-1}(s(1/1^r)) = p^{-1}(1^{(rs)}/1^r) = z_p/0.
\]

Thus

\[
1^{(rs)}/1^r = s(1/1^r) = p(z_p/0) = (z_p \lor 1^r)/1^r.
\]

So \( z_p \lor 1^r \leq 1^{(rs)} \), and then \( z_p \leq 1^{(rs)} \). Hence \( p^{-1}(s(1/1^r)) = z_p/0 \subseteq 1^{(rs)}/0 \), which implies that

\[
1^{(rs)}/0 = p^{-1}(s(1/1^r)).
\]

Thus, the linear morphism

\[
\overline{p} : 1^{(rs)}/0 \rightarrow s(1/1^r), \overline{p}(x) = p(x) = x \lor 1^r,
\]

is the restriction of the linear morphism \( p \).

Now let \( f : L \rightarrow L' \) be a linear morphism, where \( L' = 1'/0' \), and let

\[
p' : 1'/0' \rightarrow 1'/1'^r, \ p'(x') := x' \lor 1'^r, \forall x' \in L',
\]

be the surjective linear morphism with kernel \( 1'^r \). So, \( p' \circ f : 1/0 \rightarrow 1'/1'^r \) is a linear morphism, and \( (p' \circ f)(x) = p'(f(x)) = f(x) \lor 1'^r, \forall x \in L \).

On the other hand as \( r \) is a preradical, \( f(r(L)) \subseteq r(L') \). Thus \( f(1^r/0) \subseteq 1'^r/0' \), so \( f(1') \leq 1'^r \). Hence \( (p' \circ f)(1') = f(1') \lor 1'^r = 1'^r \). Because \( 1'^r \) is the least element of the lattice \( L'' = 1'/1'^r \), by [5, Proposition 0.3(3)], \( p' \circ f \) induces the linear morphism

\[
f' : 1'/1'^r \rightarrow 1'/1'^r, f'(x) = (p' \circ f)(x) = f(x) \lor 1'^r, \forall x \in 1'/1'^r,
\]

Since \( s \) is a preradical, \( s(f') : s(1/1^r) \rightarrow s(1'/1'^r) \) is the restriction of the linear morphism \( f' \).

Hence

\[
s(f')(x) = f'(x) = f(x) \lor 1'^r, \forall x \in s(1/1^r) = 1^{(rs)}/1^r.
\]

We claim that

\[
(r : s)(f) : (r : s)(L) \rightarrow (r : s)(L')
\]

is the restriction of the linear morphism \( f : L \rightarrow L' \).

Since \( (r : s)(L) = (r : s)(1/0) = 1^{(rs)}/0 \) and \( (r : s)(L') = 1^{(rs)}/0' \), it follows that \( f(1^{(rs)}/0) \subseteq 1^{(rs)}/0' \). Indeed, let \( x \in 1^{(rs)}/0 \) and consider the linear morphism
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 Proposition 2.2. For any \( T \).

 Let \( s \) morphism, then \( s \).

 It follows that \( p' f(x) = f(x) \vee 1'^r \in s (1'/1'^r) \) so, \( f(x) \in p'^{-1}(s (1'/1'^r)) \). Notice that \( p'^{-1}(s (1'/1'^r)) = 1^{(r:s)} / 0' \).

 Indeed, we have seen above in this proof, just before defining the linear morphism \( p \), that \( 1^{(r:s)} / 0 = p^{-1}(s (1')) \) for any \( r, s \in \mathcal{L}-\text{pr} \) and any \( L \in \mathcal{M}_c \). Specialize now this equality for \( L' = 1'/0' \) and the linear morphism \( p \) we considered above in this proof:

 \[ p' : 1'/0' \longrightarrow 1'/1'^r, \quad p' (x) := x \vee 1'^r, \quad \forall x \in L', \]

to obtain \( p'^{-1}(s (1'/1'^r)) = 1^{(r:s)} / 0' \), as desired.

 Resume now the proof of our claim. From the equality we have just established, we deduce that \( f(x) \in 1^{(r:s)} / 0' \), hence \( f(1^{(r:s)} / 0) \leq 1^{(r:s)} / 0' \), which proves our claim. \( \square \)

 Proposition 2.2. For any \( r, s \in \mathcal{L}-\text{pr} \) we have \( r \cdot s \leq r \lor s \leq r \lor s \leq r : s \).

 Proof. Let \( L \in \mathcal{M}_c \). As \( s (L) \subseteq L \), we have \( r (s (L)) \subseteq r (L) \), so \( r \cdot s \leq r \). Moreover, \( r (s (L)) \subseteq s (L) \) because \( r \) is a preradical, so \( r \cdot s \leq s \), and then \( r \cdot s \leq r \lor s \). Clearly \( r \lor s \leq r \) and \( r \lor s \leq r \lor s \), so \( r \lor s \leq r \lor s \).

