Bull. Math. Soc. Sci. Math. Roumanie
Tome 61 (109) No. 1, 2018, 95–105

Some algebraic invariants of edge ideal of circulant graphs

by
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Abstract

Let G be the circulant graph $C_n(S)$ with $S \subseteq \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and let $I(G)$ be its edge ideal in the ring $K[x_0, \dots, x_{n-1}]$. Under the hypothesis that n is prime we : 1) compute the regularity index of $R/I(G)$; 2) compute the Castelnuovo-Mumford regularity when $R/I(G)$ is Cohen-Macaulay; 3) prove that the circulant graphs with $S = \{1, \dots, s\}$ are sequentially S_2 . We end characterizing the Cohen-Macaulay circulant graphs of Krull dimension 2 and computing their Cohen-Macaulay type and Castelnuovo-Mumford regularity.

Key Words: Circulant graphs, Cohen-Macaulay, Serre's condition.
2010 Mathematics Subject Classification: Primary 13F55. Secondary 13H10

Introduction

Let $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $G := C_n(S)$ is a graph with vertex set $\mathbb{Z}_n = \{0, \dots, n-1\}$ and edge set $E(G) := \{\{i, j\} \mid |j-i|_n \in S\}$ where $|k|_n = \min\{|k|, n-|k|\}$.

Let $R = K[x_0, \dots, x_{n-1}]$ be the polynomial ring on n variables over a field K. The *edge ideal* of G, denoted by $I(G)$, is the ideal of R generated by all square-free monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Edge ideals of a graph have been introduced by Villarreal [11] in 1990, where he studied the Cohen–Macaulay property of such ideals. Many authors have focused their attention on such ideals (see [8], [6]). A known fact about Cohen-Macaulay edge ideals is that they are well-covered.

A graph G is said *well-covered* if all the maximal independent sets of G have the same cardinality. Recently well-covered circulant graphs have been studied (see [1], [2], [9]). In [14] and [4] the authors studied well-covered circulant graphs that are Cohen-Macaulay.

In this article we put in relation the values n, S of a circulant graph $C_n(S)$ and algebraic invariants of $R/I(G)$. In particular we study the regularity index, the Castelnuovo-Mumford regularity, the Cohen-Macaulayness and Serre's condition of $R/I(G)$.

In the first section we recall some concepts and notations and preliminary notions.

In the second section under the hypothesis that n is prime we observe that the regularity index of $R/I(G)$ is 1 obtaining as a by-product the Castelnuovo-Mumford regularity of the ring when it is Cohen-Macaulay.

In the third section we prove that each k-skeleton of the simplicial complex of the independent set of $G = C_n(S)$ is connected when n is prime. As an application we prove that the circulant graphs $C_n(\{1, \dots, s\})$ (studied in [1], [2], [4], [9], [11],[14]) are sequentially S_2 (see [7]).

In the last section we characterize the Cohen-Macaulay circulant graphs of Krull dimension 2 and compute their Cohen-Macaulay type and Castelnuovo–Mumford regularity.

1 Preliminaries

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article. Let G be a simple graph with vertex set $V(G)$ and the edge set $E(G)$. A subset C of $V(G)$ is called a *clique* of G if for all i and j belonging to C with $i \neq j$ one has $\{i,j\} \in E(G)$. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent.The *complement graph* $\bar{G}$ of G is the graph with vertex set $V(\bar{G}) = V(G)$ and edge set $E(\bar{G}) = \{\{u,v\} \in V(G)^2 \mid \{u,v\} \notin E(G)\}$.

Set $V = \{x_1,\ldots,x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$;

(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ.

