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Expansions of arithmetic functions of several variables with respect to certain modified unitary Ramanujan sums

by

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Abstract

We introduce new analogues of the Ramanujan sums, denoted by $\widetilde{c}_q(n)$, associated with unitary divisors, and obtain results concerning the expansions of arithmetic functions of several variables with respect to the sums $\widetilde{c}_q(n)$. We apply these results to certain functions associated with $\sigma^*(n)$ and $\phi^*(n)$, representing the unitary sigma function and unitary phi function, respectively.

Key Words: Ramanujan expansion of arithmetic functions, arithmetic function of several variables, multiplicative function, unitary divisor, sum of unitary divisors, unitary Euler function, unitary Ramanujan sum.
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1 Introduction

Let $c_q(n)$ denote the Ramanujan sums, defined by

$$c_q(n) = \sum_{\substack{1\le k\le q\\(k,q)=1}} \exp(2\pi ikn/q),$$

where $q,n \in \mathbb{N} = \{1,2,\ldots\}$. Let $\sigma(n)$ be, as usual, the sum of divisors of n. Ramanujan's [7] classical identity

$$\frac{\sigma(n)}{n} = \zeta(2)\sum_{q=1}^{\infty}\frac{c_q(n)}{q^2} \quad (n\in\mathbb{N}), \tag{1.1}$$

where ζ is the Riemann zeta function, can be generalized as

$$\frac{\sigma((n_1,\ldots,n_k))}{(n_1,\ldots,n_k)} = \zeta(k+1)\sum_{q_1,\ldots,q_k=1}^{\infty}\frac{c_{q_1}(n_1)\cdots c_{q_k}(n_k)}{[q_1,\ldots,q_k]^{k+1}} \quad (n_1,\ldots,n_k\in\mathbb{N}), \tag{1.2}$$

valid for any $k\in\mathbb{N}$. See the author [15, Eq. (28)]. Here $(n_1,\ldots,n_k)$ and $[n_1,\ldots,n_k]$ stand for the greatest common divisor and the least common multiple, respectively, of $n_1,\ldots,n_k$. For $k=2$ identity (1.2) was deduced by Ushiroya [18, Ex. 3.8].

By making use of the unitary Ramanujan sums $c_q^*(n)$, we also have

$$\frac{\sigma((n_1,\dots,n_k))}{(n_1,\dots,n_k)} = \zeta(k+1) \sum_{q_1,\dots,q_k=1}^{\infty} \frac{\phi_{k+1}([q_1,\dots,q_k])}{[q_1,\dots,q_k]^{2(k+1)}} c_{q_1}^*(n_1)\cdots c_{q_k}^*(n_k) \quad (n_1,\dots,n_k \in \mathbb{N}), \tag{1.3}$$

for any $k \in \mathbb{N}$. See [15, Eq. (30)]. The notations used here (and throughout the paper), which are not explained in the text, are included in Section 2.1. In fact, (1.2) and (1.3) are special cases of the following general result, which can be applied to several other special functions, as well.

Theorem 1 ([15, Th. 4.3])**.** *Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $k \in \mathbb{N}$. Assume that*

$$\sum_{n=1}^{\infty} 2^{k\,\omega(n)} \frac{|(\mu * g)(n)|}{n^k} < \infty.$$

Then for every $n_1,\dots,n_k \in \mathbb{N}$,

$$g((n_1,\dots,n_k)) = \sum_{q_1,\dots,q_k=1}^{\infty} a_{q_1,\dots,q_k} c_{q_1}(n_1)\cdots c_{q_k}(n_k),$$

$$g((n_1,\dots,n_k)) = \sum_{q_1,\dots,q_k=1}^{\infty} a^*_{q_1,\dots,q_k} c^*_{q_1}(n_1)\cdots c^*_{q_k}(n_k)$$

are absolutely convergent, where

$$a_{q_1,\dots,q_k} = \frac{1}{Q^k} \sum_{m=1}^{\infty} \frac{(\mu * g)(mQ)}{m^k},$$

$$a^*_{q_1,\dots,q_k} = \frac{1}{Q^k} \sum_{\substack{m=1\\(m,Q)=1}}^{\infty} \frac{(\mu * g)(mQ)}{m^k},$$

with the notation $Q = [q_1,\dots,q_k]$.

