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Existence results for sweeping process with almost convex perturbation

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Abstract

We consider an evolution inclusion governed by the so-called "sweeping process". The right-hand side contains a set-valued perturbation, upper semi-continuous with nonempty compact and almost convex values. We generalize first an existence result when the perturbation is with convex but not necessary bounded values, topological properties of the attainable set are also established. Then, in a particular case arising in planning procedures, the problem with almost convex perturbation is investigated in order to establish the existence of time optimal solutions.

Key Words: Normal cone, almost convex set, minimal norm, attainable set, differential inclusion, time optimal problem.
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1 Introduction

It's well known that the attainable sets plays an important role in control theory; many problems of optimization, dynamics, planning procedures in mathematical economy and game theory can be stated and solved in terms of attainable sets. In the study of existence of solutions for differential inclusions, the use of convexity assumptions is widely acknowledged; in particular to establish that the set of all solutions is closed. This property is not true, in general, when the convexity is dropped. The non-convex case has been studied by various approaches. In [9], a generalization of convexity has been defined, namely, the almost convexity of sets (see the definition below); the authors have shown the existence of solution to upper semi-continuous differential inclusions, and that the attainable set (instead of the solutions set) is closed. This almost convexity condition has been used successfully by [1] and [2]. In [2], an evolution inclusion has been considered, the right-hand side contains the classical Moreau's sweeping process and a set-valued perturbation with almost convex values. Let recall that the sweeping process is an evolution differential inclusion governed by a maximal monotone operator defined as the subdifferential of the indicator function of a convex set, and that includes, as a special case, a class of variational inequality. It has the following form

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)), \quad a.e \;\; t \in [T_0, T]; \\ x(t) \in C(t), \; \forall t \in [T_0, T], \quad x(T_0) = a. \end{cases}$$

where $C(t)$ is a time dependent subset of $\mathbb{R}^d$ and $N_{C(t)}(x(t))$ is the normal cone to $C(t)$ at $x(t)$. Such problems has been introduced and thoroughly studied in the 70's by Moreau in the setting where the sets $C(t)$ are assumed to be convex (see [13]). Generalizations of the sweeping process have been the object of many studies, see e.g. [5, 6, 8, 14, 16, 17] and the references therein. The perturbed problem appears as follows

$$(\mathcal{I}) \quad \begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + G(t, u(t)), \ \ a.e. \ \ t \in [T_0, T], \\ u(t) \in C(t)\,, \quad \forall t \in [T_0, T], \\ u(T_0) = u_0 \in C(T_0)\,, \end{cases}$$

where the perturbation $G : [T_0, T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a nonempty closed valued set-valued function, Lebesgue-measurable on $[T_0, T]$ and upper semi-continuous on $\mathbb{R}^d$.

In the present paper, we extend the results in [2] in many directions. We show that the approach in [2] can be adapted to yield the existence of solution for $(\mathcal{I})$ with a set-valued perturbation unnecessarily bounded values. Moreover, we establish on the whole interval $\mathbb{R}^+ := [0, +\infty[$, the existence of solutions of the perturbed sweeping process

$$(\mathcal{I}_{\mathbb{R}^+}) \begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + G(t, u(t)), \ \ a.e. \ \ t \in \mathbb{R}^+, \\ u(t) \in C(t)\,, \quad \forall t \in \mathbb{R}^+, \\ u(0) = a \in C(0)\,, \end{cases}$$

Making use of the result obtained, we present some topological properties of the attainable sets.

On the other hand, when the sets $C(t) := C$ are fixed and convex, one obtain the following problem arising in the study of planning procedures in mathematical economy, and studied by [11] and [12] :

$$(\mathcal{P}) \begin{cases} -\dot{u}(t) \in N_C(u(t)) + G(u(t)), \ \ a.e. \ \ t \in \mathbb{R}^+, \\ u(t) \in C\,, \quad \forall t \in \mathbb{R}^+, \\ u(0) = a \in C\,, \end{cases}$$

We investigate under the weaker assumption of almost convexity, the existence of solutions to $(\mathcal{P})$. This result lead us to obtain a solution of reaching any element of the attainable sets in a minimum time which is known as the time optimality problem.

This paper is organized as follows. In section 2, several notations and preliminaries of convex and non-smooth analysis are recalled, in section 3, we prove the existence of $(\mathcal{I}_{\mathbb{R}^+})$ when G is upper semi-continuous on $\mathbb{R}^+ \times \mathbb{R}^d$ with unnecessary bounded values, and establish a topological property on the attainable set, finally in section 4 we present an existence result of $(\mathcal{P})$ when G has compact almost convex values, and deduce a solution of the time optimality problem.

2 Notation and Preliminaries

Through the paper, we will use the following notations and definitions.

- $C_{\mathbb{R}^d}([T_0, T])$ is the Banach space of all continuous mappings from $[T_0, T]$ to $\mathbb{R}^d$ endowed with the sup-norm.
- $L^1_{\mathbb{R}^d}([T_0, T])$ is the space of all Lebesgue integrable $\mathbb{R}^d$-valued mappings defined on $[T_0, T]$.
- $\overline{B}$ is the closed unit ball of $\mathbb{R}^d$.

- If D is non empty closed subset of $\mathbb{R}^d$, then $\delta^*(x', D) = \sup_{y \in D} < x', y >$ is the support function of D at $x' \in \mathbb{R}^d$, $d(., D)$ is the usual distance function associated with D, i.e. $d(x, D) = \inf_{u \in D} ||x - u||$ for $x \in \mathbb{R}^d$, which is convex whenever D is convex. The projection of x on D is the element of D denoted by $Proj_D(x)$ and satisfying

$$Proj_D(x) = \{y \in D \ : \ d(x, D) = \|x - y\|\}.$$

We denote the element of D with minimal norm by $m(D) = Proj_D(0)$, it is unique whenever D is a closed convex subset of $\mathbb{R}^d$. In the case of set valued maps, if $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a measurable multifunction with nonempty closed convex images, then F admits a measurable selection with minimal norm $x \to m(F(x)) = Proj_{F(x)}(0)$ (see [4])
- If $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $\mathbb{R}^d$, the excess of $\mathcal{A}$ over $\mathcal{B}$ is

$$e(\mathcal{A}, \mathcal{B}) = \sup\{d(a, \mathcal{B}) \ : \ a \in \mathcal{A}\}$$

and their Hausdorff distance is $\quad H(\mathcal{A}, \mathcal{B}) = \max(e(\mathcal{A}, \mathcal{B}), e(\mathcal{A}, \mathcal{B}))$.
- For a subset $\mathcal{C} \subset \mathbb{R}^d$, $co(\mathcal{C})$ denote the convex hull of $\mathcal{C}$, and $\overline{co}(\mathcal{C})$ it's closed convex hull, which could be characterized by (see [4])

$$\overline{co}(\mathcal{C}) = \{x \in \mathbb{R}^d : \forall x' \in \mathbb{R}^d, < x', x > \leq \delta^*(x', \mathcal{C})\}$$