If Δ is a simplicial complex with facets $F_1,\ldots,F_q$, we call $\{F_1,\ldots,F_q\}$ the facet set of Δ and we denote it by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of Δ is the maximum of the dimensions of all facets in $\mathcal{F}(\Delta)$. If all facets of Δ have the same dimension, then Δ is called *pure*. Let $d-1$ the dimension of Δ and let f_i be the number of faces of Δ of dimension i with the convention that $f_{-1} = 1$. Then the f-vector of Δ is the d-tuple $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$. The h-vector of Δ is $h(\Delta) = (h_0, h_1, \ldots, h_d)$ with

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

The sum

$$\widetilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i$$

is called the reduced Euler characteristic of Δ and $h_d = (-1)^{d-1}\widetilde{\chi}(\Delta)$. Given any simplicial complex Δ on V, we can associate a monomial ideal I_Δ in the polynomial ring R as follows:

$$I_\Delta = (\{x_{j_1}x_{j_2}\cdots x_{j_r} : \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\} \notin \Delta\}).$$

R/I_Δ is called Stanley-Reisner ring and its Krull dimension is d. If G is a graph we call the *independent complex* of G by

$$\Delta(G) = \{A \subset V(G) : A \text{ is an independent set of } G\}.$$

The *clique complex* of a graph G is the simplicial complex whose faces are the cliques of G. Let $\mathbb{F}$ be the minimal free resolution of the quotient ring $R/I(G)$. Then

$$\mathbb{F}:\ 0 \to F_p \to \cdots \to F_{p-1} \to \cdots \to F_0 \to R/I(G) \to 0$$

with $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$. The numbers β_{ij} are called the Betti numbers of $\mathbb{F}$. The Castelnuovo–Mumford regularity of $R/I(G)$, denoted by $\operatorname{reg} R/I(G)$, is defined by

$$\operatorname{reg} R/I(G) = \max\{j - i : \beta_{ij} \neq 0\}.$$

A graph G is said Cohen-Macaulay if the ring $R/I(G)$, or equivalentelly $R/I_{\Delta(G)}$ is Cohen-Macaulay (over the field K) (see [3], [10], [17]). The Cohen-Macaulay type of $R/I(G)$ is equal to the last total Betti number in the minimal free resolution $\mathbb{F}$.

We end this section with the following

Remark 1. *Let $T = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and G be a circulant graph on $S \subseteq T$ with $s = |S|$, then:*

1. *$\bar{G}$ is a circulant graph on $\bar{S} = T \setminus S$;*
2. *The clique complex of $\bar{G}$ is the independent complex of G, $\Delta(G)$;*
3. $$|E(G)| = \begin{cases} ns - \frac{n}{2} & \textit{if } n \textit{ is even and } \frac{n}{2} \in S \\ ns & \textit{otherwise.} \end{cases}$$

2 Regularity and connectedness of the independent complex of circulant graphs of prime order

We recall some basic facts about the regularity index (see also [15]). Let R be standard graded ring and I be a homogeneous ideal. The *Hilbert function* $H_{R/I} : \mathbb{N} \to \mathbb{N}$ is defined by

$$H_{R/I}(k) := \dim_K (R/I)_k$$

and the Hilbert-Poincaré series of R/I is given by

$$\mathrm{HP}_{R/I}(t) := \sum_{k \in \mathbb{N}} H_{R/I}(k) t^k.$$

By Hilbert-Serre theorem, the Hilbert-Poincaré series of R/I is a rational function, that is

$$\mathrm{HP}_{R/I}(t) = \frac{h(t)}{(1-t)^n}.$$

There exists a unique polynomial such that $H_{R/I}(k) = P_{R/I}(k)$ for all $k \gg 0$. The minimum integer $k_0 \in \mathbb{N}$ such that $H_{R/I}(k) = P_{R/I}(k)\ \forall\ k \geq k_0$ is called *regularity index* and we denote it by $\mathrm{ri}(R/I)$.