Recall that d is a unitary divisor of n if $d \mid n$ and $(d, n/d) = 1$. Notation $d \parallel n$. Let $\sigma^*(n)$, defined as the sum of unitary divisors of n, be the unitary analogue of $\sigma(n)$. Properties of the function $\sigma^*(n)$, compared to those of $\sigma(n)$ were investigated by several authors. See, e.g., Cohen [1], McCarthy [6], Sitaramachandrarao and Suryanarayana [10], Sitaramaiah and Subbarao [11], Trudgian [16]. For example, one has

$$\sum_{n\le x} \sigma^*(n) = \frac{\pi^2 x^2}{12\zeta(3)} + O(x(\log x)^{5/3}).$$

In this paper we are looking for unitary analogues of formulas (1.1) and (1.2). Theorem 1 can be applied to the function $g(n) = \sigma^*(n)/n$. However, in this case $(\mu * g)(p) = 1/p$, $(\mu * g)(p^\nu) = (1-p)/p^\nu$ for any prime p and any $\nu \ge 2$. Hence the coefficients of the corresponding expansion can not be expressed by simple special functions, and we consider the obtained identities unsatisfactory.

Let $(k,n)_{**}$ denote the greatest common unitary divisor of k and n. Note that $d \parallel (k,n)_{**}$ holds true if and only if $d \parallel k$ and $d \parallel n$. Bi-unitary analogues of the Ramanujan sums may be defined as follows:

$$c_q^{**}(n) = \sum_{\substack{1\le k\le q \\ (k,q)_{**}=1}} \exp(2\pi i k n/q) \quad (q,n\in\mathbb{N}),$$

but the function $q \mapsto c_q(n)$ is not multiplicative, and its properties are not parallel to the sums $c_q(n)$ and $c_q^*(n)$. The function $c_q^{**}(q) = \phi^{**}(q)$, called bi-unitary Euler function was investigated in our paper [13].

Therefore, we introduce in Section 2.3 new analogues of the Ramanujan sums, denoted by $\widetilde{c}_q(n)$, also associated with unitary divisors, and show that

$$\frac{\sigma^*((n_1,\ldots,n_k)_{*k})}{(n_1,\ldots,n_k)_{*k}} = \zeta(k+1) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\phi_{k+1}([q_1,\ldots,q_k])}{[q_1,\ldots,q_k]^{2(k+1)}} \widetilde{c}_{q_1}(n_1)\cdots \widetilde{c}_{q_k}(n_k), \quad n_i\in\mathbb{N}, \tag{1.4}$$

where $(n_1,\ldots,n_k)_{*k}$ denotes the greatest common unitary divisor of $n_1,\ldots,n_k\in\mathbb{N}$. Now formulas (1.2), (1.3) and (1.4) are of the same shape. In the case $k=1$, identity (1.4) gives

$$\frac{\sigma^*(n)}{n} = \zeta(2)\sum_{q=1}^{\infty} \frac{\phi_2(q)}{q^4}\widetilde{c}_q(n) \quad (n\in\mathbb{N}),$$

which may be compared to (1.1).

We also deduce a general result for arbitrary arithmetic functions f of several variables (Theorem 2), which is the analogue of [15, Th. 4.1], concerning the Ramanujan sums $c_q(n)$ and their unitary analogues $c_q^*(n)$. We point out that in the case $k=1$, Theorem 2 is the analogue of the result of Delange [2], concerning classical Ramanujan sums. As applications, we consider the functions $f(n_1,\ldots,n_k) = g((n_1,\ldots,n_k)_{*k})$, where g belongs to a large class of functions of one variable, including $\sigma^*(n)/n$ and $\phi^*(n)/n$, where ϕ^* is the unitary Euler function (Theorem 3).

