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Classification of right unimodal and bimodal hypersurface singularities in positive characteristic by invariants

by
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Abstract

The complete classification of right unimodal and bimodal hypersurface singularities over a field of positive characteristic was given by H. D. Nguyen in form of a classifier, which allows the concrete classification from the given equation in a step by step procedure. The aim of this article is to characterize right unimodal and bimodal hypersurface singularities of corank ≤ 2 by means of easy computable invariants such as the multiplicity, the Milnor number of the given equation and its blowing up. We also give a description of the algorithm to compute the type of right unimodal and bimodal hypersurface singularities without computing the normal form, which we have implemented in the computer algebra system SINGULAR.

Key Words: Right equivalence, Milnor number, Blowing up.
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1 Introduction

Let $K[[x,y]]$ be the local ring of formal power series in two variables, $\mathcal{M}$ its maximal ideal, K an algebraically closed field of characteristic $p > 0$ and $\mathcal{R} = Aut(K[[x,y]])$, the set of all automorphisms of $K[[x,y]]$. Let f and $g \in \mathcal{M}$ then f is said to be right equivalent to g, $f \sim_r g$ if there exists an automorphism $\phi \in \mathcal{R}$ such that $\phi(f) = g$.

Classification of singularities started by Arnold in 1972 who gave a classification of simple singularities with respect to right equivalence over the field of complex numbers [1]. These are also the simple singularities with respect to contact equivalence. Simple hypersurface singularities in charateristic $p > 0$ were classified by Greuel and Kröning [11] with respect to contact equivalence. Greuel and Nguyen [9] classified the simple hypersurface singularities in characteristic $p > 0$ with respect to right equivalence. These classifications are characterized in [6].

In [2] Arnold and in [14] Schappert classified the unimodular plane curve singularities with respect to right and contact equivalence respectively. Drozd and Greuel (cf. [7]) introduced the notion of ideal-unimodal plane curve singularities (IUS). In characteristic zero ideal-unimodal singularities and contact unimodal singularities coincide. Nguyen in [4] introduced some invariants for IUS and also gave pre-normal forms of all IUS by using the theorem on parametrization finite determinacy. In [3] a sufficient condition for IUS in terms of certain invariants is given.

Nguyen gave the classification of right unimodal and bimodal hypersurface singularities in positive characteristic [5].
In this article we use the results of [5] in order to characterize this classification for unimodal and bimodal hypersurface singularities in terms of certain invariants. Moreover, we use the names of the singularities from [5], where normal forms are given. Also we describe our implementation of a classifier for unimodal and bimodal hypersurface singularities with respect to right equivalence in SINGULAR [12],[8]. In some cases we use blowing up as a tool to differentiate differentent types. We use the right-modality as defined in [9] and used in [5].

2 Characterization of Right Uni-Modal and Bi-Modal Hypersurface Singularities

In the following we characterize all right unimodal and bimodal hypersurface singularities of corank ≤ 2 in terms of multiplicity, Milnor number and blowing-ups. We have only to consider $m(f) \leq 4$, since f is not uni or bimodal if $m(f) > 4$ (see Theorems-71, 72 and 73 in [5]). Moreover we assume $p > 7$.

Proposition 1. *Let $f \in K[[x,y]]$ such that the multiplicity $m(f) = 2$. Then*

1. *if $\mu(f) \leq p-2$ then f is simple of type A_μ;*
2. *if $p \leq \mu(f) \leq 2p-2$ then f is unimodal of type A_μ;*
3. *if $2p \leq \mu(f) \leq 3p-2$ then f is bimodal of type A_μ;*
4. *if $\mu(f) \geq 3p$ then the right modality of f, $rmod(f) \geq 3$.*

Proof. Since $m(f) = 2$ then by using the right splitting lemma (Lemma-3.7 in [9]) we can write $f \sim_r x^2 + h(y)$, where $h(y) = y^2$ if $corank(f) = 0$ and $m(h) \geq 3$ if $corank(f) = 1$. Moreover $\mathrm{rmod}(h) = \lfloor \frac{\mu(h)}{p} \rfloor$ (Theorem-3.1 in [9]), this gives $\mathrm{rmod}(f) = \lfloor \frac{\mu(f)}{p} \rfloor$. Then clearly if $\mu(f) \leq p-2$ then f is simple of type A_μ, if $p \leq \mu(f) \leq 2p-2$ then f is unimodal of type A_μ, if $2p \leq \mu(f) \leq 3p-2$ then f is bimodal of type A_μ and if $\mu(f) \geq 3p$ then $\mathrm{rmod}(f) \geq 3$. □

