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Three lemmas on commutators

by

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Abstract

A group G, two subgroups H, K of G and a subgroup A of $Aut(G)$ are considered. One looks for conditions on G, H, K, and A ensuring that the set of all commutators $[g, a]$ with $g \in G$ and $a \in A$ is contained in the union $H \cup K$.

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1 Introduction

If G is a group and if $H, K, M \leqslant G$ such that $M \subseteq H \cup K$ it is well known that either $M \leqslant H$ or $M \leqslant K$; in particular, if H and K are proper subgroups of G, then $G \neq H \cup K$. For $x, y \in G$ we write $[x, y] = x^{-1}x^y$ for the commutator of x and y and we denote by C the set of all commutators in G. The set C is not always a subgroup of G and if $C \subseteq H \cup K$ for subgroups H, K of G one may ask whether $C \subseteq H$ or $C \subseteq K$.

The situation where C is contained in the union of a family of subgroups of G was considered before in the literature – see [3] for a recent example. We are interested here in the very particular case of a union of just two subgroups. In a letter to the author, M. Isaacs proved that if H, K are normal subgroups of G and if $C \subseteq H \cup K$, then either $C \subseteq H$ or $C \subseteq K$.

We consider here a more general situation as follows. Let $A \leqslant Aut(G)$. For $g \in G, a \in A$ let $[g, a] = g^{-1}g^a$ and $L(G, A)) = \{[g, a] \mid g \in G, a \in A\}$. The set $L(G, A)$ is an A-invariant subset of G and the fact that $C = L(G, Inn(G))$ where $Inn(G)$ is the group of the inner automorphisms of G shows that $L(G, A)$ is not always a subgroup of G.

From now on, we will consider a group G, two subgroups H, K of G and a subgroup A of $Aut(G)$. Two natural questions are addressed in this note.

Q_1 What conditions on G, H, K and A ensure that $L(G, A) \subseteq H \cup K$?

Q_2 If $L(G, A) \subseteq H \cup K$ is it true that either $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$?

Isaacs' result gives an affirmative answer for Q_2 when $A = Inn(G)$ and both H and K are A-invariant subgroups of G. If one defines $core_A(H) = \bigcap_{a \in A} H^a$, then $core_A(H) \leqslant H$ and H is A-invariant if and only if $H = core_A(H)$. Another way to verify the A-invariance for a subgroup H of G is to consider $N_A(H) = \{a \in A \mid H^a = H\}$. Observe that $N_A(H) \leqslant A$ and that $A = N_A(H)$ if and only if H is A-invariant.

When H is an A-invariant subgroup of G, consider the set G/H of the left cosets of H in G and observe that A acts on G/H via $(gH)^a = g^a H$. When G is finite the number $t_A(G/H)$ of the orbits of A in the set G/H will be important. We write $t_A(G) = t_A(G/1)$ to denote the number of orbits of A in G. For an A-invariant subgroup H of G we let $C_A(G/H)$ denote the subgroup of A consisting of those elements a of A for which $[g, a] \in H$ for all $g \in G$ and observe that $L(G, A) \subseteq H$ if and only if $A = C_A(G/H)$.

We are now ready to state the main results.

Lemma 1. *Let G be a group, let $A \leqslant Aut(G)$ and let H, K be normal A-invariant subgroups of G. If $L(G, A) \subseteq H \cup K$, then either $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$.*

Lemma 2. *Let G be a periodic group, let $A \leqslant Aut(G)$ and let $H, K \leqslant G$. If $L(G, A) \subseteq H \cup K$, then:*
i) Either H or K is A-invariant.
ii) $L(G, A) \subseteq core_A(H) \cup core_A(K)$.

Lemma 3. *Let G be a finite group, let $A \leqslant Aut(G)$ and let H, K be A-invariant subgroups of G. Then*

$$(*) \qquad |G| \geq |H| t_A(G/H) + |K| t_A(G/K) - |H \cap K| t_A(G/(H \cap K))$$

and the equality holds in $(*)$ if and only if $L(G, A) \subseteq H \cup K$.

2 The proofs of the lemmas

Proof of Lemma 1. Suppose that $L = L(G, A)$ is contained neither in H nor K. Then $C_A(G/H) \neq A \neq C_A(G/K)$ and one can pick $a \in A \setminus (C_A(G/H) \cup C_A(G/K))$ and $x, y \in G$ such that $[x, a] \notin H$ and $[y, a] \notin K$. Since $L \subseteq H \cup K$ we see that $[x, a] \in K \setminus H$ and $[y, a] \in H \setminus K$.

