Bull. Math. Soc. Sci. Math. Roumanie
Tome 61 (109) No. 3, 2018, 237–253

Bounds for the Betti numbers of graded modules with given Hilbert function in an exterior algebra via lexicographic modules

by
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Abstract

Let K be a field, V a finite dimensional K-vector space, E the exterior algebra of V, and F a finitely generated graded free E-module with all basis elements of the same degree. We prove that given any graded submodule M of F, there exists a unique lexicographic submodule L of F such that $H_{F/L} = H_{F/M}$. As a consequence, we are able to describe the possible Hilbert functions of graded E-modules of the type F/M. Finally, we state that the lexicographic submodules of F give the maximal Betti numbers among all the graded submodules of F with the same Hilbert function.

Key Words: Exterior algebra, Monomial ideal, Betti number, Hilbert function.
2010 Mathematics Subject Classification: Primary 13A02; Secondary 15A75, 18G10

1 Introduction

A classical problem in commutative algebra consists in studying minimal graded free resolutions of finitely generated graded modules over graded rings. In particular, a lot of work has been done to establish bounds for the Betti numbers (see, for instance, [4, 8, 7, 11, 12, 14, 15, 23, 22], and the reference therein).

Let K be a field, V a K-vector space with basis $e_1, \ldots, e_n$, and E the exterior algebra of V. In [4], Aramova, Herzog and Hibi found a necessary and sufficient condition for a function to be the Hilbert function of a graded K–algebra of the type E/I, with I graded ideal in E (see also [18]). Moreover, they showed that the lexicographic ideals provide an upper bound for the class of all graded ideals in E with the same Hilbert function. In this paper, we use the rank 1 case [4] to extend the result to graded submodules of a free E-module. More precisely, let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, and $m \in M$. Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \ldots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \ldots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. A monomial submodule M of F is a submodule of the form $M = \oplus_{i=1}^{r} I_i g_i$, with I_i $(i = 1, \ldots, r)$ monomial ideals in E. It is clear that a monomial submodule is in the category $\mathcal{M}$. A class of monomial submodules of F playing a relevant role in combinatorial commutative algebra is the class of lexicographic submodules (Definition 8). Such a class allows us to state a characterization of all possible Hilbert functions of graded E-modules of the form F/M (M graded submodule of F) with all basis elements of F of the same degree, and consequently we get upper bounds for the Betti numbers of the class of all graded submodules in F with the same Hilbert function.

The outline of the paper is as follows. Section 2 contains preliminary notions and results. In Section 3, we discuss both the class of almost lexicographic submodules (Definition 7) and the class of lexicographic submodules of F. We prove that the almost lexicographic submodules provide an upper bound for the Betti numbers of all graded submodules of F with the same Hilbert function (Proposition 1). Such a bound is not maximal in general. Finally, we give a characterization of the class of lexicographic submodules (Proposition 2). Section 4 is devoted to the study of the Hilbert functions of graded E–modules of the form F/M, with M graded submodule of F. We focus our attention on the case when $F = E^r$, *i.e.*, F is the free E-module with homogeneous basis $g_1, \dots, g_r$, where g_i $(i = 1, \dots, r)$ is the r-tuple whose only non zero–entry is 1 in the i–th position and such that $\deg(g_i) = 0$, for all i. Hence, we are able to give a generalization of the Kruskal–Katona theorem (Theorem 1). More precisely, if M is a graded submodule of E^r, we give a characterization of all possible Hilbert functions of graded E–modules of the form E^r/M $(r \geq 1)$ (Theorem 3). The crucial point for the statement of the theorem is the existence of a unique lexicographic submodule of E^r with the same Hilbert function as M. In Section 5, by combinatorial arguments and using the same techniques as in [4, 5], we prove that the lexicograhic submodules of E^r have the greatest Betti numbers among all the graded submodules of E^r with the same Hilbert function (Theorem 4). Finally, Section 6 contains our conclusions and perspectives.

2 Preliminaries and notations

Let K be a field. We denote by $E = K\langle e_1, \dots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$. For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$ with $i_1 < i_2 < \cdots < i_d$ we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$. The set of monomials in E forms a K-basis of E of cardinality 2^n.

In order to simplify the notation, we put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. An ideal I is called *graded* if I is generated by homogeneous elements. If I is graded, then $I = \oplus_{j\geq 0} I_j$, where I_j is the K-vector space of all homogeneous elements $f \in I$ of degree j. We denote by $\operatorname{indeg}(I)$ the *initial degree* of I, *i.e.*, the minimum s such that $I_s \neq 0$.

Let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. Note that if I is a graded ideal of E, then $I \in \mathcal{M}$ and $E/I \in \mathcal{M}$. Every E-module $M \in \mathcal{M}$ has a minimal graded free resolution $\mathbb{F}$ over E:

$$\mathbb{F} : \ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0,$$

where $F_i = \oplus_j E(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^E(M, K)_j$ are called the *graded Betti numbers* of M, whereas the numbers $\beta_i(M) = \sum_j \beta_{i,j}(M)$ are called the *Betti numbers* of M.

If $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \to \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the Hilbert function of M.

Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \dots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \dots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \oplus_{i=1}^r E g_i$. The elements of the form $e_\sigma g_i$, where $e_\sigma \in \operatorname{Mon}(E)$, are called *monomials* of F, and $\deg(e_\sigma g_i) = \deg(e_\sigma) + \deg(g_i)$.

In particular, if $F = E^r$ and $g_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears in the i-th place, we assume, as usual, $\deg(e_\sigma g_i) = \deg(e_\sigma)$, *i.e.*, $\deg(g_i) = f_i = 0$, for all i.

From now on, we will denote by $F = \oplus_{i=1}^r E g_i$ a free E-module with homogeneous basis $g_1, \dots, g_r$, where $\deg(g_i) = f_i$ ($i = 1, \dots, r$) with $f_1 \leq f_2 \leq \cdots \leq f_r$. Furthermore, when we write $F = E^r$, we mean that F is the free E-module $F = \oplus_{i=1}^r E g_i$ with homogeneous basis $g_1, \dots, g_r$, where g_i ($i = 1, \dots, r$) is the r-tuple where the unique non zero–entry is 1 in the i–th position, and such that $\deg(g_i) = 0$, for all i.