Definition 1. *Let $f \in K[[x,y]]$ of order s. The number of different linear factors of $j^s(f)$ (modulo constant factor) is denoted by $\gamma_s(f)$.*

Definition 2. *$j_{\{X^{\alpha_i}\}}(f)$, quasijet of f determined by $\{X^{\alpha_i}\}$, defined as follow:*
Let $\{\alpha_i\}$ is a system of n points defining an affine hyperplane H in $\mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}$ be the linear form defining H with $v(\alpha_i) = 1$ for all i. Then $j_{\{X^{\alpha_i}\}}(f)$ is the image of f in $K[[X]]$ modulo the ideal generated by X^α, $v(\alpha) > 1$.

Proposition 2. *Let $f \in K[[x,y]]$ such that $m(f) = 3$. Then $rmod(f) \geq 3$ if*

1. *$\gamma_3(f) = 1$, $p = 11$ and $j_{x^3,y^{11}}(f) = x^3 + y^{11}$;*
2. *$\gamma_3(f) = 1$ and $j_{x^3,y^{11}}(f) = x^3$;*

3. $\gamma_3(f) = 1$, $p > 3k$, $j_{x^3,y^{3k}}(f) = x^3 + x^2y^k$ *and* $\mu(f) \geq 2p + 3k - 2$, *where* $k = 2, 3$;
4. $\gamma_3(f) = 2$ *and* $\mu(h) \geq 3p$ *with* $f = x^2y + h(y)$.

Proof. (1) is a consequence of step 24, (2) is a consequence of step 25, (3) is a consequence of step 29 and (4) is a consequence of step 8 in the singularity determinator of [5]. □

Proposition 3. *Let* $f \in K[[x, y]]$ *such that* $m(f) = 4$. *Then* $rmod(f) \geq 3$ *if*

1. $\gamma_4(f) = 1$, $j_{x^4,y^6}(f) = (x^2 + y^3)^2$ *and* $\mu(f) \geq p + 8$;
2. $\gamma_4(f) = 1$ *and* $j_{x^4,y^7}(f) = x^4$;
3. $\gamma_4(f) = 1$, $j_{x^4,y^6}(f) = x^4 + x^2y^3$ *and* $\mu(f) \geq p + 8$;
4. $\gamma_4(f) = 2$ *and* $\mu(f_1) \geq p$ *or* $\mu(f_2) \geq 2p$, *where* f_1 *and* f_2 *are two irreducible branches of* f *with* $2 \leq \mu(f_1) \leq \mu(f_2)$;
5. $\gamma_4(f) = 2$, $j_{x^3y,y^7}(f) = y(x^3 + x^2y^2)$ *and* $\mu(f) \geq p + 8$;
6. $\gamma_4(f) = 2$ *and* $j_{x^3y,y^9}(f) = x^3y$;
7. $j^4(f) = x^4 + x^2y^2$ *and* $\mu(f) \geq 2p + 5$.

Proof. The proof follows from the Theorems-[34, 36, 52, 53, 63, 65, 70] in [5]. □

Proposition 4. *Let* $f \in K[[x, y]]$ *such that* $m(f) = 3$. *If* $j^3(f)$ *has two linear factors one of multiplicity* 1 *and one of multiplicity* 2 *then*

1. *if* $4 \leq \mu(f) \leq p - 1$ *then* f *is simple of type* D_μ;
2. *if* $\mu(f) = p$ *or* $p + 1 < \mu(f) \leq 2p - 1$ *then* f *is unimodal of type* D_μ;
3. *if* $\mu(f) = 2p$ *or* $2p + 1 < \mu(f) \leq 3p - 1$ *then* f *is bimodal of type* D_μ.