 We are now going to prove that \( r \leq r : s \) and \( s \leq r : s \). Since \( (r : s) (L) = 1^{(r:s)} / 0 \), where \( s (1'/1'^r) = 1^{(r:s)} / 1'^r \), we have \( 1'^r \leq 1^{(r:s)} \). Therefore \( 1'/0 \leq 1^{(r:s)} / 0 \). It follows that \( r (L) \subseteq (r : s) (L) \), so \( r \subseteq r : s \).

 On the other hand, if \( p : 1 / 0 \longrightarrow 1 / 1'^r \), \( p (x) := x \lor 1'^r, \forall x \in L \), is the surjective linear morphism, then \( s (p) : s (1/0) \longrightarrow s (1'/1'^r) \) is the restriction of \( p \), so \( p (s (1/0)) \subseteq s (1'/1'^r) \). It follows that \( s (L) = s (1/0) \subseteq p^{-1}(s (1'/1')) = 1^{(r:s)} / 0 = (r : s) (L) \) (see the proof of Proposition 2.1). Thus \( s (L) \subseteq (r : s) (L) \), so \( s \leq r : s \), and then \( r \lor s \leq r : s \), as desired. \( \square \)

 The concepts from the definition below are exactly the ones introduced in [5] without refereeing to the operations \( \cdot \) and \( : \) we defined at the beginning of this section.

 Definitions 2.3. Let \( r \in \mathcal{L}-\text{pr} \). We say that:

(1) \( r \) is an idempotent preradical if \( r \cdot r = r \).

(2) \( r \) is a radical if \( r : r = r \).

(3) \( r \) is a left exact (or hereditary) preradical if \( a^r = a \lor 1'^r, \forall L \in \mathcal{L} \mathcal{M}, a \in L \).

 Next we present the latticial counterparts of the properties of basic operations on \( R-\text{pr} \) (see [13], [14], [15]), as well as the relationship between them.
Lemma 2.4. Let \( r \) be a lattice radical, let \( L = 1/0 \in \mathcal{L} \mathcal{M} \), and let \( A \subseteq L \) be a subobject of \( L \), i.e., \( A = a/0 \) for some \( a \in L \).

If \( a \leq b \), where \( r(L) = 1^r/0 = b/0 \), then \( r(1/a) = b/a \).

Proof. By [5, Example 0.2(2)], the mapping
\[
p : L \rightarrow 1/a, \quad p(x) := x \lor a, \quad \forall x \in L,
\]
is a surjective linear morphism with kernel \( a \), that induces the linear morphism
\[
r(p) : b/0 = r(L) \rightarrow r(1/a)
\]
with kernel \( a \), so \( b/a \subseteq r(1/a) \).

Again by [5, Example 0.2(2)], the mapping
\[
q : 1/a \rightarrow 1/b, \quad q(x) := x \lor b, \quad \forall x \in 1/a,
\]
has the kernel \( b \) and induces the zero morphism on \( r(1/a) \). Indeed, we have
\[
r(L) = r(1/0) = 1^r/0 = b/0,
\]
and moreover \( r(1/b) = r(1/1^r) = 1^r/1^r = b/b \) since \( r \) is a radical. This shows that \( r(1/a) \subseteq b/a \), and we are done. \( \Box \)

Proposition 2.5. The following assertions hold for \( r, s, t \in \mathcal{L} \text{-pr} \) and a family \((r_i)_{i \in I}\) of elements of \( \mathcal{L} \text{-pr} \).

1. \( (\bigwedge_{i \in I} r_i) \cdot s = \bigwedge_{i \in I} (r_i \cdot s) \).
2. \( (\bigvee_{i \in I} r_i) \cdot s = \bigvee_{i \in I} (r_i \cdot s) \).
3. \( (r : s) : t = r : (s : t) \) and \( (r \cdot s) \cdot t = r \cdot (s \cdot t) \).
4. \( t : (\bigwedge_{i \in I} r_i) = \bigwedge_{i \in I} (t : r_i) \).
5. \( t : (\bigvee_{i \in I} r_i) = \bigvee_{i \in I} (t : r_i) \).
6. \( (s : t) \cdot r \leq (s \cdot r) \cdot (t : r) \), and \( r \) is a radical \( \iff (s : t) \cdot r = (s \cdot r) \cdot (t : r) \), \( \forall s, t \in \mathcal{L} \text{-pr} \).
7. \( (r : s) \cdot (r : t) \leq r : (s \cdot t) \), and \( r \) is idempotent \( \iff (r : s) \cdot (r : t) = r : (s \cdot t) \), \( \forall s, t \in \mathcal{L} \text{-pr} \).