For any non empty subset S of E (of F, respectively), we denote by $\mathrm{Mon}(S)$ the set of all monomials in S (of F, respectively), and we denote its cardinality by $|S|$.

Definition 1. *A graded submodule M of F is a monomial submodule if M is a submodule generated by monomials of F,* i.e.*,*

$$M = I_1 g_1 \oplus \cdots \oplus I_r g_r,$$

with I_i a monomial ideal of E, for each i.

Moreover, if $r = 1$ and $f_1 = 0$, a monomial submodule is a monomial ideal of E.

Let $e_\sigma = e_{i_1} \cdots e_{i_d} \neq 1$ be a monomial in E. We define

$$\mathrm{supp}(e_\sigma) = \sigma = \{i : e_i \text{ divides } e_\sigma\},$$

and we write

$$\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\} = \max\{i : i \in \sigma\}.$$

We set $\mathrm{m}(e_\sigma) = 0$, if $e_\sigma = 1$.

Definition 2. *Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$. I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.*

Definition 3. *A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F is an* almost (strongly) stable *submodule if I_i is a (strongly) stable ideal of E, for each i.*

Definition 4. *A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F is a* (strongly) stable *submodule if I_i is a (strongly) stable ideal of E, for each i, and $(e_1, \dots, e_n)^{f_{i+1} - f_i} I_{i+1} \subseteq I_i$, for $i = 1, \dots, r-1$.*

Example 1. *Let $E = K\langle e_1, e_2, e_3, e_4 \rangle$ and $F = E^2$. The submodule*

$$M = (e_1 e_2) g_1 \oplus (e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_3 e_4) g_2$$

of F is an almost strongly stable submodule; whereas

$$N = (e_1 e_2, e_1 e_3) g_1 \oplus (e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_3 e_4) g_2$$

is a strongly stable submodule.

If I is a monomial ideal in E, we denote by $G(I)$ the unique minimal set of monomial generators of I, and by $G(I)_d$ the set of all monomials $u \in G(I)$ such that $\deg(u) = d$, $d > 0$. On the contrary, for every monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F, we set

$$G(M) = \{u g_i \,:\, u \in G(I_i), i = 1, \dots, r\},$$

and

$$G(M)_d = \{ug_i \,:\, u \in G(I_i)_{d-f_i}, i = 1, \dots, r\}.$$

Now, order the monomials of F in the *degree reverse lexicographic order*, $>_{\mathrm{degrevlex}_F}$, as follows: let $e_\sigma g_i$ and $e_\tau g_j$ be monomials of F, then $e_\sigma g_i >_{\mathrm{degrevlex}_F} e_\tau g_j$ if

- $\deg(e_\sigma g_i) > \deg(e_\tau g_j)$, or
- $\deg(e_\sigma g_i) = \deg(e_\tau g_j)$, and either $e_\sigma >_{\mathrm{revlex}} e_\tau$, or $e_\sigma = e_\tau$ and $i < j$;

$>_{\mathrm{revlex}}$ is the usual reverse lexicographic order on E (see [4]).

Any element f of F is a unique linear combination of monomials with coefficients in K. The largest monomial in this presentation with respect to $>_{\mathrm{revlex}}$ is called the *initial monomial of* f and denoted by $\mathrm{in}(f)$. If M is a graded submodule of F then the submodule of initial terms of M, denoted by $\mathrm{in}(M)$, is the submodule of F generated by the initial terms of elements of M. Using the same arguments as in the polynomial case ([10, Ch. 15], [25, Ch. 8.3], [21]; see, also, [4] for the rank one case), one has that

$$H_{F/M} = H_{F/\,\mathrm{in}(M)} \tag{2.1}$$

and

$$\beta_{i,j}(F/M) \le \beta_{i,j}(F/\,\mathrm{in}(M)), \text{ for all } i, j. \tag{2.2}$$

One can observe that, since $\mathrm{in}(M)$ is a monomial submodule of F with the same Hilbert function as M, one may assume M itself to be a monomial submodule without changing the Hilbert function.

Example 2. *Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^2$. Consider the graded submodule*

$$M = (e_1e_2e_3 + e_3e_4e_5, e_1e_3 + e_4e_5, e_2e_3e_4)g_1 \oplus (e_1e_2 + e_1e_3, e_4e_5)g_2$$

of F. M is not a monomial submodule and the initial module of M is

$$\mathrm{in}(M) = (e_1e_3, e_1e_4e_5, e_2e_3e_4, e_2e_4e_5, e_3e_4e_5)g_1 \oplus (e_1e_2, e_4e_5)g_2.$$

Note that $H_{F/M} = (2, 10, 17, 7, 0, 0) = H_{F/\,\mathrm{in}(M)}$. Finally, by comparing the Betti diagrams (as displayed by the computer program Macaulay2 [17]) of M and $\mathrm{in}(M)$

total	5	20	56	123	234	404	650
2	3	4	6	8	10	12	14
3	2	16	50	115	224	392	636

Betti diagram for M

total	7	25	63	132	245	417	665
2	3	6	9	12	15	18	21
3	4	19	54	120	230	399	644

Betti diagram for $\mathrm{in}(M)$

one can verify that the inequality in (2.2) is satisfied.

We close this Section with a relevant result on the Hilbert functions of graded K-algebras of the form E/I, with I graded ideal in E.

Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots a_j \geq j \geq 1$. We define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

We also set $0^{(i)} = 0$ for all $i \geq 1$.

We quote next result from [4].

Theorem 1. ([4, Theorem 4.1]) *Let $(h_1, \ldots, h_n)$ be a sequence of non–negative integers. Then the following conditions are equivalent:*

(a) $1 + \sum_{i=1}^{n} h_i t^i$ *is the Hilbert series of a graded K-algebra E/I;*

(b) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$.

Theorem 1 is known as the *Kruskal-Katona theorem*.

From now on, if $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K-algebra E/I, the sequence $(1, h_1, \ldots, h_n)$ will be called the Hilbert sequence of E/I, with I graded ideal in E.