For background material on classical Ramanujan sums and Ramanujan expansions (Ramanujan-Fourier series) of functions of one variable we refer to the book by Schwarz and Spilker [9] and to the survey papers by Lucht [5] and Ram Murty [8]. Section 2 includes some general properties on arithmetic functions of one and several variables defined by unitary divisors, needed in the present paper.

2 Preliminaries

2.1 Notations

- $\mathbb{P}$ is the set of (positive) primes,
- the prime power factorization of $n\in\mathbb{N}$ is $n=\prod_{p\in\mathbb{P}} p^{\nu_p(n)}$, the product being over the primes p, where all but a finite number of the exponents $\nu_p(n)$ are zero,

- $(f*g)(n)=\sum_{d\mid n} f(d)g(n/d)$ is the Dirichlet convolution of the functions $f,g:\mathbb{N}\to\mathbb{C}$,
- id_s is the function $\mathrm{id}_s(n)=n^s$ $(n\in\mathbb{N}, s\in\mathbb{R})$,
- $\mathbf{1}=\mathrm{id}_0$ is the constant 1 function,
- μ is the Möbius function,
- $\omega(n)$ stands for the number of distinct prime divisors of n,
- ϕ_s is the Jordan function of order s given by $\phi_s(n)=n^s\prod_{p\mid n}(1-1/p^s)$ $(s\in\mathbb{R})$,
- $\phi=\phi_1$ is Euler's totient function,
- $d\parallel n$ means that d is a unitary divisor of n, i.e., $d\mid n$ and $(d,n/d)=1$ (we remark that this is in concordance with the standard notation $p^\nu\parallel n$ used for prime powers p^ν),
- $(k,n)_*=\max\{d : d\mid k, d\parallel n\}$,
- $c_q^*(n)=\sum_{1\le k\le q,(k,q)_*=1}\exp(2\pi ikn/q)$ are the unitary Ramanujan sums $(q,n\in\mathbb{N})$,
- $(n_1,\ldots,n_k)_{*k}$ denotes the greatest common unitary divisor of $n_1,\ldots,n_k\in\mathbb{N}$,
- $(n_1,n_2)_{**}=(n_1,n_2)_{*2}$,
- $\sigma_s^*(n)=\sum_{d\parallel n} d^s$ $(s\in\mathbb{R})$,
- $\sigma^*(n)=\sigma_1^*(n)$ is the sum of unitary divisors of n,
- $\tau^*(n)=\sigma_0^*(n)$ is the number of unitary divisors of n, which equals $2^{\omega(n)}$.

2.2 Functions defined by unitary divisors

The study of arithmetic functions defined by unitary divisors goes back to Vaidyanathaswamy [17] and Cohen [1]. The function $\sigma^*(n)$ was already defined above. The analog of Euler's ϕ function is ϕ^*, defined by $\phi^*(n)=\#\{k\in\mathbb{N}:1\le k\le n,(k,n)_*=1\}$. The functions σ^* and ϕ^* are multiplicative and $\sigma^*(p^\nu)=p^\nu+1$, $\phi^*(p^\nu)=p^\nu-1$ for any prime powers p^ν $(\nu\ge 1)$.

The unitary convolution of the functions f and g is

$$(f\times g)(n)=\sum_{d\parallel n} f(d)g(n/d)\quad (n\in\mathbb{N}),$$

it preserves the multiplicativity of functions, and the inverse of the constant 1 function under the unitary convolution is μ^*, where $\mu^*(n)=(-1)^{\omega(n)}$, also multiplicative. The set $\mathcal{A}$ of arithmetic functions forms a unital commutative ring with pointwise addition and the unitary convolution, having divisors of zero.