- $\mathcal{D}$ is called almost convex if for every $\beta \in co(\mathcal{D})$ there exist ξ_1 and ξ_2 , $0 \leq \xi_1 \leq 1 \leq \xi_2$ such that, $\xi_1 \beta \in \mathcal{D}$ and $\xi_2 \beta \in \mathcal{D}$. Note that if $0 \in co(\mathcal{D})$ then $0 \in \mathcal{D}$, also every convex set is almost convex. Concretes examples of almost convex sets are $\mathcal{D} = \partial Z$, with Z a convex set not containing the origin, or $\mathcal{D} = \{0\} \cup \partial Z$, Z a convex set containing the origin.

Let us recall some definitions about subdifferential and normal cone of closed sets.
- Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper convex continuous function on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$ with $f(x) < +\infty$, the subdifferential of f is the set

$$\partial f(x) = \{x^* \in \mathbb{R}^d \quad : \ < x^*, y - x > \leq f(y) - f(x), \ \forall y \in \mathbb{R}^d\}$$

if $f(x)$ is not finite we set $\partial f(x) = \emptyset$, $\partial f(x)$ is closed convex set if f is convex.
- Let $C \subset \mathbb{R}^d$ and $x \in C$, the normal cone to C at x is defined by

$$N_C(x) = \{y \in \mathbb{R}^d \quad : \ < y, c - x > \leq 0, \text{ for all } c \in C\}$$

and satisfies (see [10])

$$\begin{aligned} &y \in N_C(x) \Leftrightarrow x \in C \ and \ < y, x > = \delta^*(y, C). \\ &y = Proj_C(x) \Leftrightarrow x - y \in N_C(y). \\ &\partial d(x, C) = N_C(x) \cap \overline{B}. \end{aligned}$$

- Let $t \in \mathbb{R}^+$, the attainable set of $(\mathcal{I}_{\mathbb{R}^+})$ at time t is defined by

$$S_a(t) = \{u(t) \ : \ u(.) \in \Upsilon_t(a)\}$$

where $\Upsilon_t(a)$ is the set of the trajectories of the differential inclusion $(\mathcal{I}_{\mathbb{R}^+})$ on the interval $[0, t]$. We signify by $S_a = \bigcup_{t \in \mathbb{R}^+} S_a(t)$ the attainable set with unspecified end-time.

3 Sweeping process with convex unbounded perturbation

The main result of this section is an extension of previous results about the perturbed sweeping process, in particular Theorem 3.1 in [2], even when the sets C are taken non-convex, namely "uniformly r-prox regular", but here we give the result in the convex case since we deal with this case in the application witch follows. Let $G : \mathbb{R}^+ \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a multifunction with nonempty closed convex values, globally upper semi-continuous such that
(H_1) for some real $\alpha > 0$,

$$\|m(G(t,y))\| \leq \alpha, \text{ for all } (t,y) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

and let $C : \mathbb{R}^+ \rightrightarrows \mathbb{R}^d$ be a multifunction with nonempty closed convex values such that
(H_2) there exist a constant $\Lambda > 0$ satisfies

$$H(C(t_1), C(t_2)) \leq \Lambda |t_1 - t_2|, \text{ for all } t_1, t_2 \in \mathbb{R}^+.$$

The following theorem treats the existence result for the problem $(\mathcal{I}_{\mathbb{R}^+})$ when G has convex and not necessary bounded values.

Theorem 1. *Suppose that (H_1) and (H_2) are satisfied, then for every $a \in C(0)$, there exists an absolutely continuous mapping $u : \mathbb{R}^+ \to \mathbb{R}^d$ solution of $(\mathcal{I}_{\mathbb{R}^+})$ satisfying*

$$||\dot{u}(t)|| \leq \Lambda + 2\alpha \text{ , a.e. } \; t \in \mathbb{R}^+.$$

Proof. a) We need first to prove that the problem (I) admit a solution $u : [T_0, T] \to \mathbb{R}^d$. Indeed, for every $n \geq 0$, we consider a partition of $[T_0, T]$ by the points

$$t_k^n = T_0 + ke_n \text{ , } e_n = \frac{T - T_0}{n} \text{ , } k = 0, 1, 2, ..., n.$$

Step1. Construction of approximate solution.
For each $t \in [t_0^n, t_1^n[$, we defined

$$u_n(t) = \frac{t_1^n - t}{e_n} x_0^n + \frac{t - t_0^n}{e_n} x_1^n,$$

where $x_0^n = u_0 \in C(T_0)$ and

$$x_1^n = Proj_{C(t_1^n)}(x_0^n - e_n m(G(t_0^n, u_0)))$$

so that, $u_n(t_0^n) = u_0$. Then we have the estimate

$$\begin{aligned} d(x_0^n - e_n m(G(t_0^n, u_n(t_0^n))), C(t_1^n)) &\leq d(x_0^n - e_n m(G(t_0^n, u_n(t_0^n))), x_0^n) + d(x_0^n, C(t_1^n)) \\ &\leq e_n ||m(G(t_0^n, u_n(t_0^n)))|| + H(C(t_0^n), C(t_1^n)) \\ &\leq e_n \alpha + \Lambda |t_0^n - t_1^n| = e_n(\Lambda + \alpha). \end{aligned}$$