Remark 1. *From the Kruskal-Katona theorem, one can deduce that a sequence of non–negative integers $(h_0, h_1, \ldots, h_n)$ is the Hilbert sequence of a graded K–algebra E/I, with I graded ideal of E of initial degree ≥ 1, if $h_0 = 1$, $h_1 \leq n$ and condition (b) in Theorem 1 holds.*

3 (Almost) Lexicographic submodules

In this Section, we analyze two special classes of monomial submodules of F that will play a fundamental role for the development of the paper: the *almost lexicographic submodules* and the *lexicographic submodules*.

Let $\mathrm{Mon}_d(E)$ be the set of all monomials of degree $d \geq 1$ in E. Denote by $>_{\mathrm{lex}}$ the *lexicographic order* (lex order, for short) on $\mathrm{Mon}_d(E)$, *i.e.*, if $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ and $e_\tau = e_{j_1} e_{j_2} \cdots e_{j_d}$ are monomials belonging to $\mathrm{Mon}_d(E)$ with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_d \leq n$, then $e_\sigma >_{\mathrm{lex}} e_\tau$ if $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $1 \leq s \leq d$.

Definition 5. *A non empty subset M of $\mathrm{Mon}_d(E)$ is called a lexicographic segment (lex segment, for short) of degree d if for all $v \in M$ and all $u \in \mathrm{Mon}_d(E)$ such that $u >_{\mathrm{lex}} v$, we have that $u \in M$.*

Definition 6. *A monomial ideal I of E is called a lexicographic ideal (lex ideal, for short) if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\mathrm{lex}} u$, then $v \in I$.*

Equivalently, a monomial ideal I of E is called a *lexicographic ideal* if $\mathrm{Mon}_d(I)$ is a lex segment, for all d; $\mathrm{Mon}_d(I)$ is the set of all monomials of degree d in I.

Remark 2. *Every lex ideal of E is obviously a (strongly) stable ideal.*

It is well known that *if I is a graded ideal of E, then there exists a unique lexsegment ideal of E, usually denoted by I^{lex}, such that $H_{E/I} = H_{E/I^{\mathrm{lex}}}$* [4, Theorem 4.1].

Now, we give the following definition.

Definition 7. *A monomial submodule $M = \oplus_{i=1}^{r} I_i g_i$ of F is an almost lexicographic submodule (almost lex submodule, for short) if I_i is a lex ideal of E, for each i.*

Next result associates to a graded submodule M of F an almost lex submodule of F which preserves the Hilbert function and provides an upper bound for the Betti numbers of the class of all graded submodules of F with given Hilbert function.

For a monomial submodule $M = \oplus_{i=1}^{r} I_i g_i$ of F, let us denote by $\mathcal{D}(M)$ the set of all the monomial ideals I_i which appear in the direct decomposition of M.

Proposition 1. *Let M be a graded submodule of F. Then there exists an almost lex submodule $\mathcal{L}$ of F such that*

(a) $H_{F/M} = H_{F/\mathcal{L}}$;

(b) $\beta_{p,q}(F/M) \le \beta_{p,q}(F/\mathcal{L})$, *for all p, q.*

Proof. First of all, from (2.1), (2.2), we may assume that M is a monomial submodule of F.
Set $M = \oplus_{j=1}^{r} I_j g_j$, with I_j monomial ideal of E, for all j. From Theorem 1 and [4, Theorem 4.1], for every $I_j \in \mathcal{D}(M)$ $(j = 1, \ldots, r)$ there exists a unique lex ideal I_j^{lex} of E such that $H_{E/I_j} = H_{E/I_j^{\mathrm{lex}}}$ and $\beta_{p,q}(E/I_j) \le \beta_{p,q}(E/I_j^{\mathrm{lex}})$, for all p, q.

Hence, setting $\mathcal{L} = \oplus_{j=1}^{r} I_j^{\mathrm{lex}} g_j$, $\mathcal{L}$ is an almost lex submodule of F such that

$$\begin{aligned} H_{F/M}(d) &= \sum_{j=1}^{r} H_{Eg_j/I_j g_j}(d) = \sum_{j=1}^{r} H_{E/I_j}(d - f_i) = \sum_{j=1}^{r} H_{E/I_j^{\mathrm{lex}}}(d - f_i) = \\ &= \sum_{j=1}^{r} H_{Eg_j/I_j^{\mathrm{lex}} g_j}(d) = H_{F/\mathcal{L}}(d), \quad \text{for all} \;\; d, \end{aligned}$$

and

$$\begin{aligned} \beta_{p,q}(F/M) &= \sum_{j=1}^{r} \beta_{p,q}(Eg_j/I_j g_j) = \sum_{j=1}^{r} \beta_{p,q-f_j}(E/I_j) \le \\ &\le \sum_{j=1}^{r} \beta_{p,q-f_j}(E/I_j^{\mathrm{lex}}) = \sum_{j=1}^{r} \beta_{p,q}(Eg_j/I_j^{\mathrm{lex}} g_j) = \beta_{p,q}(F/\mathcal{L}), \quad \text{for all} \;\; p, q. \end{aligned}$$

The assertions (a), (b) follow. ∎

If $M = \oplus_{j=1}^{r} I_j g_j$ is a monomial submodule of F, we will denote by M^{alex} the almost lex submodule of F defined in Proposition 1, *i.e.*, $M^{\mathrm{alex}} = \oplus_{j=1}^{r} I_j^{\mathrm{lex}} g_j$. Such a monomial submodule will be called the *almost lex submodule associated* to M.

Note that Proposition 1 implies that if M is a graded submodule of F, we may assume M itself to be an almost lex submodule (hence, an almost strongly stable submodule) without changing the Hilbert function.