Proof. We may assume that $f = x^2y + \sum_{i+j\geq 4} a_{i,j}x^iy^j$. By using the transfomations

$$x \to x - \frac{a_{1,k-1}}{2}y^{k-2}$$

$$y \to y - a_{k,0}x^{k-2} - a_{k-1,1}x^{k-3}y - \cdots - a_{2,k-2}y^{k-2}$$

we can always transform f into $x^2y + \sum_{l\geq 4} a_ly^l$. Then Proposition-3.5 (i) in [9] gives if $4 \leq \mu(f) \leq p - 1$ then f is simple of type D_μ and from the Theorems 6, 7 and 8 in [5] it follows that if $\mu(f) = p$ or $p + 1 < \mu(f) \leq 2p - 1$ then f is unimodal of type D_μ and if $\mu(f) = 2p$ or $2p + 1 < \mu(f) \leq 3p - 1$ then f is bimodal of type D_μ. □

Lemma 1. *Let* $f \in K[[x, y]]$ *then* $rmod(f) \geq 2 + l$ *with* $l \geq 0$, *if either* $f \in < x, y^{3+l} >^3$ *or* $f \in < x^2, y^{3+l} >^2$.

Type	Normal form	modality
D_p	$x^2y + y^{p-1}$	1
D_k	$x^2y + ay^p + y^{k-1}$	1
E_{12}	$x^3 + y^7 + axy^5$	1
E_{13}	$x^3 + xy^5 + ay^8$	1
E_{14}	$x^3 + y^8 + axy^6$	1
$J_{10} = J_{2,0} = T_{2,3,6}$	$x^3 + y^6 + ax^2y^2$	1
$J_{2,q}$	$x^3 + ax^2y^2 + y^{6+q}$	1
D_{2p}	$x^2y + ay^p + y^{2p-1}$	2
D_k	$x^2y + a_1y^p + a_2y^{2p} + y^{k-1}$	2
E_{18}	$x^3 + y^{10} + axy^7$	2
E_{19}	$x^3 + xy^7 + ay^{11}$	2
E_{20}	$x^3 + y^{11} + axy^8$	2
$J_{10} = J_{2,0} = T_{2,3,6}$	$x^3 + bx^2y^2 + y^6 + ay^5$	2
$J_{2,q} = T_{2,3,6+q}$	$x^3 + x^2y^2 + ay^p + by^{6+q}$	2
$J_{3,0}$	$x^3 + bx^2y^3 + cxy^7 + y^9$	2
$J_{3,q}$	$x^3 + x^2y^3 + ay^{9+q}$	2

Table 1: Normal forms of unimodal and bimodal hypersurface singularities with multiplicity 3

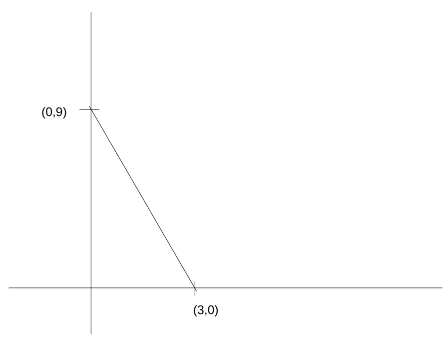

Figure 1: Newton polygon of $f \in < x, y^3 >^3$

Proof. See Lemma-5.5 in [5]. ∎

Proposition 5. *Let $f \in K[[x,y]]$ such that $f \in < x, y^3 >^3$ then $\mu(f) \geq 16$.*

Proof. The Newton polygon of f is on or above the face of Figure 1. This implies that the Newton number of f is greater or equal to the Newton number of polygon above which is 16. The theorem of Kouchnirenko ([10]) implies that the Milnor number of f is greater or equal to 16. ∎

Type	Normal form	modality
W_{12}	$x^4+y^5+ax^2y^3$	1
W_{13}	$x^4+xy^4+ay^6$	1
$X_9=X_{1,0}=T_{2,4,4}$	$x^4+y^4+ax^2y^2$	1
$X_{1,q}=T_{2,4,4+q}$	$x^4+x^2y^2+ay^{4+q}$	1
$Y_{r,s}=T_{2,4+r,4+s}$	$x^{4+r}+ax^2y^2+y^{4+s}$	1
Z_{11}	$x^3y+y^5+axy^4$	1
Z_{12}	$x^3y+xy^4+ax^2y^3$	1
Z_{13}	$x^3y+y^6+axy^5$	1
W_{17}	$x^4+xy^5+ay^7$	2
W_{18}	$x^4+y^7+ax^2y^4$	2
$W_{1,0}$	$x^4+ax^2y^3+y^6$	2
$W_{1,q}$	$x^4+x^2y^3+ay^{6+q}$	2
$W_{1,2q-1}$	$(x^2+y^3)^2+axy^{4+q}$	2
$W_{1,2q}$	$(x^2+y^3)^2+ax^2y^{3+q}$	2
Z_{17}	$x^3y+axy^6+y^8$	2
Z_{18}	$x^3y+xy^6+ay^9$	2
Z_{19}	$x^3y+y^9+axy^7$	2
$Z_{1,0}$	$x^3y+bx^2y^3+cxy^6+y^7$	2
$Z_{1,q}$	$x^3y+x^2y^3+ay^{7+q}$	2