Hence for $t \in [t_0^n, t_1^n[$, we have

$$\dot{u}_n(t) = \frac{x_1^n - x_0^n}{e_n}.$$

Using the characterization of the normal cone in term of the projection operator, we can write

$$x_0^n - e_n m(G(t_0^n, u_0)) - x_1^n \in -N_{C(t_1^n)}(x_1^n),$$

so,

$$\dot{u}_n(t) = \frac{x_1^n - x_0^n}{e_n} \in -N_{C(t_1^n)}(x_1^n) - m(G(t_0^n, u_n(t_0^n))),$$

with

$$||x_0^n - e_n m(G(t_0^n, u_0)) - x_1^n|| \leq d(x_0^n - e_n m(G(t_0^n, u_0), C(t_1^n))) \leq (\Lambda + \alpha)e_n,$$

then

$$||\frac{x_1^n - x_0^n}{e_n}|| \leq (\Lambda + \alpha) + ||m(G(t_0^n, u_0))|| \leq \Lambda + 2\alpha,$$

and

$$||\dot{u}_n(t)|| = ||\frac{x_1^n - x_0^n}{e_n}|| \leq \Lambda + 2\alpha.$$

For each $t \in [t_1^n, t_2^n[$, we defined

$$u_n(t) = \frac{t_2^n - t}{e_n} x_1^n + \frac{t - t_1^n}{e_n} x_2^n,$$

where

$$x_2^n = Proj_{C(t_2^n)}(x_1^n - e_n m(G(t_1^n, u_n(t_1^n))).$$

Then for $t \in [t_1^n, t_2^n[$, we have $x_1^n = u_n(t_1^n)$ and

$$\dot{u}_n(t) = \frac{x_2^n - x_1^n}{e_n} \in -N_{C(t_2^n)}(x_2^n) - m(G(t_1^n, u_n(t_1^n))),$$

with the estimate

$$d(x_1^n - e_n m(G(t_1^n, u_n(t_1^n))), C(t_2^n)) \leq (\Lambda + \alpha)e_n,$$

so that

$$\|\dot{u}_n(t)\| = ||\frac{x_2^n - x_1^n}{e_n}|| \leq \Lambda + 2\alpha.$$

Suppose that, (u_n) is well defined on $[t_{k-1}^n, t_k^n[$ with

$$u_n(t_k^n) = x_k^n \ \text{ and } \ ||\frac{x_k^n - x_{k-1}^n}{e_n}|| \leq \Lambda + 2\alpha,$$

for each $t \in [t_k^n, t_{k+1}^n[$, we define

$$u_n(t) = \frac{t_{k+1}^n - t}{e_n} x_k^n + \frac{t - t_k^n}{e_n} x_{k+1}^n \ ,$$

where

$$x_{k+1}^n = Proj_{C(t_{k+1}^n)}(x_k^n - e_n m(G(t_k^n, u_n(t_k^n)))).$$

Then for $t \in [t_k^n, t_{k+1}^n[$,

$$\dot{u}_n(t) = \frac{x_{k+1}^n - x_k^n}{e_n} \in -N_{C(t_{k+1}^n)}(x_{k+1}^n) - m(G(t_k^n, u_n(t_k^n))), \tag{3.1}$$

with the estimate

$$\|\dot{u}_n(t)\| = \|\frac{x_{k+1}^n - x_k^n}{e_n}\| \leq \Lambda + 2\alpha. \tag{3.2}$$

For each $t \in [T_0, T]$ and each $n \geq 1$, let $\delta_n(t) = t_k^n$, $\theta_n(t) = t_{k+1}^n$, if $t \in [t_k^n, t_{k+1}^n[$ and $\delta_n(T) = t_{n-1}^n, \theta_n(T) = T$. So by (3.1) we get

$$\dot{u}_n(t) \in -N_{C(\theta_n(t))}(u_n(\theta_n(t))) - m(G(\delta_n(t), u_n(\delta_n(t)))),\ a.e,\ \ t \in [T_0, T].$$

It is obvious that, for all $n \geq 1$ and for all $t \in [T_0, T]$ the following hold,

$$m(G(\delta_n(t), u_n(\delta_n(t)))) \in G(\delta_n(t), u_n(\delta_n(t))); \tag{3.3}$$
$$u_n(\delta_n(t)) \in C(\delta_n(t)); \tag{3.4}$$
$$u_n(\theta_n(t)) \in C(\theta_n(t)); \tag{3.5}$$
$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \theta_n(t) = t. \tag{3.6}$$

Step 2. The convergence of the sequences.
For all $n \geq 1$ and for a.e. $t \in [T_0, T]$, we have

$$\begin{aligned}
||u_n(\theta_n(t)) - u_n(t)|| &= ||u_n(t_{k+1}^n) - u_n(t)|| \\
&= ||x_{k+1}^n - \frac{t_{k+1}^n - t}{e_n} x_k^n - \frac{t - t_k^n}{e_n} x_{k+1}^n|| \\
&= ||\frac{e_n x_{k+1}^n - t_{k+1}^n x_k^n + t x_k^n - t x_{k+1}^n + t_k^n x_{k+1}^n}{e_n}|| \\
&= ||\frac{t_{k+1}^n x_{k+1}^n - t_k^n x_{k+1}^n - t_{k+1}^n x_k^n + t x_k^n - t x_{k+1}^n + t_k^n x_{k+1}^n}{e_n}|| \\
&= ||\frac{t_{k+1}^n (x_{k+1}^n - x_k^n) + t(x_k^n - x_{k+1}^n)}{e_n}|| \\
&= ||\frac{x_{k+1}^n - x_k^n}{e_n}|| |t_{k+1}^n - t|,
\end{aligned}$$

that is,

$$||u_n(\theta_n(t)) - u_n(t)|| = \|\dot{u}_n(t)\|(\theta_n(t) - t) \leq (\Lambda + 2\alpha)(\theta_n(t) - t), \tag{3.7}$$

so, by (3.6)

$$\lim_{n\to\infty} ||u_n(\theta_n(t)) - u_n(t)|| = 0.$$

As

$$\begin{aligned}
||x_k^n - x_0^n|| &\leq ||x_k^n - x_{k-1}^n|| + ||x_{k-1}^n - x_{k-2}^n|| + \ldots.. + ||x_1^n - x_0^n|| \\
&\leq e_n(\Lambda + 2\alpha) + e_n(\Lambda + 2\alpha) + \ldots + e_n(\Lambda + 2\alpha).
\end{aligned}$$

Then, for all k = 1,2,...,n

$$\|u_n(t_k^n)\| = \|x_k^n\| \leq k e_n(\Lambda + 2\alpha) + \|u_0\| \leq (T - T_0)(\Lambda + 2\alpha) + \|u_0\|$$

and,

$$||u_n(t)|| - ||u_n(\theta_n(t))|| \le ||u_n(t) - u_n(\theta_n(t))|| \le (\Lambda + 2\alpha)(\theta_n(t) - t) \le (\Lambda + 2\alpha)(T - T_0)$$

so, we get

$$||u_n(t)|| \le (\Lambda + 2\alpha)(T - T_0) + ||u_n(t_k^n)|| \le 2(\Lambda + 2\alpha)(T - T_0) + \|u_0\|,$$

we conclude that the sequence $(u_n(t))_n$ is relatively compact.
In the other hand, for all $t_1, t_2 \in [T_0, T]$ such that $t_1 \le t_2$ we have

$$\|u_n(t_2) - u_n(t_1)\| = \| \int_{t_1}^{t_2} \dot{u}_n(s)ds\| \le (\Lambda + 2\alpha)(t_2 - t_1)$$