2.3 Modified unitary Ramanujan sums

For $q,n\in\mathbb{N}$ we introduce the functions $\widetilde{c}_q(n)$ by the formula

$$\sum_{d\parallel q}\widetilde{c}_d(n)=\begin{cases} q, & \text{if } q\parallel n,\\ 0, & \text{if } q\nparallel n.\end{cases}\tag{2.1}$$

It follows that $\widetilde{c}_q(n)$ is multiplicative in q,

$$\widetilde{c}_{p^\nu}(n)=\begin{cases} p^\nu-1, & \text{if } p^\nu\parallel n,\\ -1, & \text{if } p^\nu\nparallel n,\end{cases}\tag{2.2}$$

for any prime powers p^ν $(\nu \geq 1)$ and

$$\widetilde{c}_q(n) = \sum_{d \| (n,q)_{**}} d\mu^*(q/d) \quad (q, n \in \mathbb{N}).$$

We will need the following result.

Proposition 1. *For any $q, n \in \mathbb{N}$,*

$$\sum_{d\|q} |\widetilde{c}_d(n)| = 2^{\omega(q/(n,q)_{**})}(n,q)_{**}, \tag{2.3}$$

$$\sum_{d\|q} |\widetilde{c}_d(n)| \leq 2^{\omega(q)} n. \tag{2.4}$$

Proof. If $q = p^\nu$ $(\nu \geq 1)$ is a prime power, then we have by (2.2),

$$\sum_{d\|p^\nu} |c_d^*(n)| = |c_1^*(n)| + |c_{p^\nu}^*(n)| = \begin{cases} 1 + p^\nu - 1 = p^\nu, & \text{if } p^\nu \parallel n, \\ 1 + 1 = 2, & \text{otherwise.} \end{cases}$$

Now (2.3) follows at once by the multiplicativity in q of the involved functions, while (2.4) is its immediate consequence. ∎

For classical Ramanujan sums the inequality corresponding to (2.4) is crucial in the proof of the theorem of Delange [2], while the identity corresponding to (2.3) was pointed out by Grytczuk [3]. In the case of unitary Ramanujan sums the counterparts of (2.3) and (2.4) were proved by the author [15, Prop. 3.1].

Proposition 2. *For any $q, n \in \mathbb{N}$,*

$$\widetilde{c}_q(n) = \frac{\phi^*(q)\mu^*(q/(n,q)_{**})}{\phi^*(q/(n,q)_{**})}. \tag{2.5}$$

Proof. Both sides of (2.5) are multiplicative in q. If $q = p^\nu$ $(\nu \geq 1)$ is a prime power, then

$$\frac{\phi^*(p^\nu)\mu^*(p^\nu/(n,p^\nu)_{**})}{\phi^*(p^\nu/(n,p^\nu)_{**})} = \begin{cases} \frac{\phi^*(p^\nu)\mu^*(1)}{\phi^*(1)} = p^\nu - 1, & \text{if } p^\nu \parallel n, \\ \frac{\phi^*(p^\nu)\mu^*(p^\nu)}{\phi^*(p^\nu)} = -1, & \text{otherwise.} \end{cases} = c_{p^\nu}(n),$$

by (2.2). ∎

For the Ramanujan sums $c_q(n)$ the identity similar to (2.5) is usually attributed to Hölder, but was proved earlier by Kluyver [4]. In the case of the unitary Ramanujan sums $c_q^*(n)$ the counterpart of (2.5) was deduced by Suryanarayana [12].

Basic properties (including those mentioned above) of the classical Ramanujan sums $c_q(n)$, their unitary analogues $c_q^*(n)$ and the modified sums $\widetilde{c}_q(n)$ can be compared by the next table.