Table 2: Normal forms of unimodal and bimodal hypersurface singularities with multiplicity 4

Proposition 6. *Let $f \in K[[x,y]]$ such that $m(f)=3$. If $j^3(f)$ has only one linear factor of multiplicity 3 and $9<\mu(f)\leq 15$ then f is unimodal.*

Proof. We may assume that $f = x^3+\sum_{i+j\geq 4} a_{i,j}x^iy^j$. Since $9<\mu(f)\leq 15$ then from the Proposition 5 it follows that $f \notin <x,y^3>^3$. An analysis of the proofs of Theorem-9 to Theorem-30 in [5] shows that f is unimodal. ∎

In the following propositions $f^{(n)}$ denote the strict transformation of n-th blow-up of f.

Proposition 7. *Let $f \in K[[x,y]]$ such that $m(f)=3$. If $j^3(f)$ has only one linear factor of multiplicity 3 and $12\leq\mu(f)\leq 14$ then*

1. *if $\mu(f^{(2)})=\mu(f)-11$ then f is of type $J_{2,\mu(f)-10}$;*
2. *if $\mu(f^{(2)})=\mu(f)-12$ then f is of type E_μ.*

Proof. We may assume that $f = x^3+\sum_{i+j\geq 4} a_{i,j}x^iy^j$. Since $12\leq\mu(f)\leq 14$ then from the proofs of Theorem-9 to Theorem-30 in [5] it follows that f is of type E_μ or $J_{2,\mu(f)-10}$. Since $j^3(f)\sim_r x^3$ the transformations bringing f to normal form has 1-jet the identity. This implies that Milnor number of the strict transform of the blowing up of f does not change under this transformation. If we compute the Milnor number of the blowing up of the normal form of E_μ or $J_{2,\mu(f)-10}$ we obtain $\mu(f^{(2)}) = \mu(f) - 12$ or $\mu(f^{(2)}) = \mu(f) - 11$ respectively. □

Proposition 8. *Let $f \in K[[x,y]]$ such that $m(f) = 3$ and $\mu(f) \geq 16$. If $j^3(f)$ has only one linear factor of multiplicity 3 then*

1. *if $\mu(f) < p + 4$ and the $j^3(f^{(1)})$ has two linear factors one of multiplicity 1 and one of multiplicity 2 then f is unimodal of type $J_{2,\mu(f)-10}$;*
2. *if $p + 4 < \mu(f) < 2p + 4$ and the $j^3(f^{(1)})$ has two linear factors one of multiplicity 1 and one of multiplicity 2 then f is bimodal of type $J_{2,\mu(f)-10}$;*
3. *if $\mu(f) < p + 7$ and the $j^3(f^{(1)})$ has only one linear factor of multiplicity 3 then f is bimodal of type $J_{3,\mu(f)-16}$;*

Proposition 9. *Let $f \in K[[x,y]]$ such that $m(f) = 3$. If $j^3(f)$ has only one linear factor of multiplicity 3 and $18 \leq \mu(f) \leq 20$ then*

1. *if $\mu(f^{(2)}) = \mu(f) - 12$ and $\mu(f^{(3)}) = \mu(f) - 17$ then f is bimodal of type $J_{3,\mu(f)-16}$;*
2. *if $\mu(f^{(2)}) = \mu(f) - 12$ and $\mu(f^{(3)}) = \mu(f) - 18$ then f is bimodal of type E_μ.*

The proof of Proposition 8 and Proposition 9 is similar to the proof of Proposition 7.

Proposition 10. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has two linear factors of multiplicity 2 then the Milnor number of its branches can be computed by using blowing ups.*

Proof. Assume $f = x^2y^2 + \sum_{i+j\geq 5} a_{i,j}x^iy^j$ then $f = f_1f_2$ such that $m(f_1) = 2 = m(f_2)$. After using the transformations

$$x \to x - \frac{1}{2}[a_{1,k-1}y^{k-3} + a_{2,k-2}xy^{k-4} + \cdots + a_{k-3,3}x^{k-4}y]$$

$$y \to y - \frac{1}{2}[a_{k-1,1}x^{k-3} + a_{k-2,2}x^{k-4}y]$$

we can assume

$$f(x,y) = x^2y^2 + \sum_{r\geq 1} a_rx^{r+4} + \sum_{s\geq 1} b_sy^{s+4}$$

with $r \leq s$. Consider the blowing up in the first chart defined by $x \to xy$, $y \to y$.