Observe that $[xy, a] = [x, a]^y [y, a]$. If $[xy, a] \in H$, then $[x, a]^y = [xy, a][y, a]^{-1} \in H$ and the normality of H forces $[x, a] \in H$, a contradiction. Similarly, if $[xy, a] \in K$, one obtains the contradiction $[x, a] \notin K$ and the proof is complete. □

Proof of Lemma 2. Let $L = L(G, A)$ and assume that neither H nor K is A-invariant. Then $N_A(H) \neq A \neq N_A(K)$ and one can pick $a \in A \setminus (N_A(H) \cup N_A(K)), h \in H, k \in K$ such that $[h, a] \notin H$ and $[k, a] \notin K$. Thus $[h, a] \in K \setminus H, [k, a] \in H \setminus K$ and one obtains that $[hk, a] = [h, a]^k [k, a] \in H \setminus K$.

Suppose next that for some positive integer m we have $[(hk)^m, a] \in H \setminus K$. Then $[(hk)^m h, a] = [(hk)^m, a]^h [h, a] \in K \setminus H$ and this implies that $[(hk)^{m+1}, a] = [(hk)^m hk, a] = [(hk)^m h, a]^k [k, a] \in H \setminus K$. By induction it follows that $[(hk)^n, a] \in H \setminus K$ for every positive integer n. Since G is periodic, choose n such that $(hk)^n = 1$ to obtain the contradiction $1 = [1, a] = [(hk)^n, a] \in H \setminus K$. This proves part i).

To prove ii), note that one may suppose without loss that only H is A-invariant, so $H = core_A(H)$. Since L is an A-invariant subset of G, the inclusion $L \subseteq H \cup K$ implies that $L \subseteq \bigcap_{a\in A}(H\cup K)^a = H \cup core_A(K) = core_A(H) \cup core_A(K)$ and this completes the proof. □

Proof of Lemma 3. For $g \in G$ let $m(g) = |\{(x,a) \in G \times A \mid [x,a] = g\}|$ and note that $m(g) \neq 0$ if and only if $g \in L = L(G,A)$. For a nonempty subset subset S of G define $m(S)$ to be the sum of all integers $m(s)$ for $s \in S$ and let also $m(\emptyset) := 0$. Observe that $m(G) = m(L) = |G||A|$ and that $m(S) = m(G)$ if and only if $L \subseteq S$.

Also, it was shown in [1] that if E is an A-invariant subgroup of G then $m(E) = |E||A|t_A(G/E)$. The map $g \to m(g)$ is a measure on G which depends, of course, on the choice of A. These preparations lead to a short and conceptual proof as follows.

By hypothesis the subgroups H, K and $H \cap K$ are all A-invariant and then we have

$$\begin{aligned} |G||A| &= m(G) \\ &\geq m(H\cup K) \\ &= m(H) + m(K) - m(H\cap K) \\ &= |H||A|t_A(G/H) + |K||A|t_A(G/K) - |H\cap K||A|t_A(G/(H\cap K)). \end{aligned}$$

The inequality $(*)$ follows after cancelling $|A|$ and the equality holds in $(*)$ if and only if $m(G) = m(H\cup K)$, i.e. if and only if $L \subseteq H \cup K$. □

3 Applications

The first application is a slight extension of Isaacs' observation:

Corollary 1. *Let G be a group and let $H, K \leqslant G$ such that either H or K is normal in G. If $C \subseteq H \cup K$, then either $C \subseteq H$ or $C \subseteq K$.*

Proof: We let $A = Inn(G)$ in Lemma 2. Observe that the hypothesis ensures that at least one of the subgroups H, K is A-invariant and derive, as in the proof of Lemma 2 ii), that $C = L(G, Inn(G)) \subseteq core_G(H) \cup core_G(K)$. Applying now Lemma 1 for $A = Inn(G)$ completes the proof. □

When G is periodic and A is large enough then Q_2 has an affirmative answer:

Corollary 2. *Let G be periodic, let $H, K \leqslant G$ and let $Inn(G) \leqslant A \leqslant Aut(G)$. Then $L(G,A) \subseteq H \cup K$ if and only if $L(G,A) \subseteq H$ or $L(G,A) \subseteq K$.*

Proof: The hypothesis implies that the A-invariant subgroups of G are automatically normal in G and the result follows by combining Lemma 2 with Lemma 1. □

In particular, if G is periodic and if $H, K \leqslant G$, then $C \subseteq H \cup K$ if and only if H is contained in either H or K. This extends Isaacs' result in the case of periodic groups.

Corollary 3. *Let G be a finite group, let $Inn(G) \leqslant A \leqslant Aut(G)$ and let H, K be A-invariant subgroups of G. Then $L(G, A) \subseteq H \cup K$ if and only if $t_A(G/(H \cap K))$ is equal to either $|H/(H \cap K)| t_A(G/H)$ or $|K/(H \cap K)| t_A(G/K)$.*

Proof: Let $L = L(G, A)$. By Lemma 3, $L \subseteq H \cup K$ if and only if we have equality in $(*)$ and by Corollary 2 this happens if and only if L is contained in either H or K. If, for example, $L \subseteq H$, then clearly $t_A(G/H) = |G/H|$ and the equality in $(*)$ reduces to $|K/(H \cap K)| t_A(G/K) = t_A(G/(H \cap K))$. ∎

Yet one more application uses two of the lemmas to show that if G is periodic then one can eliminate the A-invariance condition in Lemma 1.