$c_q(n) = \sum_{d\mid(n,q)} d\mu(q/d)$	$c_q^*(n) = \sum_{d\mid(n,q)_*} d\mu^*(q/d)$	$\widetilde{c}_q(n) = \sum_{d\Vert(n,q)_{**}} d\mu^*(q/d)$
$c_q(n) = \dfrac{\phi(q)\mu(q/(n,q))}{\phi(q/(n,q)}$	$c_q^*(n) = \dfrac{\phi^*(q)\mu^*(q/(n,q)_*)}{\phi^*(q/(n,q)_*)}$	$\widetilde{c}_q(n) = \dfrac{\phi^*(q)\mu^*(q/(n,q)_{**})}{\phi^*(q/(n,q)_{**})}$
$c_{p^\nu}(n) = \begin{cases} p^\nu - p^{\nu-1}, & \text{if } p^\nu \mid n, \\ -p^{\nu-1}, & \text{if } p^{\nu-1} \Vert n, \\ 0, & \text{if } p^{\nu-1} \nmid n \end{cases}$	$c_{p^\nu}^*(n) = \begin{cases} p^\nu - 1, & \text{if } p^\nu \mid n, \\ -1, & \text{if } p^\nu \nmid n \end{cases}$	$\widetilde{c}_{p^\nu}(n) = \begin{cases} p^\nu - 1, & \text{if } p^\nu \Vert n, \\ -1, & \text{if } p^\nu \not\Vert n \end{cases}$
$\sum_{d\mid q} c_d(n) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n \end{cases}$	$\sum_{d\Vert q} c_d^*(n) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n \end{cases}$	$\sum_{d\Vert q} \widetilde{c}_d(n) = \begin{cases} q, & \text{if } q \Vert n, \\ 0, & \text{if } q \not\Vert n \end{cases}$
$\sum_{d\mid q} \lvert c_d(n)\rvert = 2^{\omega(q/(n,q))}(n,q)$	$\sum_{d\Vert q} \lvert c_d^*(n)\rvert = 2^{\omega(q/(n,q)_*)}(n,q)_*$	$\sum_{d\Vert q} \lvert\widetilde{c}_d(n)\rvert = 2^{\omega(q/(n,q)_{**})}(n,q)_{**}$

Table: Properties of $c_q(n)$, $c_q^*(n)$ and $\widetilde{c}_q(n)$

2.4 Arithmetic functions of several variables

For every fixed $k \in \mathbb{N}$ the set $\mathcal{A}_k$ of arithmetic functions $f : \mathbb{N}^k \to \mathbb{C}$ of k variables is a unital commutative ring with pointwise addition and the unitary convolution defined by

$$(f \times g)(n_1, \ldots, n_k) = \sum_{d_1\Vert n_1, \ldots, d_k \Vert n_k} f(d_1, \ldots, d_k) g(n_1/d_1, \ldots, n_k/d_k), \tag{2.6}$$

the unity being the function δ_k, where

$$\delta_k(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if } n_1 = \cdots = n_k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The inverse of the constant 1 function under (2.6) is μ_k^*, given by

$$\mu_k^*(n_1, \ldots, n_k) = \mu^*(n_1) \cdots \mu^*(n_k) = (-1)^{\omega(n_1)+\cdots+\omega(n_k)} \quad (n_1, \ldots, n_k \in \mathbb{N}).$$

A function $f \in \mathcal{A}_k$ is said to be multiplicative if it is not identically zero and

$$f(m_1n_1, \ldots, m_kn_k) = f(m_1, \ldots, m_k) f(n_1, \ldots, n_k)$$

holds for any $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbb{N}$ such that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$.

If f is multiplicative, then it is determined by the values $f(p^{\nu_1}, \ldots, p^{\nu_k})$, where p is prime and $\nu_1, \ldots, \nu_k \in \mathbb{N} \cup \{0\}$. More exactly, $f(1, \ldots, 1) = 1$ and for any $n_1, \ldots, n_k \in \mathbb{N}$,

$$f(n_1, \ldots, n_k) = \prod_{p\in\mathbb{P}} f(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_k)}).$$

Similar to the one dimensional case, the unitary convolution (2.6) preserves the multiplicativity of functions. See our paper [14], which is a survey on (multiplicative) arithmetic functions of several variables.

3 Main results

First we prove the following general result.