then the sequence $(u_n(.))$ is equi-continuous, by the Ascoli-Arzelà theorem (see [3], Theorem 0.3.1) we conclude that $(u_n(.))$ is relatively compact in $C_{\mathbb{R}^d}([T_0, T])$, since $\|\dot{u}_n(t)\| \le \Lambda + 2\alpha$, a.e. on $[T_0, T]$, we conclude by the consequence of Ascoli-Arzelà theorem (see [3], Theorem 0.3.4) that there exists a subsequence (again denote by) $(u_n(.))$ converging to an absolutely continuous mapping $u(.)$ and $(\dot{u}_n(.))$ converges $\sigma(L^1_{\mathbb{R}^d}([T_0, T]), L^\infty_{\mathbb{R}^d}([T_0, T]))$ to $\dot{u}(.)$. Then

$$u(t) = \lim_{n\to\infty} u_n(t) = u_0 + \lim_{n\to\infty} \int_{T_0}^{t} \dot{u}_n(s)ds = u_0 + \int_{T_0}^{t} \dot{u}(s)ds.$$

Now, we put $(m(G(\delta_n(.), u_n(\delta_n(.)))))_n = (g_n(.))_n$, for all $n \ge n_0$ and for all $t \in [T_0, T]$, then $\|g_n(t)\| \le \alpha$. So $(g_n(.))$ is bounded in $L^\infty_{\mathbb{R}^d}([T_0, T])$, taking a subsequence if necessary we may conclude that $(g_n(.))$ converges $\sigma(L^\infty_{\mathbb{R}^d}([T_0, T]), L^1_{\mathbb{R}^d}([T_0, T]))$ to some mapping $g \in L^\infty_{\mathbb{R}^d}([T_0, T])$. Consequently, for all $v(.) \in L^1_{\mathbb{R}^d}([T_0, T])$, we have

$$\lim_{n\to\infty} < g_n(.), v(.) >=< g(.), v(.) > .$$

Let $y(.) \in L^\infty_{\mathbb{R}^d}([T_0, T]) \subset L^1_{\mathbb{R}^d}([T_0, T])$ then

$$\lim_{n\to\infty} < g_n(.), y(.) >=< g(.), y(.) > .$$

This shows that $(g_n(.))$ converges $\sigma(L^1_{\mathbb{R}^d}([T_0, T]), L^\infty_{\mathbb{R}^d}([T_0, T]))$ to $g(.)$, with $\|g(t)\| \le \alpha$ a.e.

Step 3. $\dot{u}(t) + g(t) \in -N_{C(t)}(u(t))$
First we show that $u(t) \in C(t)$, $\forall t \in [T_0, T]$. Indeed, for every $t \in [T_0, T]$, and for every $n \ge 1$ by (3.5)

$$\begin{aligned} d(u_n(t), C(t)) &\le d(u_n(t), u_n(\theta_n(t))) + d(u_n(\theta_n(t)), C(t)) \\ &\le ||u_n(t) - u_n(\theta_n(t))|| + H(C(\theta_n(t)), C(t)) \\ &\le ||u_n(t) - u_n(\theta_n(t))|| + \Lambda|\theta_n(t) - t| \end{aligned}$$

Since $\lim_{n\to\infty} ||u_n(t) - u_n(\theta_n(t))|| = 0$, $\lim_{n\to\infty} |\theta_n(t) - t| = 0$ and $C(t)$ is closed, by passing to the limit in the preceding inequality, we get $u(t) \in C(t)$.
In the other hand, we have

$$\|\dot{u}_n(t) + g_n(t)\| \le \|\dot{u}_n(t)\| + \|g_n(t)\| \le \Lambda + 3\alpha = \gamma$$

that is

$$\dot{u}_n(t) + g_n(t) \in \gamma \overline{B},$$

since

$$\dot{u}_n(t) + g_n(t) \in -N_{C(\theta_n(t))}(u_n(\theta_n(t)))$$

by the properties of the convex normal cone, we get

$$\dot{u}_n(t) + g_n(t) \in -\gamma \partial d(u_n(\theta_n(t)), C(\theta_n(t))).$$

An application of the Mazur's Theorem to $(\dot{u}_n + g_n, g_n)$ provide a sequence (w_n, v_n) with

$$w_n \in co\{\, \dot{u}_m + g_m \ :\ m \geq n\,\} \ \ and \ \ v_n \in co\{\, g_m \ :\ m \geq n\,\}$$

such that (w_n, v_n) converges strongly in $L^1_{\mathbb{R}^d \times \mathbb{R}^d}([T_0, T])$ to $(\dot{u} + g, g)$. We can extract from (w_n, v_n) a subsequence which converges a.e. to $(\dot{u} + g, g)$. Then, there is a Lebesgue negligible set $N \subset [T_0, T]$ such that for every $t \in [T_0, T] \setminus N$

$$\dot{u}(t) + g(t) \in \bigcap_{n \geq 0} \overline{\{w_k(t) \ :\ k \geq n\}} \subset \bigcap_{n \geq 0} \overline{co}\{\dot{u}_k(t) + g_k(t) \ :\ k \geq n\}. \tag{3.8}$$

and,

$$g(t) \in \bigcap_{n \geq 0} \overline{\{v_k(t) \ :\ k \geq n\}} \subset \bigcap_{n \geq 0} \overline{co}\{g_k(t) \ :\ k \geq n\}. \tag{3.9}$$

Fix any $t \in [T_0, T] \setminus N$ and $\mu \in \mathbb{R}^d$ the relation (3.8) gives

$$< \mu, \dot{u}(t) + g(t) > \leq \limsup_{n \to \infty} \delta^*(\mu, -\gamma \partial d(u_n(\theta_n(t)), C(\theta_n(t))) \leq \delta^*(\mu, -\gamma \partial d(u(t), C(t)))$$

where the second inequality follows from Theorem 3.1 in [5]. Since $\partial d(u(t), C(t))$ is closed convex set we obtain

$$\dot{u}(t) + g(t) \in -\gamma \partial d(u(t), C(t)) \subset -N_{C(t)}(u(t)).$$

Further, the relation (3.9) and the upper semi-continuity of G give

$$< \mu, g(t) > \leq \limsup_{n \to \infty} \delta^*(\mu, G(\delta_n(t), u_n(\delta_n(t))) \leq \delta^*(\mu, G(t, u(t))),$$

So $g(t) \in G(t, u(t))$, because G has closed convex values, and the proof of the point (a) is complete.