Proof. (1) Let \( L \in \mathcal{M}_c \). Then \( (r_i \cdot s)(L) = r_i(s(L)) = r_i(s(1/0)) = r_i(1^\vee/0) = (1^\vee)^{r_i}/0 \).
Thus \( (\bigwedge_{i \in I} (r_i \cdot s))(L) = (\bigwedge_{i \in I} (1^\vee)^{r_i})/0 \).
Now
\[
((\bigwedge_{i \in I} r_i) \cdot s)(L) = (\bigwedge_{i \in I} r_i)(s(L)) \left( (\bigwedge_{i \in I} r_i)(1^\vee/0) = (\bigwedge_{i \in I} (1^\vee)^{r_i})/0 \right).
\]
It follows that \( (\bigwedge_{i \in I} (r_i \cdot s))(L) = (\bigwedge_{i \in I} (r_i \cdot s))(L) \).
Therefore \( (\bigwedge_{i \in I} r_i) \cdot s = \bigwedge_{i \in I} (r_i \cdot s) \).

(2) Let \( L \in \mathcal{M}_c \). Then \( (r_i \cdot s)(L) = r_i(s(L)) = r_i(s(1/0)) = r_i(1^\vee/0) = (1^\vee)^{r_i}/0 \).
Thus \( (\bigvee_{i \in I} (r_i \cdot s))(L) = (\bigvee_{i \in I} (1^\vee)^{r_i})/0 \), so
\[
((\bigvee_{i \in I} r_i) \cdot s)(L) = (\bigvee_{i \in I} r_i)(s(L)) = (\bigvee_{i \in I} r_i)(1^\vee/0) = (\bigvee_{i \in I} (1^\vee)^{r_i})/0.
\]
It follows that \((\bigvee_{i \in I} (r_i \cdot s))(L) = ((\bigvee_{i \in I} r_i) \cdot s)(L)\), so \((\bigvee_{i \in I} r_i) \cdot s = \bigvee_{i \in I} (r_i \cdot s)\).

(3) Let \(L = 1/0 \in \mathcal{M}_c\). In order to calculate \((r : (s : t))(1/0)\), we have first to do it for \((s : t)(1/1')\). By definition, \(s(1/1') = 1^{(r:s)}/1'\). Remember that when calculating \((s : t)(1/1')\), we are working inside the lattice \(1/1'\), so \(t(1/1^{(r:s)}) = 1^{(r:s)}/1^{(r:s)}\), and then
\[(s : t)(1/1') = 1^{(r:s)}/1'.\]

By definition,
\[(r : (s : t))(1/0) = 1^{(r:s)}/0 \quad \text{and} \quad ((r : s) : t)(1/0) = 1^{(r:s)}/0,
\]
so \((r : (s : t))(1/0) = ((r : s) : t)(1/0)\). Therefore \((r : s) : t = r : (s : t)\).

(4) For \(L \in \mathcal{M}_c\), we have \((t : r_i)(L) = 1^{(t : r_i)}/0\), with \(r_i(1/1') = 1^{(t : r_i)}/1'\) by definition, hence \((\bigwedge_{i \in I} r_i)(1/1') = (\bigwedge_{i \in I} 1^{(t : r_i)})/1'\), and then
\[(t : (\bigwedge_{i \in I} r_i))(L) = (\bigwedge_{i \in I} 1^{(t : r_i)})/0.\]

Indeed, if we denote \(u := \bigwedge_{i \in I} r_i\), then \((t : (\bigwedge_{i \in I} r_i))(L) = (t : u)(L) = 1^{(t : u)}/0\), where \(1^{(t : u)}/1' = u(1/1') = (\bigwedge_{i \in I} r_i)(1/1') = (\bigwedge_{i \in I} 1^{(t : r_i)})/1'\), so \(1^{(t : u)} = \bigwedge_{i \in I} 1^{(t : r_i)}\). Thus
\[(t : (\bigwedge_{i \in I} r_i))(L) = (t : u)(L) = 1^{(t : u)}/0 = (\bigwedge_{i \in I} 1^{(t : r_i)})/0.\]

On the other hand, since \((t : r_i)(L) = 1^{(t : r_i)}/0\), we have
\[(\bigwedge_{i \in I} (t : r_i))(L) = (\bigwedge_{i \in I} 1^{(t : r_i)})/0.\]
Hence \((t : (\bigwedge_{i \in I} r_i))(L) = (\bigwedge_{i \in I} (t : r_i))(L), \forall L \in \mathcal{M}_c\), and so \(t : (\bigwedge_{i \in I} r_i) = \bigwedge_{i \in I} (t : r_i)\).