Example 3. *Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = E^3$. Consider the monomial submodule*

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3$$

of F. Set $I_1 = (e_1e_2,\ e_3e_4)$, $I_2 = (e_1e_2,\ e_2e_3e_4)$ and $I_3 = (e_2e_3e_4)$.
If one considers the monomial ideal I_1, one has $H_{E/I_1} = (1,4,4,0,0)$ and consequently $I_1^{\mathrm{lex}} = (e_1e_2, e_1e_3, e_2e_3e_4)$. Furthermore, $H_{E/I_2} = (1,4,5,1,0)$ and $I_2^{\mathrm{lex}} = (e_1e_2, e_1e_3e_4)$; whereas, $H_{E/I_3} = (1,4,6,3,0)$ and $I_3^{\mathrm{lex}} = (e_1e_2e_3)$. Therefore,

$$M^{\mathrm{alex}} = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3,$$

and $H_{F/M^{\mathrm{alex}}} = (3,12,15,4,0) = H_{F/M}$. Finally, if we compare the Betti diagrams of M and M^{alex}

total	5	14	29	52	85	130
2	3	6	9	12	15	18
3	2	8	20	40	70	112

Betti diagram for M

total	6	18	38	68	110	166
2	3	7	12	18	25	33
3	3	11	26	50	85	133

Betti diagram for M^{alex}

the inequalities on the Betti numbers of Proposition 1 (b) are verified.

Remark 3. *It is worthy to be highlighted that if M is a graded submodule of F, then almost lex submodules which are not equal to M^{alex} but with the same Hilbert function as M could exist. Indeed, let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$ and*

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3.$$

The almost lex submodule of F associated to M is

$$M^{\mathrm{alex}} = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3,$$

and $H_{F/M^{\mathrm{alex}}} = (3,12,15,4,0) = H_{F/M}$. The following submodule

$$N = (e_1e_2, e_1e_3, e_1e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3,$$

of F is an almost lex submodule (different from M^{alex}) such that $H_{F/N} = (3,12,15,4,0) = H_{F/M}$.

Now, for every d, let F_d be the part of degree d of $F = \oplus_{i=1}^{r} Eg_i$, *i.e.*, the K-vector space of homogeneous elements of F of degree d. Denote by $\mathrm{Mon}_d(F)$ the set of all monomials of degree d of F. We order such a set by the ordering $>_{\mathrm{lex}_F}$ defined as follows:

if ug_i and vg_j are monomials of F such that $\deg(ug_i) = \deg(vg_j)$, then $ug_i >_{\text{lex}_F} vg_j$ if $i < j$ or $i = j$ and $u >_{\text{lex}} v$.

For example, if $E = K\langle e_1, e_2, e_3\rangle$, the monomials of degree 2 of $F = E^2$ are ordered as follows (with respect to $>_{\text{lex}_F}$):

$$e_1e_2g_1 >_{\text{lex}_F} e_1e_3g_1 >_{\text{lex}_F} e_2e_3g_1 >_{\text{lex}_F} e_1e_2g_2 >_{\text{lex}_F} e_1e_3g_2 >_{\text{lex}_F} e_2e_3g_2.$$

Definition 8. *Let $\mathcal{L}$ be a monomial submodule of F. $\mathcal{L}$ is a lexicographic submodule (lex submodule, for short) if for all $u, v \in \text{Mon}_d(F)$ with $u \in \mathcal{L}$ and $v >_{\text{lex}_F} u$, one has $v \in \mathcal{L}$, for every d.*

Let us give the following definition.

Definition 9. *A non empty subset N of $\text{Mon}_d(F)$ is called a lexicographic segment of F (lex$_F$ segment, for short) of degree d if for all $v \in N$ and all $u \in \text{Mon}_d(F)$ such that $u >_{\text{lex}_F} v$, then $u \in N$.*

Example 4. *Let $E = K\langle e_1, e_2, e_3\rangle$ and $F = E^2$. The subset $N = \{e_1e_2g_1, e_1e_3g_1, e_2e_3g_1, e_1e_2g_2\}$ is a lex$_F$ segment of degree 2 of F; on the contrary, $N' = \{e_1e_2g_1, e_1e_3g_1, e_1e_2g_2\}$ is not a lex$_F$ segment of degree 2. Indeed, the monomial $e_2e_3g_1 >_{\text{lex}_F} e_1e_2g_2$ does not belong to N'.*

Remark 4. *A monomial submodule $\mathcal{L}$ of F is a lexicographic submodule if $\text{Mon}_d(\mathcal{L})$ is a lex$_F$ segment of degree d, for each degree d; $\text{Mon}_d(\mathcal{L})$ is the set of all monomials of degree d of $\mathcal{L}$.*

Next characterization holds.

Proposition 2. *Let $\mathcal{L}$ be a graded submodule of F. Then $\mathcal{L}$ is a lex submodule of F if and only if*

(i) *$\mathcal{L} = \oplus_{i=1}^r I_i g_i$, with I_i lex ideals of E , for $i = 1, \dots, r$, and*

(ii) *$(e_1, \dots, e_n)^{\rho_i + f_i - f_{i-1}} \subseteq I_{i-1}$, for $i = 2, \dots, r$, with $\rho_i = \text{indeg} I_i$.*

Proof. The proof is verbatim the same as [15, Proposition 3.8]. We include it to make the paper self contained.

Let $\mathcal{L}$ be a lex submodule of F.

(i) Since $\mathcal{L}$ is a monomial submodule of F, one has $\mathcal{L} = \oplus_{i=1}^r I_i g_i$, with I_i monomial ideal of E, for every i. Let $u, v \in \text{Mon}_d(E)$ with $u \in I_i$ and $v >_{\text{lex}} u$. It follows that $ve_i >_{\text{lex}_F} ue_i$. Since $ug_i \in I_ig_i$ and $\mathcal{L}$ is a lex submodule of F, $vg_i \in I_ig_i$, and so $v \in I_i$, *i.e.*, I_i is a lex ideal of E for every i.

(ii) Since I_i is a lex ideal of E, then $e_1e_2\cdots e_{\rho_i} \in I_i$, $\rho_i = \text{indeg} I_i$, and consequently $e_1e_2\cdots e_{\rho_i} g_i \in I_ig_i$. On the other hand, $\mathcal{L}$ is a lex submodule of F, then for all $u \in (e_1, \dots, e_n)^{\rho_i + f_i - f_{i-1}}$, we have that $ug_{i-1} >_{\text{lex}_F} e_1e_2\cdots e_{\rho_i} g_i$. Hence, $ug_{i-1} \in I_{i-1}g_{i-1}$, *i.e.*, $u \in I_{i-1}$.

Conversely, let $\mathcal{L}$ be a graded submodule of F satisfying (i) and (ii).