b) Since $\mathbb{R}^+ = \bigcup_{k \in \mathbb{N}} [k, k+1]$. For all $k \in \mathbb{N}$, applying the step (a) on $[k, k+1]$, so there exists an absolutely continuous mapping $u_k : [k, k+1] \to \mathbb{R}^d$ solution of the problem

$$\begin{cases} -\dot{u}_k(t) \in N_{C(t)}(u_k(t)) + G(t, u_k(t)) \ \ a.e.\ \ t \in [k, k+1], \\ u_k(t) \in C(t)\,, \quad \forall t \in [k, k+1], \\ u_k(k) \in C(k)\,, \end{cases}$$

Considering the mapping $u : \mathbb{R}^+ \to \mathbb{R}^d$ defined by $u(t) = u_k(t)$ for $t \in [k, k+1]$ and $k \in \mathbb{N}$, then it is easy to conclude that u is an absolutely continuous solution of the problem $(\mathcal{I}_{\mathbb{R}^+})$. This completes the proof of the theorem. $\square$

Remark 1. *This result is still valid in the case of nonconvex uniformly r-prox regular sets $C(t)$ since our assumptions guarantee the existence of the (unique) projection necessary for the construction of approximates solutions. The infinite dimension setting in a separable Hilbert space is also obtained by adding the ball-compactness condition (see [8]).*

The next Corollary characterize the attainable set, which will be used later in solving the minimum-time problem.

Corollary 1. *Suppose that (H_1) and (H_2) are satisfied, then for every $a \in C(0)$, and for all $t \in \mathbb{R}^+$,*

1. *the set of the trajectories of the differential inclusion $(I_{\mathbb{R}^+})$ on the interval $[0,t]$, $\Upsilon_t(a)$ is nonempty compact;*
2. *the multifunction $t \mapsto S_a(t)$ is upper semi-continuous with nonempty compact values.*

Proof. 1) By the Theorem 2.1, we have $\Upsilon_t(a) \neq \emptyset$. Let $(u_n)_n$ be a sequence of trajectories in $\Upsilon_t(a)$, then, for each $n \in \mathbb{N}, (u_n)_n$ is an absolutely continuous solution of (I) with

$$||\dot{u}_n(\tilde{t})|| \leq \Lambda + 2\alpha \;\; a.e.\ \tilde{t} \in [0,t].$$

and

$$||u_n(\tilde{t})|| \leq ||a|| + \int_0^t ||\dot{u}_n(s)||ds \leq ||a|| + \int_0^t (\Lambda + 2\alpha)ds \leq ||a|| + (\Lambda + 2\alpha)t.$$

Then the sequence $(u_n(\tilde{t}))_n$ is relatively compact. In addition, it is equi-continuous.
By the Ascoli-Arzelà theorem we conclude that $(u_n(.))_n$ is relatively compact in $C_{\mathbb{R}^d}([0,t])$, since $||\dot{u}_n(\tilde{t})|| \leq \Lambda + 2\alpha$, a.e. $\tilde{t} \in [0,t]$, we conclude by Theorem 0.3.4 in [3] that there exists a subsequence of $(u_n(.))_n$ (denoted again $(u_n(.))_n$) converges uniformly to an absolutely continuous mapping $u(.)$ from $[0,t]$ to $\mathbb{R}^d$, and $(\dot{u}_n(.))_n$ converges $\sigma(L^1_{\mathbb{R}^d}([0,t]), L^\infty_{\mathbb{R}^d}([0,t]))$ to $\dot{u}(.)$ with $||\dot{u}(\tilde{t})|| \leq \Lambda + 2\alpha$, a.e. $\tilde{t} \in [0,t]$ and

$$u(\tilde{t}) = \lim_{n\to\infty} u_n(\tilde{t}) = a + \lim_{n\to\infty} \int_0^{\tilde{t}} \dot{u}_n(s)ds = a + \int_0^{\tilde{t}} \dot{u}(s)ds.$$

For each $n \in \mathbb{N}$, let $g_n : [0,t] \to \mathbb{R}^d$ be the measurable selection of G with minimal norm, by the hypothesis (H_2), we get for every $\tilde{t} \in [0,t]$ $\|g_n(\tilde{t})\| \leq \alpha$, so $(g_n)_n$ is bounded in $L^\infty_{\mathbb{R}^d}([0,t])$, taking a subsequence if necessary, we may conclude that $(g_n)_n$ converges $\sigma(L^\infty_{\mathbb{R}^d}([0,t]), L^1_{\mathbb{R}^d}([0,t]))$ to some mapping $g \in L^\infty_{\mathbb{R}^d}([0,t])$. Consequently $(g_n(.))_n$ converges $\sigma(L^1_{\mathbb{R}^d}([0,t]), L^\infty_{\mathbb{R}^d}([0,t]))$ to $g(.)$. Let us prove now that u is a solution of (I). Since (u_n) is a sequence of solutions of (I) and $g_n(\tilde{t}) \in G(\tilde{t}, u_n(\tilde{t}))$ for each $n \in \mathbb{N}$, we can write

$$-\dot{u}_n(\tilde{t}) - g_n(\tilde{t}) \in N_{C(\tilde{t})}(u_n(\tilde{t}))$$

and we have

$$\|\dot{u}_n(\tilde{t}) + g_n(\tilde{t})\| \leq \Lambda + 3\alpha = \gamma.$$

So we get

$$\dot{u}_n(\tilde{t}) + g_n(\tilde{t}) \in -\gamma\partial d(u_n(\tilde{t}), C(\tilde{t})).$$

An application to Mazur's Theorem to $(\dot{u}_n + g_n, g_n)$ provide a sequence (r_n, s_n) with

$$r_n \in co\{\,\dot{u}_m + g_m \;:\; m \geq n\,\} \;\; and \;\; s_n \in co\{\,g_m \;:\; m \geq n\,\}$$

(r_n, s_n) converges strongly in $L^1_{\mathbb{R}^d\times\mathbb{R}^d}([0,t])$ to $(\dot{u}+g, g)$, we can extract from (r_n, s_n) a subsequence which converges a.e. to $(\dot{u}+g, g)$. Then, there is a Lebesgue negligible set $N \subset [0,t]$ such that for every $\tilde{t} \in [0,t] \setminus N$

$$\dot{u}(\tilde{t}) + g(\tilde{t}) \in \bigcap_{n\geq 0} \overline{\{r_m(\tilde{t}) \;:\; m \geq n\}} \subset \bigcap_{n\geq 0} \overline{co}\{\dot{u}_m(\tilde{t}) + g_m(\tilde{t}) \;:\; m \geq n\}. \tag{3.10}$$

$$g(\tilde{t}) \in \bigcap_{n\geq 0} \overline{\{s_m(\tilde{t}) \;:\; m \geq n\}} \subset \bigcap_{n\geq 0} \overline{co}\{g_m(\tilde{t}) \;:\; m \geq n\}. \tag{3.11}$$