(5) The proof is similar with that in (4) just by replacing \(\bigwedge\) with \(\bigvee\). Indeed, as above, for \(L \in \mathcal{M}_c\) we have \((t : r_i)(L) = 1^{(t : r_i)}/0\), so \(r_i(1/1') = 1^{(t : r_i)}/1'\), and then \((\bigvee_{i \in I} r_i)(1/1') = (\bigvee_{i \in I} 1^{(t : r_i)})/1'\). As in (4), we have
\[(t : (\bigvee_{i \in I} r_i))(L) = (\bigvee_{i \in I} 1^{(t : r_i)})/0.\]
Since \((t : r_i)(L) = 1^{(t : r_i)}/0\), we deduce that
\[(\bigvee_{i \in I} (t : r_i))(L) = (\bigvee_{i \in I} 1^{(t : r_i)})/0,
\]
so
\[(t : (\bigvee_{i \in I} r_i))(L) = (\bigvee_{i \in I} (t : r_i))(L), \forall L \in \mathcal{M}_c.\]
Therefore \(t : (\bigvee_{i \in I} r_i) = \bigvee_{i \in I} (t : r_i)\), as desired.

(6) We claim that if \(r \in \mathcal{L}_{pr}\), \(a \in L = 1/0\) with \(a \preceq 1'\), then \(1'/a \preceq r(1/a)\). Indeed if \(p_a : 1/0 \to 1/a, p_a(x) := x \lor a, \forall x \in L,\)
is the surjective linear morphism with kernel \(a\), then \(r(p_a) : 1'/0 \to r(1/a)\) is the restriction of \(p_a\). As \(a \preceq 1'\), the kernel of \(r(p_a)\) is \(a\). So \(r(p_a)\) induces the linear morphism \(h : 1'/a \to r(1/a), h(x) = (r(p_a))(x) = p_a(x) = x \lor a, \forall x \in 1'/a.\)
Hence we have \(h(1'/a) \subseteq r(1/a)\). As \(x \succeq a\), then \(h(x) = x\). Thus \(1'/a = h(1'/a) \subseteq r(1/a)\).

On the other hand, consider the surjective linear morphism
\[ p : 1/0 \rightarrow 1/1', \ p(x) := x \lor 1', \ \forall x \in L, \]

with kernel \( k = 1' \). Since \( a \leq 1' \), we have \( p(a) = a \lor 1' = 1' \), hence \( p \) induces the linear morphism

\[ g : 1/a \rightarrow 1/1', \ g(x) := p(x) = x \lor 1', \ \forall x \in 1/a. \]

Moreover, if \( r \) is radical, then \( r(g) : r(1/a) \rightarrow r(1/1') = 1'/1' \), so \( r(g)(x) = x \lor 1' = 1' \). Consequently, by [5, Proposition 0.3(2)], we have \( x \leq k = 1', \forall x \in r(1/a) \), so \( r(1/a) \subseteq 1'/a \). Therefore \( r(1/a) = 1'/a \).

We are now going to prove the inequality

\[ (s : t) \cdot r \leq (s \cdot r) : (t \cdot r). \]

We have \((s : t) \cdot r)(1/0) = (s : t)(1'/0) = (1')(s : t)/0, \) with \( t(1'(1')) = (1'(1'))(s : t)/(1'). \)

As \( (1') = 1' \), we have \( 1'(1') \subseteq r(1'(1')) \). Indeed we have proved above that for any \( a \in L = 1/0 \) with \( a \leq 1' \), we have \( 1'/a \subseteq r(1/a) \). Now take \( a = (1') \); then \( a = (1') \subseteq 1' \), hence \( 1'(1') = 1'/a \subseteq r(1/a) = r(1'(1')). \) Thus \( 1'(1') \subseteq r(1'(1')). \)

It follows that \( t(1'(1')) \subseteq t(r(1'(1'))) \), and then

\[ (1')(1') = t(r(1'(1'))) = (t \cdot r)(1'(1')) = (s : t)(s : t) : (t \cdot r)/(1'). \]

Consequently \( (1')(1') \subseteq (s : t)/(t \cdot r) \), and then \( (1')(1')/0 \subseteq 1'(1')/(t \cdot r)/0. \) Therefore

\[ (s \cdot t) : (s \cdot r) \cdot (t \cdot r). \]

Note that in case \( r \) is a radical, then \( 1'(1') \subseteq r(1'(1')) \) by Lemma 2.4 with \( a = (1') \).

We deduce that \( t(1'(1')) = t(r(1'(1')) \) and so \((s : t) \cdot r = (s \cdot r) : (t \cdot r) \).

Conversely, assume that \((s : t) \cdot r = (s \cdot r) \cdot (t \cdot r), \ \forall s, t \in L-pr. \) If we take \( s = t = 1 \), then, as one can easily check, we have \( 1 : 1 = 1 \) and \( 1 \cdot r = r \), so

\[ (1 : 1) \cdot r = (1 \cdot r) : (1 \cdot r) = r : r, \]

i.e., \( r \) is a radical, as desired.