Since every ideal I_i is a lex ideal, we have only to prove that for any pair (i, j) of integers with $1 \le i < j \le r$, if $ug_i, vg_j \in \text{Mon}_d(\mathcal{L})$, then $vg_j \in \mathcal{L}$ implies $ug_i \in \mathcal{L}$, where $\text{Mon}_d(\mathcal{L})$ is the set of all monomials of degree d of $\mathcal{L}$.

(Case 1). $i = j-1$. Let $ug_{j-1}, ve_j \in \mathrm{Mon}_d(\mathcal{L})$ with $vg_j \in \mathcal{L}$.
Since $d = \deg ug_{j-1} = \deg vg_j$, it follows that $\deg u = \deg v + f_j - f_{j-1} \geq \rho_j + f_j - f_{j-1}$ and so $u \in (e_1, \ldots, e_n)^{\rho_j + f_j - f_{j-1}} \subseteq I_{j-1}$.
(Case 2). $i \leq j-2$. Let $ug_i >_{\mathrm{lex}_F} vg_j$ with $ug_i, vg_j \in \mathrm{Mon}_d(\mathcal{L})$ and $vg_j \in \mathcal{L}$.
For $t = i+1, \ldots, j-1$, set $w_t = e_1 e_2 \cdots e_{d-f_t}$. It is

$$ug_i >_{\mathrm{lex}_F} w_{i+1}g_{i+1} >_{\mathrm{lex}_F} w_{i+2}g_{i+2} >_{\mathrm{lex}_F} \cdots >_{\mathrm{lex}_F} w_{j-1}g_{j-1} >_{\mathrm{lex}_F} vg_j.$$

Since $d = \deg ug_i = \deg w_t g_t = \deg vg_j$, for $t = i+1, \ldots, j-1$, then, from (Case 1), $w_{j-1} \in I_{j-1}$, $w_{j-2} \in I_{j-2}$, …, $w_{i+1} \in I_{i+1}$ and finally $u \in I_i$. □

The result above immediately yields that every lex submodule of F is a strongly stable submodule (see [15, Proposition 3.9]). Moreover, it is clear that a lex submodule is an almost lex submodule. The converse does not hold, as next example illustrates.

Example 5. (1) *Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^3$. The submodule*

$$\begin{aligned} M = &(e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)g_1 \oplus (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)g_2 \oplus \\ &(e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)g_3 \end{aligned}$$

of F is not a lex submodule of F even if the ideals $(e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)$, $(e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)$, $(e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)$ are lex ideals of E. In fact, $e_1e_2g_2 \in M_2$ but $e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2$ and $e_2e_3g_1 \notin M_2$. Observe that $(e_1, e_2, e_3, e_4, e_5)^2 \not\subseteq (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)$. M is an almost lex submodule of F.

(2) *Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^3$. The submodule*

$$\begin{aligned} \mathcal{L} = &(e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4, e_2e_3e_5, e_2e_4e_5, e_3e_4e_5)g_1 \oplus \\ &(e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4e_5, e_2e_3e_4e_5)g_2 \oplus (e_1e_2e_3e_4, e_1e_2e_3e_5, e_1e_2e_4e_5, e_1e_3e_4e_5)g_3 \end{aligned}$$

is a lex submodule of F.

4 A generalization of the Kruskal-Katona theorem

Let M be a graded submodule of F. The purpose of this Section is to describe the possible Hilbert functions of F/M when $F = E^r$, $r \geq 1$.

Theorem 2. ([4, Theorem 4.2]) *Let J be a lex ideal generated in degree s with $\dim_K E_s/J_s = a$. Then $\dim_K E_{s+1}/J_{s+1} = a^{(s)}$.*

For a graded submodule $M = \oplus_{j \geq 0} M_j$ of F, the *initial degree* of M, denoted by $\mathrm{indeg}(M)$, is the minimum s such that $M_s \neq 0$.

Let $F = E^r$, $r \geq 1$. If M is a set of monomials of degree $d < n$ of F, we denote by $\mathrm{Shad}(M)$ the following set of monomials of degree $d+1$ of F:

$$\mathrm{Shad}(M) = \{(-1)^{\alpha(\sigma,j)} e_j e_\sigma g_i : e_\sigma g_i \in M,\ j \notin \mathrm{supp}(e_\sigma),\ j = 1, \ldots, n,\ i = 1, \ldots r\},$$

$\alpha(\sigma, j) = |\{t \in \sigma : t < j\}|$. Such a set is called the *shadow* of M (see [14], for the r=1 case).

Furthermore, if M is a monomial submodule of F and M_d ($d \geq 1$) is the K-vector space generated by all monomials of degree d belonging to M, we set $\mathrm{Shad}(M_d) = \mathrm{Shad}(\mathrm{Mon}(M_d))$ and by $E_1 M_d$ the K-vector space spanned by $\mathrm{Shad}(M_d)$.

For istance, if $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^2$ and $M = (e_1e_2, e_1e_3)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2$, then $M_2 = \langle e_1e_2g_1, e_1e_3g_1, e_1e_2g_2\rangle$ (as K-vector space) and $E_1M_2 = \langle \mathrm{Shad}(M_2)\rangle = \langle e_1e_2e_3g_1$, $e_1e_2e_4g_1$, $e_1e_3e_4g_1, e_1e_2e_3g_2, e_1e_2e_4g_2\rangle$ (as K-vector space).

Next result generalizes the Kruskal-Katona theorem (Theorem 1).