Fix any $\mu \in \mathbb{R}^d$ the relation (3.10) gives

$$< \mu, \dot{u}(\tilde{t}) + g(\tilde{t}) > \leq \limsup_{n\to\infty} \delta^*(\mu, -\gamma\partial d(u_n(\tilde{t})), C(\tilde{t})) \leq \delta^*(\mu, -\gamma\,\partial d(u(\tilde{t}), C(\tilde{t})))$$

Since $\partial d(u(\tilde{t}), C(\tilde{t}))$ is closed convex set we obtain

$$\dot{u}(\tilde{t}) + g(\tilde{t}) \in -\gamma\,\partial d(u(\tilde{t}), C(\tilde{t})) \subset -N_{C(\tilde{t})}(u(\tilde{t})).$$

Moreover by the upper semi-continuity of G we have

$$< \mu, g(\tilde{t}) > \leq \limsup_{n\to\infty} \delta^*(\mu, G(\tilde{t}, u_n(\tilde{t})) \leq \delta^*(\mu, G(\tilde{t}, u(\tilde{t}))),$$

which implies that $g(\tilde{t}) \in G(\tilde{t}, u(\tilde{t}))$, then we get $-\dot{u}(\tilde{t}) \in N_{C(\tilde{t})}(u(\tilde{t})) + G(\tilde{t}, u(\tilde{t}))$, and we conclude that $\Upsilon_t(a)$ is compact.
2) From the compactness of $\Upsilon_t(a)$ we have that of $S_a(t)$. Showing now the upper semi-continuity of the multifunction $S_a(.)$ on $\mathbb{R}^+$. Consider the graph of $S_a(.)$ defined by

$$Gph(S_a) = \{(t,z) \in \mathbb{R}^+ \times \mathbb{R}^d \;:\; z \in S_a(t)\}$$

Let (t_n, z_n) be a sequence of $Gph(S_a)$ converges to (t,z), so, for all $n \geq 0$ there exists an absolutely continuous mapping $u_n(.) \in \Upsilon_t(a)$ satisfies $u_n(t_n) = z_n$, since $\Upsilon_t(a)$ is a compact, we can extract from $(u_n(.))$ a subsequence that we do not relabel converges uniformly to an absolutely continuous mapping $u(.) \in \Upsilon_t(a)$, which gives that $u(t) = z \in S_a(t)$. Consequently $Gph(S_a)$ is closed. Then $S_a(.)$ is upper semi-continuous. □

4 Sweeping process with almost convex valued perturbation

Now we are going to announce the main theorem of the paper, which is an existence result for the first order perturbed sweeping process, when the perturbation G has an almost convex values, but before that we need the following preliminary result.

Theorem 2. *Let $C \subset \mathbb{R}^d$ be a nonempty closed convex subset, $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be an upper semi-continuous multifunction with nonempty compact values. Let $u_0 \in C$, and $x : \mathbb{R}^+ \to \mathbb{R}^d$ be an absolutely continuous solution of*

$$(\mathcal{P}_{co}) \begin{cases} -\dot{u}(t) \in N_C(u(t)) + co\big(G(u(t))\big), & a.e.\ t \in \mathbb{R}^+, \\ u(t) \in C, \quad \forall t \in \mathbb{R}^+, \\ u(0) = a \in C. \end{cases}$$

Assume that there exist integrable functions $\xi_1(.)$ and $\xi_2(.)$ defined on $\mathbb{R}^+$, satisfying $0 \le \xi_1(t) \le 1 \le \xi_2(t)$, and

$$(H_3) \quad \xi_1(t)\, m\big(co\big(G(x(t))\big)\big) \in G(x(t)) \quad and \quad \xi_2(t)\, m\big(co\big(G(x(t))\big)\big) \in G(x(t)), a.e\ t \in \mathbb{R}^+.$$

Then, the problem $(\mathcal{P})$ admits an absolutely continuous solution.

Proof. We will show first that for all $t \in [T_0, T]$, there exists $t = t(\tau)$, a nondecreasing absolutely continuous map from the interval $[T_0, T]$ into itself, such that the map $\tilde{x}(\tau) = x(t(\tau))$ is a solution of the problem $(\mathcal{P})$ on the interval $[T_0, T]$. Moreover, $\tilde{x}(T_0) = x(T_0)$ and $\tilde{x}(T) = x(T)$.

a) By using the same procedure as in the proof of Theorem 3.3. in [2] and Theorem 2. in [9] we conclude that for all $[a, b] \subset [T_0, T]$ there exist two measurable subsets of $[a, b]$, having characteristic functions Ψ_1 and Ψ_2 such that $\Psi_1 + \Psi_2 = \Psi_{[T_0,T]}$, and an absolutely continuous function $s : [a, b] \to [a, b]$, such that

$$\dot{s}(\tau) = \Psi_1(\tau)\frac{1}{\xi_1(\tau)} + \Psi_2(\tau)\frac{1}{\xi_2(\tau)} \text{ and } s(b) - s(a) = b - a$$

b) Consider the closed set $K = \{\tau \in [T_0, T] : m\big(co(G(x(\tau)))\big) = 0\}$.
If $K = \emptyset$, in this case $\xi_1(\tau) > 0$ on a set of positive measure, and a) can be applied to the interval $[T_0, T]$. Set $s(\tau) = \int_{T_0}^{\tau} \dot{s}(\omega) d\omega$, s is increasing and we have $s(T_0) = T_0$ and $s(T) = T$, that is, s defined from $[T_0, T]$ into itself. Let $t : [T_0, T] \longrightarrow [T_0, T]$ be its inverse, then $t(T_0) = T_0$, $t(T) = T$ and

$$\dot{t}(\tau) = \frac{1}{\dot{s}(t(\tau))} = \xi_1\big(t(\tau)\big)\Psi_1\big(t(\tau)\big) + \xi_2\big(t(\tau)\big)\Psi_2\big(t(\tau)\big).$$