Theorem 2. *Let $f : \mathbb{N}^k \to \mathbb{C}$ be an arithmetic function ($k \in \mathbb{N}$). Assume that*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} 2^{\omega(n_1)+\cdots+\omega(n_k)} \frac{|(\mu_k^* \times f)(n_1,\ldots,n_k)|}{n_1\cdots n_k} < \infty. \tag{3.1}$$

Then for every $n_1,\ldots,n_k \in \mathbb{N}$,

$$f(n_1,\ldots,n_k) = \sum_{q_1,\ldots,q_k=1}^{\infty} \widetilde{a}_{q_1,\ldots,q_k} \widetilde{c}_{q_1}(n_1)\cdots\widetilde{c}_{q_k}(n_k), \tag{3.2}$$

where

$$\widetilde{a}_{q_1,\ldots,q_k} = \sum_{\substack{m_1,\ldots,m_k=1 \\ (m_1,q_1)=1,\ldots,(m_k,q_k)=1}}^{\infty} \frac{(\mu_k^* \times f)(m_1q_1,\ldots,m_kq_k)}{m_1q_1\cdots m_kq_k}, \tag{3.3}$$

the series (3.2) being absolutely convergent.

Proof. We have for any $n_1,\ldots,n_k \in \mathbb{N}$, by using property (2.1),

$$f(n_1,\ldots,n_k) = \sum_{d_1\|n_1,\ldots,d_k\|n_k} (\mu_k^* \times f)(d_1,\ldots,d_k)$$

$$= \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{(\mu_k^* \times f)(d_1,\ldots,d_k)}{d_1\cdots d_k} \sum_{q_1\|d_1} \widetilde{c}_{q_1}(n_1)\cdots \sum_{q_k\|d_k} \widetilde{c}_{q_k}(n_k)$$

$$= \sum_{q_1,\ldots,q_k=1}^{\infty} \widetilde{c}_{q_1}(n_1)\cdots\widetilde{c}_{q_k}(n_k) \sum_{\substack{d_1,\ldots,d_k=1 \\ q_1\|d_1,\ldots,q_k\|d_k}}^{\infty} \frac{(\mu_k^* \times f)(d_1,\ldots,d_k)}{d_1\cdots d_k},$$

leading to expansion (3.2) with the coefficients (3.3), by denoting $d_1 = m_1q_1,\ldots,d_k = m_kq_k$. The rearranging of the terms is justified by the absolute convergence of the multiple series, shown hereinafter:

$$\sum_{q_1,\ldots,q_k=1}^{\infty} |\widetilde{a}_{q_1,\ldots,q_k}||\widetilde{c}_{q_1}(n_1)|\cdots|\widetilde{c}_{q_k}(n_k)|$$

$$\le \sum_{\substack{q_1,\ldots,q_k=1 \\ m_1,\ldots,m_k=1 \\ (m_1,q_1)=1,\ldots,(m_k,q_k)=1}}^{\infty} \frac{|(\mu_k^* \times f)(m_1q_1,\ldots,m_kq_k)|}{m_1q_1\cdots m_kq_k} |\widetilde{c}_{q_1}(n_1)|\cdots|\widetilde{c}_{q_k}(n_k)|$$

$$= \sum_{t_1,\ldots,t_k=1}^{\infty} \frac{|(\mu_k^* \times f)(t_1,\ldots,t_k)|}{t_1\cdots t_k} \sum_{\substack{m_1q_1=t_1 \\ (m_1,q_1)=1}} |\widetilde{c}_{q_1}(n_1)|\cdots \sum_{\substack{m_kq_k=t_k \\ (m_k,q_k)=1}} |\widetilde{c}_{q_k}(n_k)|$$

$$\leq n_1 \cdots n_k \sum_{t_1,\ldots,t_k=1}^{\infty} 2^{\omega(t_1)+\cdots+\omega(t_k)} \frac{|(\mu_k^* \times f)(t_1,\ldots,t_k)|}{t_1\cdots t_k} < \infty,$$

by using inequality (2.4) and condition (3.1). ∎

Next we consider the case $f(n_1,\ldots,n_k) = g((n_1,\ldots,n_k)_{*k})$. The following result is the analogue of Theorem 1.