$$f(xy,y) = x^2y^4 + \sum_{r\geq 1} a_rx^{r+4}y^{r+4} + \sum_{s\geq 1} b_sy^{s+4}.$$

The strict transform is

$$\frac{f(xy,y)}{y^4} = x^2 + \sum_{r\geq 1} a_rx^{r+4}y^r + \sum_{s\geq 1} b_sy^s.$$

Which has obviously Milnor number $s - 1$. Similarly we can obtain $r - 1$ from the other chart. Then $\mu(f_1) = r + 1$ and $\mu(f_2) = s + 1$. □

Proposition 11. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has two linear factors of multiplicity 2 and $r = \mu(f_1) - 1 \leq s = \mu(f_2) - 1$, where f_1 and f_2 are two branches of f. Then*

1. *if $\mu(f_2) < p$ then f is unimodal of type $Y_{r,s}$;*
2. *if $\mu(f_1) < p$ and if $p \leq \mu(f_2) < 2p$ then f is bimodal of type $Y_{r,s}$;*

Proof. By using Proposition 10 we can find the Milnor number of each branch of f and hence the type of f. □

Proposition 12. *Let $f \in K[[x,y]]$ such that $f \in < x^2, y^3 >^2$ then $\mu(f) \geq 15$.*

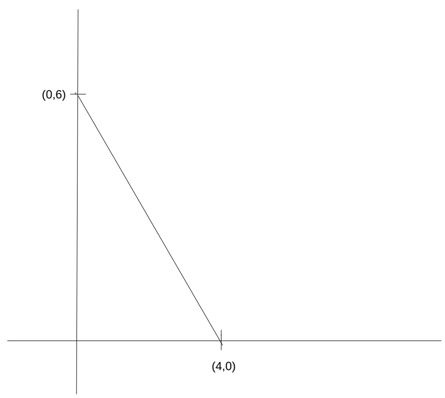

Figure 2: Newton polygon of $f \in < x^2, y^3 >^2$

Proof. The Newton polygon of f is on or above the face of Figure 2. Then similar arguments as in Proposition 5 give $\mu(f) \geq 15$. □

Proposition 13. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has only one linear factor of multiplicity 4 and $\mu(f) = 12$ or $\mu(f) = 13$ then f is unimodal of type W_μ.*

Proof. We may assume that $f = x^4 + \sum_{i+j\geq 5} a_{i,j} x^i y^j$. Since $\mu(f) = 12$ or $\mu(f) = 13$ then from the Proposition 12 it follows that $f \notin < x^2, y^3 >^2$. An analysis of the proofs of Theorem-54 to Theorem-60 in [5] gives f is unimodal of type W_μ. □

Proposition 14. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has four linear factors and $\mu(f) = 9$ then f is unimodal of type $X_{1,0}$.*

Proposition 15. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has one linear factor of multiplicity 2 and two linear factors of multiplicity 1 and $9 < \mu(f) < p + 5$ then f is unimodal of type $X_{1,\mu(f)-9}$.*

The proof of the Propositions 14 and 15 directly follows from the proofs of Theorem-31 to Theorem-34 in [5].

Proposition 16. *Let $f \in K[[x,y]]$ such that $m(f) = 4$ and $\mu(f) = 16$. If $j^4(f)$ has only one linear factor of multiplicity 4. Then*

1. *if $j^2(f^{(1)})$ has only one linear factor of multiplicity 2 then f is bimodal of type $W^\sharp_{1,1}$,*
2. *if $j^2(f^{(1)}) = 0$ and $j^3(f^{(1)})$ has three linear factors each of multiplicity 1 then f is bimodal of type $W_{1,1}$.*

Proof. The proof is similar to the proof of Proposition 7. ∎

Proposition 17. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has only one linear factor of multiplicity 4. Then*