Let $\tilde{x} : [T_0, T] \longrightarrow \mathbb{R}^d$ the mapping defined by $\tilde{x}(\tau) = x\left(t(\tau)\right)$, we have

$$-\frac{d\tilde{x}(\tau)}{d\tau} = -\dot{x}\big(t(\tau)\big)\Big(\xi_1\big(t(\tau)\big)\Psi_1\big(t(\tau)\big) + \xi_2\big(t(\tau)\big)\Psi_2\big(t(\tau)\big)\Big),$$

using the steps of the proof of the Theorem 2.1, we get

$$-\frac{d\tilde{x}(\tau)}{d\tau} \in \Big(\xi_1\big(t(\tau)\big)\Psi_1\big(t(\tau)\big) + \xi_2\big(t(\tau)\big)\Psi_2\big(t(\tau)\big)\Big) \Big(N_C\big(x\big(t(\tau)\big)\big) + m\big(co\big(G\big(x\big(t(\tau)\big)\big)\big)\big)\Big),$$

by the properties of the normal cone and hypothesis (H_3), we obtain

$$-\frac{d\tilde{x}(\tau)}{d\tau} \in N_C\big(x\big(t(\tau)\big)\big) + G\big(x\big(t(\tau)\big)\big) = N_C\big(\tilde{x}(\tau)\big) + G\big(\tilde{x}(\tau)\big).$$

Now assume that $K \neq \emptyset$. Let $l = sup\{\tau, \tau \in K\}$, since K is closed we get $l \in K$. The complement of K is open relative to $[T_0, T]$, it consists of at most countably many overlapping open interval $]a_i, b_i[$ with the possible exception of the form $[a_{i_i}, b_{i_i}[$ with $a_{i_i} = T_0$ and one of the form $]a_{i_f}, b_{i_f}]$ with $a_{i_f} = l$. For each i apply a) to the interval $]a_i, b_i[$, there existence two subsets of $]a_i, b_i[$ with characteristic functions $\Psi_{i,1}(.)$ and$\Psi_{i,2}(.)$ such that $\Psi_{i,1}(.) + \Psi_{i,2}(.) = \Psi_{]a_i,b_i[}(.)$. Setting

$$\dot{s}(\tau) = \frac{1}{\xi_1(\tau)}\Psi_{i,1}(\tau) + \frac{1}{\xi_2(\tau)}\Psi_{i,2}(\tau),$$

we get

$$\int_{a_i}^{b_i} \dot{s}(\omega)d\omega = b_i - a_i.$$

For $\tau \in [T_0, l]$, set

$$\dot{s}(\tau) = \frac{1}{\xi_2(\tau)}\Psi_K(\tau) + \sum_i \left(\frac{1}{\xi_1(\tau)}\Psi_{i,1}(\tau) + \frac{1}{\xi_2(\tau)}\Psi_{i,2}(\tau)\right)$$

where the sum is over all intervals contained in $[T_0, l]$, we have $\int_{T_0}^{l} \dot{s}(\omega)d\omega = p \leq l - T_0$, since $\xi_2(\tau) \geq 1$; and $\int_{a_i}^{b_i} \dot{s}(\omega)d\omega = b_i - a_i$. Setting $s(\tau) = T_0 + \int_{T_0}^{\tau} \dot{s}(\omega)d\omega$, we obtain that $s(.)$ is an invertible map from $[T_0, l]$ to $[T_0, p]$.

Now, let define $t : [T_0, p] \longrightarrow [T_0, l]$ to be the inverse of $s(.)$, then extend $t(.)$ to an absolutely continuous map $\tilde{t}(.)$ on $[T_0, l]$ by setting $\dot{\tilde{t}}(\tau) = 0$ for $\tau \in]p, l]$. We claim that the mapping $\tilde{x}(\tau) = x\big(\tilde{t}(\tau)\big)$ is a solution of the problem (P) on the interval $[k, l]$ and satisfies $\tilde{x}(l) = x(l)$. Observe that, for $\tau \in [T_0, p[$, $\tilde{t}(\tau) = t(\tau)$ is invertible and

$$\dot{t}(\tau) = \xi_2\big(t(\tau)\big)\Psi_K\big(t(\tau)\big) + \sum_i \Big(\xi_1\big(t(\tau)\big)\Psi_{i,1}\big(t(\tau)\big) + \xi_2\big(t(\tau)\big)\Psi_{i,2}\big(t(\tau)\big)\Big).$$

then we get

$$\begin{aligned}
-\frac{d\tilde{x}(\tau)}{d\tau} &= -\dot{x}\big(t(\tau)\big)\left(\xi_2\big(t(\tau)\big)\Psi_K\big(t(\tau)\big) + \sum_i \Big(\xi_1\big(t(\tau)\big)\Psi_{i,1}\big(t(\tau)\big) + \xi_2\big(t(\tau)\big)\Psi_{i,2}\big(t(\tau)\big)\Big)\right) \\
&\in \left(N_C\big(x(t(\tau))\big) + m\big(co\big(G\big(x(t(\tau))\big)\big)\big)\right) \\
&\quad \times \left(\xi_2(t(\tau))\Psi_K(t(\tau)) + \sum_i \big(\xi_1(t(\tau))\Psi_{i,1}(t(\tau)) + \xi_2(t(\tau))\Psi_{i,2}(t(\tau))\big)\right) \\
&\in N_C\big(x\big(t(\tau)\big)\big) + G\big(x\big(t(\tau)\big)\big) = N_C\big(\tilde{x}(\tau)\big) + G\big(\tilde{x}(\tau)\big).
\end{aligned}$$

In particular, from $t(p) = l$ and $\dot{\tilde{t}}(\tau) = 0$ for all $\tau \in]p, l]$, we obtain $\tilde{t}(\tau) = \tilde{t}(p) = t(p)$, for all $\tau \in]p, l]$, so, $\tilde{x}(l) = x(l)$, $\tilde{x}$ is constant on $]p, l]$ and we have

$$-\frac{d\tilde{x}(\tau)}{d\tau} = 0 \in co\big(G(x(l))\big) = co\big(G(\tilde{x}(\tau))\big), \quad \forall \tau \in]p, l] \tag{4.1}$$

As $0 \in N_C(\tilde{x}(\tau))$, using (4.1) and (H_3) we conclude that for $\tau \in]p, l]$

$$-\frac{d\tilde{x}(\tau)}{d\tau} = 0 \in N_C(\tilde{x}(\tau)) + G(\tilde{x}(\tau))$$

This proves our claim. With the same arguments we find a solution to the problem $(\mathcal{P})$ on $[l, T]$ because $\xi_1(\tau) > 0$. Consequently, on each $[k, k+1], k \in \mathbb{N}$, there exist a sequence of absolutely continuous mapping $\tilde{x}_k$ solution of $(\mathcal{P})$ on $[k, k+1]$. Set $\tilde{x}(.)$ the mapping from $\mathbb{R}^+$ to $\mathbb{R}^d$ satisfies $\tilde{x}(\tau) = \tilde{x}_k(\tau)$ for all $\tau \in [k, k+1]$, under this definition $\tilde{x}$ is an absolutely continuous solution of $(\mathcal{P})$. This completes the proof of our theorem. □

Now we are able to give our main result in this section.