Theorem 3. *Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $k \in \mathbb{N}$. Assume that*

$$\sum_{n=1}^{\infty} 2^{k\,\omega(n)} \frac{|(\mu^* \times g)(n)|}{n^k} < \infty.$$

Then for every $n_1,\ldots,n_k \in \mathbb{N}$,

$$g((n_1,\ldots,n_k)_{*k}) = \sum_{q_1,\ldots,q_k=1}^{\infty} \widetilde{a}_{q_1,\ldots,q_k} \widetilde{c}_{q_1}(n_1)\cdots\widetilde{c}_{q_k}(n_k),$$

is absolutely convergent, where

$$\widetilde{a}_{q_1,\ldots,q_k} = \frac{1}{Q^k} \sum_{\substack{m=1\\(m,Q)=1}}^{\infty} \frac{(\mu^* \times g)(mQ)}{m^k}, \tag{3.4}$$

with the notation $Q = [q_1,\ldots,q_k]$.

Proof. We apply Theorem 2. Taking into account the identity

$$g((n_1,\ldots,n_k)_{*k}) = \sum_{d\|n_1,\ldots,d\|n_k} (\mu^* \times g)(d)$$

we see that now

$$(\mu_k^* \times f)(n_1,\ldots,n_k) = \begin{cases} (\mu^* \times g)(n), & \text{if } n_1 = \cdots = n_k = n,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore the coefficients of the expansion are

$$\begin{aligned}\widetilde{a}_{q_1,\ldots,q_k} &= \sum_{\substack{n=1\\ m_1q_1=\cdots=m_kq_k=n\\ (m_1,q_1)=1,\ldots,(m_k,q_k)=1}}^{\infty} \frac{(\mu_k^* \times f)(m_1q_1,\ldots,m_kq_k)}{m_1q_1\cdots m_kq_k}\\ &= \sum_{\substack{n=1\\ q_1\|n,\ldots,q_k\|n}}^{\infty} \frac{(\mu^* \times g)(n)}{n^k},\end{aligned}$$

and we use that $q_1 \parallel n,\ldots,q_k \parallel n$ holds if and only if $[q_1,\ldots,q_k] = Q \parallel n$, that is, $n = mQ$ with $(m,Q) = 1$. ∎

Corollary 1. *For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:*

$$\frac{\sigma_s^*((n_1,\ldots,n_k)_{*k})}{(n_1,\ldots,n_k)_{*k}^s} = \zeta(s+k) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\phi_{s+k}(Q)\widetilde{c}_{q_1}(n_1)\cdots \widetilde{c}_{q_k}(n_k)}{Q^{2(s+k)}} \quad (s \in \mathbb{R}, s+k>1), \tag{3.5}$$

$$\tau^*((n_1,\ldots,n_k)_{*k}) = \zeta(k) \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\phi_k(Q)\widetilde{c}_{q_1}(n_1)\cdots \widetilde{c}_{q_k}(n_k)}{Q^{2k}} \quad (k \ge 2). \tag{3.6}$$

Proof. Apply Theorem 3 to $g(n) = \sigma_s^*(n)/n^s$. Here

$$\mu^* \times g = \mu^* \times \frac{\mathbf{1} \times \mathrm{id}_s}{\mathrm{id}_s} = (\mu^* \times \mathbf{1}) \times \frac{\mathbf{1}}{\mathrm{id}_s} = \frac{\mathbf{1}}{\mathrm{id}_s},$$

hence $(\mu^* \times g)(n) = 1/n^s$ $(n \in \mathbb{N})$. We deduce by (3.4) that

$$\widetilde{a}_{q_1,\ldots,q_k} = \frac{1}{Q^{s+k}} \sum_{\substack{m=1\\(m,Q)=1}}^{\infty} \frac{1}{m^{s+k}} = \zeta(s+k)\frac{\phi_{s+k}(Q)}{Q^{2(s+k)}},$$

which completes the proof. ∎

In the case $s=1$ identity (3.5) reduces to (1.4). Now let consider the function $\phi_s^*(n) = \prod_{p^\nu \| n}(p^{s\nu}-1)$, representing the unitary Jordan function of order s. Here $\phi_s^* = \mu^* \times \mathrm{id}_s$, and $\phi_1^* = \phi^*$ is the unitary Euler function, already mentioned in Section 2.2.