Corollary 4. *Let G be a periodic group, let $A \leqslant Aut(G)$ and let H, K be normal subgroups of G such that $L(G, A) \subseteq H \cup K$. If $L(G, A) \subseteq H \cup K$, then either $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$.*

Proof: By Lemma 2, $L = L(G, A) \subseteq core_A(H) \cup core_A(K)$. Since the terms of this union are normal subgroups of G one applies Lemma 1 to derive that $L \subseteq H$ or $L \subseteq K$. ∎

The complete solution for problem Q_1 in the case when G is finite is now at hand:

Corollary 5. *Let G be a finite group, let $A \leqslant Aut(G)$ and let H, K be subgroups of G with $X = core_A(H)$ and $Y = core_A(K)$. Then $L(G, A) \subseteq H \cup K$ if and only if*

$$|G| = |X| t_A(G/X) + |Y| t_A(G/Y) - |X \cap Y| t_A(G/(X \cap Y)).$$

Proof: By Lemma 2 we get that $L = L(G, A) \subseteq H \cup K$ if and only if $L \subseteq X \cup Y$. The result now follows from Lemma 3 since X and Y are A-invariant subgroups of G. ∎

The above results are immediate consequences of the lemmas. The next application is a bit surprising because it does not seem to be related to the problem at hand. It was suggested by the following very particular situation which presents some independent interest.

Let G be a finite group and let A be a subgroup of $Aut(G)$ of odd prime order p. Let $F = C_G(A)$ denote the subgroup of G consisting of all fixed points of A in G. It is well-known that the number $t_A(G)$ of the orbits of A in G is equal to $|F| + \frac{|G|-|F|}{p}$. If one assumes that $t_A(G)$ is odd, then it is an easy exercise to show that $|G|$ is odd. And so, if one is ready to apply the deep Odd Order Theorem of Feit and Thompson, it follows that in fact G is solvable.

In the case when G is finite and $t_A(G)$ is odd we have the following:

Corollary 6. *Let G be a finite group and let $A \leqslant Aut(G)$ such that $t_A(G)$ is odd. If H is a normal subgroup of G and if $H \leqslant C_G(A)$, then either H has odd order or H has nontrivial center.*

Proof: Let $C_G(H)$ denote the centralizer of H in G, so $H \cap C_G(H) = Z(H)$ is the center of H. It is clear that H is an A-invariant normal subgroup of G and the same is true for $C_G(H)$. A short calculation or an appeal to Problem 4C.3 of [2] shows that $L = L(G, A) \subseteq C_G(H)$. Thus $L \subseteq H \cup C_G(H)$ and if one takes in Corollary 3 $K := C_G(H)$ one obtains that $|H/Z(H)|t_A(G/H) = t_A(G/Z(H))$. If $Z(H) = 1$, then $|H|$ divides $t_A(G/1) = t_A(G)$ and so H has odd order, as asserted. □

Remarks. 1) The results in this note remain valid if one replaces the subgroup A of $Aut(G)$ with a group A that acts on G via automorphisms.

2) The normality condition in Corollary 6 is a bit irritating. It could be replaced by the requirement that the number $t_A(N_G(H))$ of orbits of A in the A-invariant subgroup $N_G(H)$ of G is odd. In this way one obtains a local version of Corollary 6. Also , combining the Odd Order Theorem with Corollary 6 shows that if $t_A(G/Z(X))$ is odd for $X = core_G(C_G(A))$ then X is solvable.

3) The measure $g \to m(g)$ introduced in the proof of Lemma 2 can be used to obtain a more general conditional identity. Indeed, if G is a finite group, if A is a subgroup of $Aut(G)$ and if $H_1, H_2, \ldots, H_n$ are A -invariant subgroups of G , the inclusion-exclusion principle gives an inequality similar to $(*)$. The equality occurs if and only if $L(G, A)$ is contained in the union of the mentioned A -invariant subgroups, so the algebraic information is captured by a number-theoretical identity involving invariants.

4) A few questions are left open. The necessity of the conditions in Lemma 1 is an open problem. Similarly, it is not clear if the periodicity is really needed in Lemma 2. Either a better proof which generalizes Lemma 2, or an example of a (non - periodic) group for which Lemma 2 does not apply would be of interest.

References

[1] Marian Deaconescu, Gary Walls, Remarks on finite group actions, *Bull. Math. Soc. Sci. Math. Roumanie* **59**, 225-231 (2016).

[2] I.M. Isaacs, *Finite Group Theory*, A. M. S. Graduate Studies Series, vol. **92** (2008).

[3] Cristina Acciarri, Pavel Shumyatsky, Coverings of commutators in profinite groups, *Rend. Semin. Math. Univ. Padova* **137**, 237-257 (2017)

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