1. *if $\mu(f) = 17$, $j^2(f^{(1)}) = 0$, $j^3(f^{(1)})$ has two linear factors, one of multiplicity 1 and one of multiplicity 2 and f has two branches f_1, f_2 such that $m(f_1) = 1$, $m(f_2) = 3$ and $\mu(f_1) = 0$, $\mu(f_2) = 8$ then f is bimodal of type W_{17},*
2. *if $\mu(f) = 17$, $j^2(f^{(1)}) = 0$, $j^3(f^{(1)})$ has two linear factors, one of multiplicity 1 and one of multiplicity 2 and f has two branches f_1, f_2 such that $m(f_1) = 2$, $m(f_2) = 2$ and $\mu(f_1) = 2$, $\mu(f_2) = 4$ then f is bimodal of type $W_{1,2}$,*
3. *if $\mu(f) = 17$, $j^2(f^{(1)})$ has only one linear factors of multiplicity 2 and f has two branches f_1, f_2 such that $m(f_1) = 2$, $m(f_2) = 2$ and $\mu(f_1) = 2$, $\mu(f_2) = 2$ then f is bimodal of type $W^\sharp_{1,2}$.*

Proof. We may assume that $f = x^4 + \sum_{i+j\geq 5} a_{i,j}x^i y^j$. If $\mu(f) = 17$ then from the proof of the classification follows that f is of type W_{17}, $W_{1,2}$ or $W^\sharp_{1,2}$. Moreover we can differentiate these types by computing the second and third jet of the strict transform of first blow up, number of branches, multiplicity of each branch and their Milnor numbers. ∎

Proposition 18. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has only one linear factor of multiplicity 4. Then*

1. *if $\mu(f) = 18$, f is irreducible, $j^2(f^{(1)})$ has one linear factor of multiplicity 2 then f is bimodal of type $W^\sharp_{1,3}$,*
2. *if $\mu(f) = 18$, f is irreducible, $j^2(f^{(1)}) = 0$ and $j^3(f^{(1)})$ has only one linear factor of multiplicity 3 then f is bimodal of type W_{18}.*
3. *if $\mu(f) = 18$, f has three branches, two of multiplicity 1 and one of multiplicity 2, $j^2(f^{(1)}) = 0$ and $j^3(f^{(1)})$ has two linear factors one of multiplicity 1 and one of multiplicity 2 then f is bimodal of type $W_{1,3}$.*

Proof. The proof is similar to the proof of Proposition 17. ∎

Proposition 19. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has only one linear factor of multiplicity 4. Then*

1. *if $19 \leq \mu(f) < p+9$, $j^2(f^{(1)}) = 0$ and $j^3(f^{(1)})$ has two linear factors one of multiplicity 2 and one of multiplicity 1 then f is bimodal of type $W_{1,\mu(f)-15}$,*
2. *if $19 \leq \mu(f) < 2p+6$ and $j^2(f^{(1)})$ has one linear factor of multiplicity 2 then f is bimodal of type $W^{\sharp}_{1,\mu(f)-15}$.*

Proof. The proof is similar to the proof of Proposition 17. □

Proposition 20. *Let $f \in K[[x,y]]$ such that $m(f) = 4$. If $j^4(f)$ has two linear factors one of multiplicity 1 and one of multiplicity 3 then*

1. *if $11 \leq \mu(f) \leq 13$ then f is unimodal of type Z_{μ},*
2. *if $\mu(f) = 15$ then f is bimodal of type $Z_{1,0}$,*
3. *if $16 \leq \mu(f) < p+8$ and $\mu(f^{(2)}) = \mu(f) - 16$ then f is bimodal of type $Z_{1,\mu(f)-15}$,*
4. *if $17 \leq \mu(f) \leq 19$ and $\mu(f^{(2)}) = \mu(f) - 17$ then f is bimodal of type Z_{μ}.*

Proof. The proof is similar to the proof of Proposition 17. □

3 Singular Examples:

We have implemented the Algorithm in the computer algebra system SINGULAR [8]. Code can be download from mathcity.org/junaid.

```
LIB"classifyReq.lib";

> ring R=11,(x,y),ds;
> poly f=4x2+xy-2y2+4x11-4y11+x15-4x14y-5x13y2-4x12y3+x11y4-x4y11
         +4x3y12+5x2y13+4xy14-y15;
> uniBimodalReq(f);
  f is unimodal of type A_14:x2+3y11+y15


> ring R=13,(x,y),ds;
> poly f=-4x4-x3y+x2y2-2xy3-3y4-x6+4x5y+2x4y2+4x3y3-2x2y4+4xy5
         +y6-5x15+4x14y-6x13y2+2x2y13+xy14+5y15;
> uniBimodalReq(f);
  f is bimodal of type Y_2,11:-3x2y2+x6+y15
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