(7) We have

\[ 1'(s : t)/1' = (s : t)(1/1') = s(t(1/1')) = s(1'(1')). \]

As \( 1'(1') \subseteq 1' \subseteq 1'(1') \), we deduce that \( 1'(1') \subseteq 1'(1') \), and then

\[ s(1'(1') \subseteq s(1'(1') \subseteq 1'(1') \)

because \( s \) is a preradical. By definition, we have \( s(1'(1')/(1'(1')) \) and \( (r : s)(1'(1')/0) = (r : s)(1'(1')/0) = (r : s)(1'(1')/0) = (r : s)(1'(1')/0). \) It follows that

\[ 1'(s : t)/1' = s(1'(1') \subseteq s(1'(1') \subseteq 1'(1') \)

where \( s \) is a preradical. By definition, we have \( s(1'(1')/(1'(1')) \) and \( (r : s)(1'(1')/0) = (r : s)(1'(1')/0) = (r : s)(1'(1')/0). \) It follows that

\[ 1'(s : t)/1' = s(1'(1') \subseteq s(1'(1') \subseteq 1'(1') \)
Then \(1^{r(s+t)}/1^r \subseteq 1^{(r \cdot s) \cdot (r \cdot t)}/0\), and then \(1^{r(s+t)}/0 \subseteq 1^{(r \cdot s) \cdot (r \cdot t)}/0\). Thus \(1^{r(s+t)}/0 \subseteq 1^{(r \cdot s) \cdot (r \cdot t)}/0\). We deduce that \((r : (s \cdot t))(0/1) \subseteq ((r : s) \cdot (r : t))(0/1)\). Therefore \(r : (s \cdot t) \subseteq (r : s) \cdot (r : t)\).

We claim that if \(r\) is an idempotent preradical, then \(r = r \cdot t\) for any preradical \(t\) such that \(r \leq t\). Indeed, \(r \cdot t \leq r\) by Proposition 2.2. For any \(L \in M_c\) we have \(r(L) \subseteq t(L)\), and so, \(r(r(L)) \subseteq r(t(L))\). Then \(r(L) = (r \cdot r)(L) \subseteq (r \cdot t)(L)\) because \(r\) is idempotent. It follows that \(r \leq r \cdot t\). Consequently \(r = r \cdot t\), which proves our claim.

Suppose now that \(r\) is an idempotent preradical. Since \(r \leq r : t\), then \(r = r \cdot (r : t)\) by using the claim above. It follows that

\[1^r/0 = r(1/0) = (r \cdot (r : t))(1/0) = 1^{r(r : t)}/0.\]

We deduce that \(1^{r(r : t)}/1^r = 1^{(r : t)}/1^{r(r : t)}\), and so

\[s(1^{r(r : t)}/1^r) = s(1^{(r : t)}/1^{r(r : t)}).\]

But \(1^{r(s+t)}/1^r = s(1^{r(r : t)}/1^r)\) and \(s(1^{r(r : t)}/1^{r(r : t)}) = 1^{(r : s) \cdot (r : t)}/0\), hence

\[1^{r(s+t)}/1^r = 1^{(r : s) \cdot (r : t)}/0.\]

This implies that

\[1^{r(s+t)} = 1^{(r : s) \cdot (r : t)}.\]

Indeed, we have proved above that \(1^{r(s+t)}/1^r \subseteq 1^{(r : s) \cdot (r : t)}/0\), so \(1^{r(s+t)} \leq 1^{(r : s) \cdot (r : t)}\). Now, \(1^{r(s+t)}/1^r = 1^{(r : s) \cdot (r : t)}/0\) implies that \(1^{(r : s) \cdot (r : t)} \subseteq 1^{r(s+t)}/1^r\), and then \(1^{(r : s) \cdot (r : t)} \leq 1^{r(s+t)}\).

Consequently

\[1^{r(s+t)} = 1^{(r : s) \cdot (r : t)}.\]

Then \((r : (s \cdot t))(1/0) = ((r : s) \cdot (r : t))(1/0)\), i.e., \(r : (s \cdot t) = (r : s) \cdot (r : t)\).

For the converse take \(s = t = 0\). Then \(r = r : 0 = r : (0 \cdot 0) = (r : 0) \cdot (r : 0) = r \cdot r\), so \(r\) is idempotent, as desired.

\[\square\]

**Remarks 2.6.** (1) The operations \(\lor\) and \(\land\) are associative, commutative, and order-preserving.

(2) The operations \(\cdot\) and \(\lor\) are order preserving, but not necessarily commutative. \(\square\)

**Proposition 2.7.** If \((r_i)_{i \in I}\) is any family of idempotents lattice preradicals, then so is \(\lor_{i \in I} r_i\).

**Proof.** Remember that we have proved in Proposition 2.5(7) that if \(r\) is an idempotent preradical, then \(r = r \cdot t\) for any preradical \(t\) such that \(r \leq t\).