Theorem 3. *Let $(h_1, \ldots, h_n)$ be a sequence of non negative integers. Then the following conditions are equivalent:*

(a) *$r + \sum_{i=1}^n h_i t^i$ is the Hilbert series of a graded E-module E^r/M, $r \geq 1$;*

(b) *$h_i = \sum_{j=1}^r h_{i,j}$, for $i = 1, \ldots, n$, and $(h_{1,j}, h_{2,j}, \ldots, h_{n,j})$ is an n-tuple of non negative integers such that $0 < h_{i+1,j} \leq h_{i,j}^{(i)}$, for $0 < i \leq n-1$ and $j = 1, \ldots, r$, $r \geq 1$;*

(c) *there exists a unique lexicographic submodule $\mathcal{L}$ of F such that $r + \sum_{i=1}^n h_i t^i$ is the Hilbert series of $E^r/\mathcal{L}$, $r \geq 1$.*

Proof. First of all, note that for $r = 1$ the required equivalences follow from Theorem 1. Hence, let $r > 1$.
(a) $\Rightarrow$ (b). From (2.1), we may assume that M is a monomial submodule of E^r. Set $M = \oplus_{j=1}^r I_j g_j$, with I_j ($j = 1, \ldots, r$) monomial ideals of E; $g_j = (0, \ldots, 0, 1, \ldots, 0)$, with 1 in the j-th position. By assumption, $\mathrm{indeg}(M) \geq 1$ and $h_i = H_{E^r/M}(i)$ ($1 \leq i \leq n$). On the other hand, the additivity of the Hilbert function implies the existence of an r-tuple of integers $(h_{i,1}, \ldots, h_{i,r})$ such that

$$h_i = \sum_{j=1}^r h_{i,j}, \qquad 1 \leq i \leq n.$$

More in details, $h_{i,j} = H_{E/I_j}(i)$ ($1 \leq j \leq r$), *i.e.*, $(1, h_{1,j}, h_{2,j}, \ldots, h_{n,j})$ is the Hilbert sequence of E/I_j. Hence, from Theorem 1 (b), $h_{i+1,j} \leq h_{i,j}^{(i)}$ ($0 < i \leq n-1$, $1 \leq j \leq r$). The assertion follows.
(b) $\Rightarrow$ (c). Set $F = E^r$. We construct a lexicographic submodule $\mathcal{L}$ of F such that $H_{F/\mathcal{L}} = r + \sum_{i=1}^n h_i t^i$.
Assume $\dim_K F_1 = h_1$ and let $\mathcal{L}_1 = 0$. Hence, $h_1 = rn$. Suppose $\mathcal{L}_k$, $k \leq i$, has already been constructed. Let s be the smallest integer such that $\mathcal{L}_s \neq \emptyset$. One has

$$\mathrm{Mon}(\mathcal{L}_{s+p}) = \mathrm{Shad}(\mathcal{L}_{s+p-1}) \cup T_{s+p}, \quad p = 1, \ldots, i-s,$$

where $\mathrm{Mon}(\mathcal{L}_{s+p})$ is the set of all the monomial generators of $\mathcal{L}_{s+p}$ (as a K-vector space) and T_{s+p} is a subset of $\mathrm{Mon}_{s+p}(F)$ such that $T_{s+p} \cap \mathrm{Shad}(\mathcal{L}_{s+p-1}) = \emptyset$. It is worth to underline that the sets $\mathrm{Shad}(\mathcal{L}_{s+p-1}) \cup T_{s+p} \neq \emptyset$ are lex_F segments of degree $s+p$, $p = 1, \ldots, i-s$.

Let $\mathcal{L}'$ be the monomial submodule of F, with minimal set of monomial generators defined as follows:

$$G(\mathcal{L}') = \mathrm{Mon}(\mathcal{L}_s) \cup \left(\cup_{p=1}^{i-s} T_{s+p}\right).$$

We can write $\mathcal{L}'$ as
$$\mathcal{L}' = I_1 g_1 \oplus \cdots \oplus I_r g_r,$$
with

- I_j lexicographic ideals, for $j = 1, \ldots, r$,
- $\min\{\deg(ug_j) : u \in G(I_j), j = 1, \ldots, r\} = s$,
- $\max\{\deg(ug_j) : u \in G(I_j), j = 1, \ldots, r\} = i$.

Note that $h_i = H_{F/\mathcal{L}}(i) = H_{F/\mathcal{L}'}(i) = \sum_{j=1}^{r} H_{E/I_j}(i)$. Setting $h_{i,j} = H_{E/I_j}(i)$ $(j = 1, \ldots, r)$, from Theorem 2, one has:

$$h_{i+1,j} = H_{E/I_j}(i+1) \leq h_{i,j}^{(i)} = \dim_K E_{i+1}/E_1 I_{j,i},$$

where $I_{j,i}$ is the part of degree i of the monomial ideal I_j. Hence, $E_1 I_{j,i} \subseteq I_{j,i+1}$ and consequently $E_1 \mathcal{L}_i = E_1 \mathcal{L}'_i \subseteq \mathcal{L}_{i+1}$. It follows that $\mathcal{L} = \oplus_{p \geq s} \mathcal{L}_p$ is a submodule of F which is lex. Finally, the uniqueness is clear by the definition of lex submodules.
(c)$\Rightarrow$ (a). It follows immediately. □

If M is a submodule of E^r, we will denote by M^{lex} the unique lex submodule of E^r such that $H_{E^r/M} = H_{E^r/M^{\mathrm{lex}}}$.

Example 6. *Let $E = K\langle e_1, \ldots, e_4 \rangle$. Consider the following submodule of E^3:*

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3.$$

M is a monomial submodule with $H_{E^3/M} = (3, 12, 15, 4, 0)$. One has that

$$M^{\mathrm{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

We can notice that $H_{E^3/M} = H_{E^3/M^{\mathrm{lex}}}$. Finally, one can observe that $M^{\mathrm{lex}} \neq M^{\mathrm{alex}}$ (Example 3).

5 Upper bounds for the Betti numbers

In this Section, we show that lex submodules give upper bounds for the Betti numbers of any graded submodule of E^r with the same Hilbert function by generalizing the techniques of the rank one case [4, 5]. More precisely, we prove that a lex submodule of E^r has the largest Betti numbers among all graded submodules of E^r with the same Hilbert function.

Let us introduce some notations. For a monomial $e_\sigma g_i$ of $F = \oplus_{i=1}^{r} E g_i$, let

$$\mathrm{m}_F(e_\sigma g_i) = \mathrm{m}(e_\sigma), \qquad 1 \leq i \leq r,$$

and if M is a monomial submodule of F, let us define

$$G(M : j) = \{e_\sigma g_i \in G(M) : \mathrm{m}_F(e_\sigma g_i) = j, i = 1, \ldots, r\}, \tag{5.1}$$

$$\mathrm{m}_j^F(M) = |G(M:j)|,\ 1 \le j \le n, \qquad \mathrm{m}_{\le t}^F(M) = \sum_{j=1}^{t} \mathrm{m}_j^F(M),\ 1 \le t \le n. \tag{5.2}$$

One can observe that $\mathrm{m}_{\le n}^F(M) = |G(M)|$.