Theorem 3. *Let C be a nonempty closed convex subset of $\mathbb{R}^d$ and $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be an almost convex compact valued multifunction, upper semi-continuous on $\mathbb{R}^d$. Then, for each $a \in C$;*

1. *the problem $(\mathcal{P})$ admit at least an absolutely continuous solution;*
2. *for $t \in \mathbb{R}^+$, the attainable set at t, $S_a(t)$ coincides with $S_a^{co}(t)$, the attainable set at t of the problem $(\mathcal{P}_{co})$.*

Proof. 1. In view of Theorem 2.1, and since $co(G) : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a multifunction with closed and bounded values, upper semi-continuous, there exists an absolutely continuous solution $u(.) : \mathbb{R}^+ \to \mathbb{R}^d$ of $(\mathcal{P})$ satisfying $||\dot{u}(t)|| \leq \Lambda + 2\alpha$, a.e. $t \in \mathbb{R}^+$.

Let us prove now that there exist two integrable functions $\xi_1(.)$ and $\xi_2(.)$ defined on $\mathbb{R}^+$ and satisfying $0 \leq \xi_1(t) \leq 1 \leq \xi_2(t)$ such that for almost every $t \in \mathbb{R}^+$

$$\xi_1(t)m\big(co\big(G(u(t))\big)\big) \in G(u(t)) \quad \text{and} \quad \xi_2(t)m\big(co\big(G(u(t))\big)\big) \in G(u(t)).$$

We have, for all $t \in \mathbb{R}^+$, $G(t)$ is almost convex, then there exist nonempty sets $\Gamma_1(t)$ and $\Gamma_2(t)$ such that

$$\Gamma_1(t) = \{\xi_1 \in [0,1] \ : \ \xi_1 m\big(co\big(G(u(t))\big)\big) \in G(u(t))\}$$

and

$$\Gamma_2(t) = \{\xi_2 \in [1, +\infty[\ : \ \xi_2 m\big(co\big(G(u(t))\big)\big) \in G(u(t))\}.$$

The multifunction Γ_1 is measurable, since the graph

$$\begin{aligned} Gph(\Gamma_1) &= \{(t, \xi_1) \in \mathbb{R}^+ \times [0,1] \ : \ \xi_1 m\big(co(G(u(t)))\big) \in G(u(t)\big)\} \\ &= \{(t, \xi_1) \in \mathbb{R}^+ \times [0,1] \ : \ d\big(\xi_1 m(co(G(u(t)))), G(u(t))\big) = 0\} \\ &= \varphi^{-1}(\{0\}) \cap (\mathbb{R}^+ \times [0,1]), \end{aligned}$$

is measurable ($\varphi : (t, \xi_1) \to d\big(\xi_1 m(co(G(u(t)))), G(u(t))\big)$ is measurable mapping). We conclude that there exists $\xi_1(.)$ integrable selection of Γ_1 satisfying $0 \leq \xi_1(t) \leq 1$ and $\xi_1(t)m\big(co\big(G(u(t))\big)\big) \in G(u(t))$ for all $t \in \mathbb{R}^+$. The exists of $\xi_2(.)$ integrable selection of Γ_2, satisfying $\xi_2(t) \geq 1$ and $\xi_2(t)m\big(co\big(G(u(t))\big)\big) \in G(u(t))$ for all $t \in \mathbb{R}^+$ can be proved using the same reasoning as above with the fact that $G(u(t))$ is bounded. Applying Theorem 3.1, we obtain the existence of mapping $\tilde{u}(.)$ solution of the problem $(\mathcal{P})$.

2. For all $t \in \mathbb{R}^+$, $G(t) \subset co(G(t))$, so $S_a(t) \subset S_a^{co}(t)$. It left to prove the converse inclusion. Let $u(t) \in S_a^{co}(t)$, then $u(.)$ is an absolutely solution of $(\mathcal{P})$ on $[0,t]$, so the proof of Theorem 3.1 can be repeated on $[0,t]$ and we find a solution $\tilde{u}(.) : [0,t] \to \mathbb{R}^d$ of the problem $(\mathcal{P})$ such that $\tilde{u}(t) = u(t) \in S_a(t)$. Hence we get the needed coincidence. ∎

Inspired by [15], we present in the following corollary an interesting consequence concerning time-optimality of the problem $(\mathcal{P})$ where we apply the topological properties of the attainable set.

Corollary 2. *Let C be a nonempty closed convex subset of $\mathbb{R}^d$, $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be an almost convex compact valued multifunction, upper semi-continuous on $\mathbb{R}^d$ and $a \in C$. For every $z \in S_a$, there exists a solution of the problem $(\mathcal{P})$ for which z is attainable at minimum time.*

Proof. Consider

$$\mathcal{M} = \{t \in [0, +\infty[: \; u(t) = z \text{ such that } u(.) \in \Upsilon_t(a)\}.$$

By hypothesis $\mathcal{M} \neq \emptyset$. We put $\tau = \inf \mathcal{M}$, then there exists a decreasing sequence (τ_n) in $\mathbb{R}_+$ converges to τ, and a mapping $u_n(.)$ solution of

$$-\dot{u}(t) \in N_C(u(t)) + G(u(t)), \text{ a.e. } \; t \in [0, \tau_n],$$

for all $n \geq 1$, such that $u_n(\tau_n) = z$, and $u_n(0) = a$, it is known that $u_n(.)$ is solution of

$$-\dot{u}(t) \in N_C(u(t)) + co(G(u(t))), \; a.e. \; t \in [0, \tau_n],$$

for all $n \geq 1$. Let $w_n(t) = u_n(t)$ for $t \in [0, \tau]$ and $n \geq 1$, $w_n(.) \in \Upsilon_\tau(a)$, by Corollary 2.1 this set is compact, then by extracting a subsequence if necessary we may conclude that $(w_n(.))$ converges to $w(.) \in \Upsilon_\tau(a)$. On the other hand, we have $z = u_n(\tau_n) \in S_a^{co}(\tau_n)$, by Corollary 2.1 again, the multifunction $S_a^{co}(.)$ is upper semi-continuous with nonempty compact values, we obtain $\limsup_{n\to\infty} S_a^{co}(\tau_n) = S_a^{co}(\tau)$. Then, $z \in S_a^{co}(\tau) = S_a(\tau)$. Consequently $w(.)$ is the desired time optimal solution and τ is the value of the minimum time. ∎

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