We are now going to prove the equality \((\lor_{i \in I} r_i) \cdot (\lor_{i \in I} r_i) = \lor_{i \in I} r_i\). Since \(r_i \leq \lor_{i \in I} r_i\) and \(r_i\) is idempotent, we have \(r_i = r_i \cdot (\lor_{i \in I} r_i)\), \(\forall i \in I\). By Proposition 2.5(2) we deduce that

\[(\lor_{i \in I} r_i) \cdot (\lor_{i \in I} r_i) = \lor_{i \in I} r_i \cdot (\lor_{i \in I} r_i).\]

It follows that \((\lor_{i \in I} r_i) \cdot (\lor_{i \in I} r_i) = \lor_{i \in I} r_i\), i.e., \(\lor_{i \in I} r_i\) is a preradical, as desired. \(\square\)
Proposition 2.8. If \( (r_i)_{i \in I} \) is any family of lattice radicals, then so is \( \bigwedge_{i \in I} r_i \).

Proof. If \( r,s,t \in \mathcal{L}\)-pr are such that \( r \leq s \) then \( r : t \leq s : t \) by Remarks 2.6(2). Since \( \bigwedge_{i \in I} r_i \leq r_j \), \( \forall j \in I \), we have \( (\bigwedge_{i \in I} r_i) : r_j \leq r_j : r_j \), and so

\[
(\bigwedge_{i \in I} r_i) : (\bigwedge_{i \in I} r_i) = (\bigwedge_{i \in I} (r_i : r_i)) \leq \bigwedge_{i \in I} r_i
\]

by Proposition 2.5(4).

On the other hand, for any \( u,v \in \mathcal{L}\)-pr, we have \( u \leq v : u \), so \( r_j \leq (\bigwedge_{i \in I} r_i) : r_j \), \( \forall j \in I \).

We deduce that

\[
\bigwedge_{i \in I} r_i \leq \bigwedge_{i \in I} (\bigwedge_{i \in I} r_i) : r_i = (\bigwedge_{i \in I} r_i) : (\bigwedge_{i \in I} r_i).
\]

Consequently \( (\bigwedge_{i \in I} r_i) : (\bigwedge_{i \in I} r_i) = (\bigwedge_{i \in I} r_i) \), i.e., \( (\bigwedge_{i \in I} r_i) \) is a radical, as desired. \( \square \)

We define now the latticial counterparts of the module preradicals \( \alpha^M_N \) and \( \omega^M_N \) defined in [13] for any module \( M_R \) and any fully invariant submodule \( N \) of \( M \).

Definitions 2.9. Let \( L \in \mathcal{M}_c \) and \( a \in FI(L) \). For any \( K \in \mathcal{M}_c \) we set

\[
\alpha^L_a(K) := \left( \bigvee \{ f(a) \mid f \in \text{Hom}_{\mathcal{L}M}(L,K) \} \right) / 0
\]

and

\[
\omega^L_a(K) := \left( \bigwedge \{ f^{-1}(a/0) \mid f \in \text{Hom}_{\mathcal{L}M}(K,L) \} \right) / 0.
\]

\( \square \)

For any \( f \in \text{Hom}_{\mathcal{L}M}(K,L) \) we denote \( z_f := \bigvee_{x \in f^{-1}(a/0)} x \). We claim that

\[
z_f / 0 = f^{-1}(a/0).
\]

Indeed, let \( y \in f^{-1}(a/0) \). Then \( f(y) \in a/0 \), so \( y \leq \bigvee_{x \in f^{-1}(a/0)} x = z_f \), i.e., \( y \in z_f / 0 \). It follows that \( f^{-1}(a/0) \subseteq z_f / 0 \).

For the opposite inclusion, let \( y \in z_f / 0 \). Then \( y \leq z_f = \bigvee_{x \in f^{-1}(a/0)} x \), so

\[
f(y) \leq f(\bigvee_{x \in f^{-1}(a/0)} x) = \bigvee_{x \in f^{-1}(a/0)} f(x) \leq a,
\]

i.e., \( y \in f^{-1}(a/0) \). This shows that \( z_f / 0 \subseteq f^{-1}(a/0) \) and proves our claim.

Consequently

\[
\omega^L_a(K) = \left( \bigwedge_{f \in \text{Hom}_{\mathcal{L}M}(K,L)} z_f \right) / 0.
\]

Proposition 2.10. For any \( L \in \mathcal{M}_c \) and \( a \in FI(L) \) we have \( \alpha^L_a \in \mathcal{L}\)-pr, \( \omega^L_a \in \mathcal{L}\)-pr, and

\[
\alpha^L_a(L) = a/0 = \omega^L_a(L).
\]

Proof. Let \( K \in \mathcal{M}_c \). By definition, \( \alpha^L_a(K) \) is clearly a subobject of \( K \). Let \( h : K \to K' \) be a linear morphism of lattices. For any \( f \in \text{Hom}_{\mathcal{L}M}(L,K) \) we have \( h \circ f \in \text{Hom}_{\mathcal{L}M}(L,K') \).