Corollary 2. *For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:*

$$\frac{\phi_s^*((n_1,\ldots,n_k)_{*k})}{(n_1,\ldots,n_k)_{*k}^s} = \zeta(s+k) \prod_{p \in \mathbb{P}} \left(1 - \frac{2}{p^{s+k}}\right) \times \tag{3.7}$$

$$\times \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\mu^*(Q)\phi_{s+k}(Q)\widetilde{c}_{q_1}(n_1)\cdots \widetilde{c}_{q_k}(n_k)}{Q^{2(s+k)}\prod_{p\mid Q}(1-2/p^{s+k})} \quad (s \in \mathbb{R}, s+k>1),$$

$$\frac{\phi^*((n_1,\ldots,n_k)_{*k})}{(n_1,\ldots,n_k)_{*k}} = \zeta(k+1) \prod_{p \in \mathbb{P}} \left(1 - \frac{2}{p^{k+1}}\right) \times$$

$$\times \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\mu^*(Q)\phi_{k+1}(Q)\widetilde{c}_{q_1}(n_1)\cdots \widetilde{c}_{q_k}(n_k)}{Q^{2(k+1)}\prod_{p\mid Q}(1-2/p^{k+1})} \quad (k \ge 1).$$

Proof. Apply Theorem 3 to $g(n) = \phi_s^*(n)/n^s$. Here

$$\mu^* \times g = \mu^* \times \frac{\mu^* \times \mathrm{id}_s}{\mathrm{id}_s} = (\mu^* \times \mathbf{1}) \times \frac{\mu^*}{\mathrm{id}_s} = \frac{\mu^*}{\mathrm{id}_s},$$

that is, $(\mu^* \times g)(n) = \mu^*(n)/n^s$ $(n \in \mathbb{N})$. We deduce by (3.4) that

$$\widetilde{a}_{q_1,\ldots,q_k} = \frac{1}{Q^{s+k}} \sum_{\substack{m=1\\(m,Q)=1}}^{\infty} \frac{\mu^*(mQ)}{m^{s+k}} = \frac{\mu^*(Q)}{Q^{s+k}} \sum_{\substack{m=1\\(m,Q)=1}}^{\infty} \frac{\mu^*(m)}{m^{s+k}}$$

$$= \frac{\mu^*(Q)}{Q^{s+k}}\zeta(s+k)\prod_{p\in\mathbb{P}}\left(1-\frac{2}{p^{s+k}}\right)\prod_{p\mid Q}\left(1-\frac{1}{p^{s+k}}\right)\left(1-\frac{2}{p^{s+k}}\right)^{-1},$$

leading to (3.7). ∎

For $m \in \mathbb{N}$, $m \geq 2$ consider the function $g(n) = m^{\omega(n)}$, which is the unitary analogue of the Piltz divisor function $\tau_m(n)$. Here $m^{\omega(n)} = \sum_{d\|n}(m-1)^{\omega(d)}$ for any $n \in \mathbb{N}$. We obtain by similar arguments:

Corollary 3. *For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series is absolutely convergent:*

$$m^{\omega((n_1,\ldots,n_k)_{*k})} = \zeta(k)\prod_{p\in\mathbb{P}}\left(1+\frac{m-2}{p^k}\right)\times \tag{3.8}$$

$$\times \sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\phi_k(Q)(m-1)^{\omega(Q)}\widetilde{c}_{q_1}(n_1)\cdots\widetilde{c}_{q_k}(n_k)}{Q^{2k}\prod_{p\mid Q}(1+(m-2)/p^k)} \quad (m,k\geq 2),$$

For $m = 2$ identity (3.8) reduces to (3.6).

Remark 1. It is possible to formulate the results of Theorem 2 in the case of multiplicative functions f of k variables, and Theorem 3 in the case of multiplicative functions g of one variable. Note that if g is multiplicative, then $f(n_1,\ldots,n_k) = g((n_1,\ldots,n_k)_{*k})$ is multiplicative, viewed as a function of k variables. See also Delange [2] and the author [15]. Furthermore, it is possible to apply the above results to other special (multiplicative) functions. We do not go into more details.

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