If M is a monomial ideal of E, then (5.1) and (5.2) can be rewritten as follows ([4]):

$$G(M:j) = \{e_\sigma \in G(M) : \mathrm{m}(e_\sigma) = j\},$$

$$\mathrm{m}_j(M) = |G(M:j)|,\ 1 \le j \le n, \qquad \mathrm{m}_{\le t}(M) = \sum_{j=1}^{t} \mathrm{m}_j(M),\ 1 \le t \le n.$$

Next remark will be crucial for the proof of the main result in this Section.

Remark 5. *Recall that a set T of monomials of degree $d \ge 1$ in E is called* strongly stable *(of degree d) if for each monomial $e_\sigma \in T$ and each $j \in \sigma$ one has that $(-1)^{\alpha(\sigma,i)} e_i e_{\sigma\setminus\{j\}} \in T$, where $\alpha(\sigma, i) = |\{t \in \sigma : t < i\}|$, for all $i < j$.*

Due to the nature of a lex segment, one can quickly verify that if T is a strongly stable set of degree d and L is a lex segment of degree d such that $|T| = |L|$, then $\mathrm{m}_{\le i}(L) = |\{e_\sigma \in L : \mathrm{m}(e_\sigma) \le i\}| \le \mathrm{m}_{\le i}(T) = |\{e_\sigma \in T : \mathrm{m}(e_\sigma) \le i\}|$, for all $i \le n$.

One can also note that $|L| = \mathrm{m}_{\le n}(L) = \mathrm{m}_{\le n}(T) = |T|$.

As as consequence, one has that if I is a strongly stable ideal of E generated in degree d and J is a lex ideal of E generated in degree d with $|G(I)| = |G(J)|$, then $|G(J)| = \mathrm{m}_{\le n}(J) = \mathrm{m}_{\le n}(I) = |G(I)|$, and $\mathrm{m}_{\le i}(J) \le \mathrm{m}_{\le i}(I)$, for all $i \le n-1$.

Let us consider an almost strongly stable submodule $M = \oplus_{i=1}^r I_i g_i$ of E^r generated in degree d, and let $\mathcal{L} = \oplus_{i=1}^r L_i g_i$ be the lex submodule of E^r generated in degree d such that $\dim M_d = \dim \mathcal{L}_d$. Every monomial ideal $I_i \in \mathcal{D}(M)$ is generated by a strongly stable set T_i $(i = 1, \dots n)$ of degree d. The construction of the lex submodule $\mathcal{L}$ of F with $|G(\mathcal{L})| = |G(M)|$ implies that $\mathrm{m}_n^F(\mathcal{L}) \ge \mathrm{m}_n^F(M)$. Hence, $\mathrm{m}_{\le i}^F(\mathcal{L}) \le \mathrm{m}_{\le i}^F(M)$, for all $i \le n-1$.

By using combinatorial arguments one can quickly prove the following lemma.

Lemma 1. *Let M be an almost strongly stable submodule of E^r generated in degree d. If $M_{\langle d+1\rangle}$ is the submodule of E^r generated by the elements of M_{d+1}, then*

$$\mathrm{m}_i^F(M_{\langle d+1\rangle}) = \mathrm{m}_{\le i-1}^F(M), \ \textit{for all } i.$$

If $M = \oplus_{i=1}^r I_i g_i$ is an (almost) stable submodule of F, then we can use the *Aramova-Herzog-Hibi formula* [4, Corollary 3.3] for computing the graded Betti numbers of M:

$$\beta_{k,k+\ell}(M) = \sum_{i=1}^{r} \beta_{k,\,k+\ell}(I_i g_i) = \sum_{i=1}^{r} \left[\sum_{u \in G(I_i)_{\ell - f_\ell}} \binom{\mathrm{m}(u)+k-1}{\mathrm{m}(u)-1} \right], \quad \text{for all } k \ge 0.$$

Theorem 4. *Let M be a graded submodule of E^r. Then*

$$\beta_{i,j}(M) \le \beta_{i,j}(M^{\mathrm{lex}}), \quad \textit{for all } i, j.$$

Proof. From Proposition 1, we may assume that M is an almost lex submodule, and consequently an almost stable submodule, of E^r. Therefore, setting $F = E^r$, one has:

$$\beta_{i,i+j}(M) = \sum_{u \in G(M)_j} \binom{\mathrm{m}_F(u)+i-1}{\mathrm{m}_F(u)-1}, \quad \text{for all } i \geq 1. \tag{5.3}$$

Since $G(M)_j = G(M_{\langle j\rangle}) - G(M_{\langle j-1\rangle})\{e_1,\dots,e_n\}$, due to Lemma 1, the above sum can be written as a difference $\beta_{i,i+j}(M) = C - D$, with

$$\begin{aligned}
C &= \sum_{u \in G(M_{\langle j\rangle})} \binom{\mathrm{m}_F(u)+i-1}{\mathrm{m}_F(u)-1} \\
&= \sum_{t=1}^{n} \sum_{u \in G(M_{\langle j\rangle};t)} \binom{t+i-1}{t-1} = \sum_{t=1}^{n} \mathrm{m}_t^F(M_{\langle j\rangle}) \binom{t+i-1}{t-1} \\
&= \sum_{t=1}^{n} (\mathrm{m}_{\leq t}^F(M_{\langle j\rangle}) - \mathrm{m}_{\leq t-1}^F(M_{\langle j\rangle})) \binom{t+i-1}{t-1} \\
&= \mathrm{m}_{\leq n}^F(M_{\langle j\rangle}) \binom{n+i-1}{n-1} \\
&+ \sum_{t=1}^{n-1} \mathrm{m}_{\leq t}^F(M_{\langle j\rangle}) \left[\binom{t+i-1}{t-1} - \binom{t+1+i-1}{t}\right] \\
&= \mathrm{m}_{\leq n}^F(M_{\langle j\rangle}) \binom{n+i-1}{n-1} - \sum_{t=1}^{n-1} \mathrm{m}_{\leq t}^F(M_{\langle j\rangle}) \binom{t+i-1}{t},
\end{aligned}$$

and

$$\begin{aligned}
D &= \sum_{u \in G(M_{\langle j-1\rangle})\{e_1,\dots,e_n\}} \binom{\mathrm{m}_F(u)+i-1}{\mathrm{m}_F(u)-1} \\
&= \sum_{t=2}^{n} m_{\leq t-1}^F(M_{\langle j-1\rangle}) \binom{t+i-1}{t-1}.
\end{aligned}$$