Since \( h \) commutes with arbitrary joins,

\[
h(\bigvee_{f \in \text{Hom}_{\mathcal{L}M}(L,K)} f(a)) = \bigvee_{f \in \text{Hom}_{\mathcal{L}M}(L,K)} (h \circ f)(a) \leq \bigvee_{g \in \text{Hom}_{\mathcal{L}M}(L,K')} g(a).
\]
Because $h$ is an increasing mapping, we have

$$h\left(\bigvee \{f(a) \mid f \in \text{Hom}_{\mathcal{L}}(L, K)\}\right)/0 \subseteq \bigvee \{g(a) \mid g \in \text{Hom}_{\mathcal{L}}(L, K')\}/0',$$

i.e., $h(\alpha^L_a(K)) \subseteq \alpha^L_a(K')$. So, by Lemma 1.1 we deduce that the restriction of $h$ to $\alpha^L_a(K)$ is a linear morphism of lattices. Therefore $\alpha^L_a \in \mathcal{L}$-pr.

In order to prove that $\omega^L_a \in \mathcal{L}$-pr, let $K \in \mathcal{M}_e$. Then

$$\omega^L_a(K) = \bigwedge_{f \in \text{Hom}_{\mathcal{L}}(K, L)} z_f/0$$

is clearly a subobject of $K$. Let $h : K \longrightarrow K'$ be a linear morphism of lattices. For any $g \in \text{Hom}_{\mathcal{L}}(K', L)$ we have $g \circ h \in \text{Hom}_{\mathcal{L}}(K, L)$. Let $x \in \omega^L_a(K)$. Then

$$x \leq \bigwedge_{g \in \text{Hom}_{\mathcal{L}}(K', L)} z_g.$$

Hence $x \leq z_{goh} = \bigwedge_{y \in (goh)^{-1}(a/0)} y$, so $x \leq y$ for all $y \in (g \circ h)^{-1}(a/0)$. As $g \circ h$ is an increasing mapping, $(g \circ h)(x) \leq (g \circ h)(y) \in a/0$. Thus $(g \circ h)(x) \in a/0$, so $h(x) \in g^{-1}(a/0)$. It follows that $h(x) \leq z_g$, $\forall g \in \text{Hom}_{\mathcal{L}}(K', L)$.

Therefore $h(x) \leq \bigwedge_{g \in \text{Hom}_{\mathcal{L}}(K', L)} z_g/0$, so

$$h(\omega^L_a(K)) \subseteq \bigwedge_{g \in \text{Hom}_{\mathcal{L}}(K', L)} z_g/0.$$

Consequently $h(\omega^L_a(K)) \subseteq \omega^L_a(K')$. So, by Lemma 1.1, we deduce that the restriction of $h$ to $\omega^L_a(K)$ is a linear morphism of lattices. Therefore $\omega^L_a \in \mathcal{L}$-pr.

We are now going to prove the equality $\alpha^L_a(L) = a/0 = \omega^L_a(L)$. Since the identity mapping $1_L$ on $L$ is in $\text{Hom}_{\mathcal{L}}(L, L)$ and $f(a) \leq a$, $\forall f \in \text{Hom}_{\mathcal{L}}(L, L)$, we have

$$\bigvee \{f(a) \mid f \in \text{Hom}_{\mathcal{L}}(L, L)\} = a,$$

i.e., $\alpha^L_a(L) = a/0$.

Let $x \in \omega^L_a(L)$. Then $f(x) \leq a$, $\forall f \in \text{Hom}_{\mathcal{L}}(L, L)$. As the identity mapping $1_L$ of $L$ is in $\text{Hom}_{\mathcal{L}}(L, L)$, it follows that $x = 1_L(x) \leq a$. Thus $\omega^L_a(L) \subseteq a/0$. Now, if $x \leq a$ and $f \in \text{Hom}_{\mathcal{L}}(L, L)$, then $f(x) \leq f(a)$. As $a \in FI(L)$, we have $f(x) \leq f(a) \leq a$. Hence $a/0 \subseteq \omega^L_a(L)$. We conclude that $a/0 = \omega^L_a(L)$, as desired.

**Corollary 2.11.** Let $L \in \mathcal{M}_e$ and $a \in L$. Then $a \in FI(L) \iff \exists r \in \mathcal{L}$-pr with $r(L) = a/0$.

**Proof.** $\implies$ follows at once from Proposition 2.10.

$\iff$ Let $r \in \mathcal{L}$-pr be such that $r(L) = r(1/0) = 1'/0 = a/0$. Then $a = 1'$. Let $f \in \text{Hom}_{\mathcal{L}}(L, L)$. As $r$ is a lattice preradical, we have $r(f) : r(L) \longrightarrow r(L)$, so

$$f(r(L)) = f(a/0) \subseteq r(L) = a/0,$$
and hence \( f(a) \leq a \), i.e., \( a \in FI(L) \).

Proposition 2.12. Let \( L \in \mathcal{M}_c \), \( r \in \mathcal{L}_{-pr} \), and \( a \in FI(L) \). Then

\[
r(L) = a/0 \iff \alpha_a^L \leq r \leq \omega_a^L.
\]

Proof. \( \implies \) For any \( K \in \mathcal{M}_c \), we have \( f(r(L)) = f(a/0) \subseteq r(K) \), \( \forall f \in \text{Hom}_{\mathcal{L}_{-pr}}(L, K) \). Thus \( f(a) \in r(K) \), \( \forall f \in \text{Hom}_{\mathcal{L}_{-pr}}(L, K) \), hence

\[
(\bigvee \{ f(a) \mid f \in \text{Hom}_{\mathcal{L}_{-pr}}(L, K) \})/0 \subseteq r(K).
\]

Indeed, \( r(K) \) is a subobject of \( K \), so \( r(K) = b/0 \) for some \( b \in K \). Since \( f(a) \in r(K) \), we have \( f(a) \leq b \), and then \( f(a)/0 \subseteq b/0 = r(K) \). So \( \alpha_a^L(K) \subseteq r(K) \), i.e., \( \alpha_a^L \leq r \).

Since

\[
g(r(K)) \subseteq r(L) = a/0, \forall g \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L),
\]

it follows that \( g(x) \in a/0, \forall x \in r(K) \), and \( g \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L) \). Therefore

\[
x \in g^{-1}(a/0), \forall g \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L).
\]

We claim that \( x \in \omega_a^L(K) \), which will clearly imply \( r \leq \omega_a^L \), and so, it will prove the implication \( \implies \).

In order to prove our claim, remember that, just before the statement of Proposition 2.10, we established the relation

\[
\omega_a^L(K) = (\bigwedge_{f \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L)} z_f)/0,
\]

where \( z_f := \bigvee_{x \in f^{-1}(a/0)} x \), and showed that \( z_f/0 = f^{-1}(a/0) \).

Since \( x \in g^{-1}(a/0), \forall g \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L) \), we have \( x \leq z_g, \forall g \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L) \). It follows that \( x \leq \bigwedge_{f \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L)} z_f \), and then \( x \in (\bigwedge_{f \in \text{Hom}_{\mathcal{L}_{-pr}}(K, L)} z_f)/0 = \omega_a^L(K) \), as claimed.

\( \iff \) As \( \alpha_a^L \leq r \leq \omega_a^L \), we have \( \alpha_a^L(L) \leq r(L) \leq \omega_a^L(L) \). By Proposition 2.10, we deduce that \( a/0 \leq r(L) \leq a/0 \). Therefore \( r(L) = a/0 \), and we are done.

Our final aim in this paper is to discuss a latticial counterpart of [13, Theorem 7] saying that \( R_{-pr} \) is an atomic big lattice having \( \{ \alpha_S^{E_R(S)} \mid S \in \text{R-simp} \} \) as the set of all atoms, where \( R_{-simp} \) is an irredundant representative set of the class of all simple right \( R \)-modules, and \( E_R(S) \) is the injective hull of \( S \), as well as saying that \( R_{-pr} \) is a coatomic big lattice having \( \{ \omega_I^R \mid I \text{ is a maximal ideal of } R \} \) as its set of coatoms.

Remarks 2.13. (1) When passing from \( R_{-pr} \) to \( \mathcal{L}_{-pr} \) we should consider simple lattices instead of simple right \( R \)-modules, and for any simple lattice \( S \), one should consider the injective hull \( \mathcal{E}(S) \) of \( S \) (see [3, Section 3]), in case it exists. Moreover, notice that a lattice may have no simple subobjects, as one can see, e.g., by considering the lattice of all subobjects of any non-zero object of the Grothendieck category \( \mathcal{G} \) presented in [10, p. 1539, \( \ell \) 1-5]. So, the set \( \mathcal{A}(\mathcal{L}_{-pr}) \) could be empty, and then, clearly the class \( \mathcal{L}_{-pr} \) cannot be atomic.
We guess that the subclass of $\mathcal{L}$-pr consisting of all lattice preradicals on the class $\mathcal{M}_{cg}$ of all compactly generated modular lattices is a coatomic big lattice. The reason to consider the subclass $\mathcal{M}_{cg}$-pr instead of the class $\mathcal{L}$-pr is that we need to involve compactly generated lattices, that allow us to obtain coatoms by using the Krull’s Lemma (see, e.g., [5, Remarks 4.2]).}

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