On the other hand, since the number of generators of $M_{\langle d\rangle}$ and $M_{\langle d\rangle}^{\mathrm{lex}}$ are equal for all d, we have $\mathrm{m}_{\leq n}^F(M_{\langle d\rangle}) = \mathrm{m}_{\leq n}^F(M_{\langle d\rangle}^{\mathrm{lex}})$. Moreover, from Remark 5, $\mathrm{m}_{\leq i}^F(M_{\langle d\rangle}^{\mathrm{lex}}) \leq \mathrm{m}_{\leq i}^F(M_{\langle d\rangle})$ for $1 \leq i \leq n-1$. Hence,

$$\begin{aligned}
\beta_{i,i+j}(M) &= \mathrm{m}_{\leq n}^F(M_{\langle j\rangle}) \binom{n+i-1}{n-1} - \sum_{t=1}^{n-1} \mathrm{m}_{\leq t}^F(M_{\langle j\rangle}) \binom{t+i-1}{t} \\
&- \sum_{t=2}^{n} m_{\leq t-1}^F(M_{\langle j-1\rangle}) \binom{t+i-1}{t-1} \\
&\leq \mathrm{m}_{\leq n}^F(M_{\langle j\rangle}^{\mathrm{lex}}) \binom{n+i-1}{n-1} - \sum_{t=1}^{n-1} \mathrm{m}_{\leq t}^F(M_{\langle j\rangle}^{\mathrm{lex}}) \binom{t+i-1}{t} \\
&- \sum_{t=2}^{n} m_{\leq t-1}^F(M_{\langle j-1\rangle}^{\mathrm{lex}}) \binom{t+i-1}{t-1} = \beta_{i,i+j}(M^{\mathrm{lex}}).
\end{aligned}$$

□

Finally, from Proposition 1 and Theorem 4 the next result follows.

Corollary 1. *Let M be a graded submodule of E^r. Then*

$$\beta_{i,j}(M) \leq \beta_{i,j}(M^{\text{alex}}) \leq \beta_{i,j}(M^{\text{lex}}), \quad \textit{for all } i, j.$$

Example 7. Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^3$. The submodule

$$M = (e_1e_2, e_1e_4, e_3e_4e_5)g_1 \oplus (e_1e_3, e_1e_4e_5, e_2e_3e_4)g_2 \oplus (e_1e_2e_4, e_1e_3e_5)g_3$$

of F is not an almost lex submodule of F. It is sufficient to observe that the ideal $(e_1e_2, e_1e_4, e_3e_4e_5)$ is not a lex ideals of E. Consider the almost lex submodule

$$\begin{aligned} M^{\text{alex}} =&(e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)g_1 \oplus (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)g_2 \oplus \\ &(e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)g_3, \end{aligned}$$

which is not a lex submodule of F (see Example 5), and the lex submodule

$$\begin{aligned} M^{\text{lex}} =&(e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4, e_2e_3e_5, e_2e_4e_5, e_3e_4e_5)g_1 \oplus \\ &(e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4e_5, e_2e_3e_4e_5)g_2 \\ &\oplus (e_1e_2e_3e_4, e_1e_2e_3e_5, e_1e_2e_4e_5, e_1e_3e_4e_5)g_3 \end{aligned}$$

One can quickly verify that $H_{F/M} = (3, 15, 27, 17, 1, 0) = H_{F/M^{\text{alex}}} = H_{F/M^{\text{lex}}}$.

Moreover, using the computer program *Macaulay2*, if one compares the Betti diagrams of the submodules above considered, one has the Corollary 1:

total	8	26	59	113	195	313	476
2	3	7	12	18	25	33	42
3	5	18	42	80	135	210	308
4	–	1	5	15	35	70	126

Betti diagram for M

total	11	43	113	243	460	796	1288
2	3	7	12	18	25	33	42
3	5	21	56	120	225	385	616
4	3	15	45	105	210	378	630

Betti diagram for M^{alex}

total	16	69	190	419	805	1406	2289
2	3	9	19	34	55	83	119
3	7	31	86	190	365	637	1036
4	6	29	85	195	385	686	1134

Betti diagram for M^{lex}

6 Conclusions and perspectives

In this paper, we have discussed some classes of monomial submodules of a finitely generated graded free E–module F. Functions for computing monomial ideals in a polynomial ring are available in many computer algebra system, CAS, (for instance, CoCoA [1], *Macaulay2* [17] and Singular [16]); on the contrary, to the best of our knowledge, specific packages for manipulating classes of monomial ideals in an exterior algebra have not been implemented yet. Forced by this situation, the authors of this paper have developed a new package [2], written for *Macaulay2* [17], for manipulating classes of monomial ideals in an exterior algebra of a finite dimensional vector space over a field. Currently, the authors are trying to implement such a package for monomial submodules over an exterior algebra.

Furthermore, Theorem 3 describes the possible Hilbert functions of graded E-modules of the type F/M with M graded submodule of F, when $F \simeq E^r$, $r > 1$. It would be nice to generalize such a result in the case when the basis elements of the finitely generated free E-module F have different degrees. The following question is currently under investigation.

Open 1. *Let $F = \oplus_{i=1}^r Eg_i$ be a finitely generated graded free E-module with homogeneous basis $g_1, \dots, g_r$ such that $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_r$ and let $H = (h_1, \dots, h_t)$, $t \geq n$, be a sequence of non negative integers. Under which conditions for the h_i's does there exist a graded submodule M of F such that $H = H_M$?*

A generalization of the Kruskal–Katona Theorem for finitely generated modules can be found in [19, Theorem 4.3].

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Received: 05.03.2018
Accepted: 21.06.2018

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