Remark 2. *Let R/I_Δ be a Stanley-Reisner ring. Then*

$$\mathrm{ri}(R/I_\Delta) = \begin{cases} 0 & \textit{if } h_d = 0 \\ 1 & \textit{if } h_d \neq 0 \end{cases}$$

Proof. By the hypothesis the Hilbert series can be represented by the reduced rational function

$$\frac{h(t)}{(1-t)^d}$$

where d is the Krull dimension of R/I_Δ and $h(t) = \sum_{i=0}^{d} h_i t^i$ where h_i are the entries of the h-vector of Δ. We observe that $\mathrm{ri}(R/I) = \max(0, \deg h(t) - d + 1)$. If $\mathrm{ri}(R/I_\Delta) > 0$ then $\deg h(t) > d - 1$. But since $\deg h(t) \leq d$ we have $\deg h(t) = d$. Therefore $h_d \neq 0$ and $\mathrm{ri}(R/I_\Delta) = 1$. The other case follows by the same argument. □

Lemma 1. *Let G be a circulant graph on S with n prime. Then the entries of the f-vector of $\Delta(G)$ are*

$$f_i = nf'_i$$

with $0 \leq i \leq d-1$ and $f'_i = f_{i,0}/(i+1) \in \mathbb{N}$ where $f_{i,0}$ is the number of faces of dimension i containing the vertex 0.

Proof. Call $\mathcal{F}_i \subset \Delta$ the set of faces of dimension i, that is

$$\mathcal{F}_i = \{F_1, \dots, F_{f_i}\}.$$

Let $f_{i,j}$, number of faces in $\mathcal{F}_i$ containing a given vertex $j = 0, \dots, n-1$. Since G is circulant

$$f_{i,j} = f_{i,0} \text{ for all } j \in \{0, \dots, n-1\}.$$

Let $A \in \mathbb{F}_2^{f_i \times n} = (a_{jk})$ be the incidence matrix with $a_{jk} = 1$ if the vertex $k-1$ belongs to the facet F_j and 0 otherwise. We observe that each row has exactly $i+1$ 1-entries. Hence summing the entries of the matrix we have $(i+1)f_i$. Moreover each column has exactly $f_{i,j}$ non zero entries. That is

$$nf_{i,0} = (i+1)f_i.$$

Since n is prime the assertion follows. □

Theorem 1. *Let G be a circulant graph on S with n prime. Then*

$$\mathrm{ri}(R/I(G)) = 1.$$

Proof. By Remark 2 it is sufficient to show that h_d is different from 0. Since

$$|h_d| = |\sum_{i=0}^{d}(-1)^i f_{i-1}| \neq 0,$$

it is sufficient to show that the reduced Euler formula is different from 0, that is

$$\sum_{i=1}^{d}(-1)^i f_{i-1} \neq 1.$$

By Lemma 1 we obtain

$$\sum_{i=1}^{d}(-1)^i f_{i-1} = n\sum_{i=1}^{d}(-1)^i f'_{i-1}$$

since n is prime and the assertion follows. □

Remark 3. *In the proof of Theorem 1 we are giving a partial positive answer to the Conjecture 5.38 of [9] that states that for all circulant graphs $\widetilde{\chi}(\Delta) \neq 0$. In the article [12] we found other families of circulant graphs satisfying the previous property. In the same article we found a counterexample that disprove the conjecture in general.*

Corollary 1. *Let G be a circulant graph on S with n prime that is Cohen-Macaulay. Then* $\operatorname{reg} R/I(G) = \operatorname{depth} R/I(G)$.

Proof. By Corollary 4.8 of [5] since $\mathrm{ri}(R/I) = 1$ the assertion follows. □

3 Sequentially S_2 circulant graphs of prime order and connectedness

In this section we study good properties of the independent complex $\Delta(G)$ of a circulant graph G that have prime order. We start by the following

Definition 1. *Let Δ be a simplicial complex then we define the pure simplicial complexes $\Delta^{[k]}$ whose facets are*

$$\mathcal{F}(\Delta^{[k]}) = \{F \in \Delta : \dim(F) = k\}, \qquad 0 \leq k \leq \dim(\Delta).$$

One interesting property of Cohen-Macaulay ring R/I_Δ is that the each simplicial complex $\Delta^{[k]}$ is connected. Hence the following Lemma is of interest.

Lemma 2. *Let G be a circulant graph on S with n prime. Then the k-skeleton of the simplicial complex Δ, $\Delta^{[k]}$ is connected for every $k \geq 1$.*

Proof. To prove the claim we find a Hamiltonian cycle connecting all the vertices in $V = \{0, \ldots, n-1\}$ of the 1-skeleton of $\Delta^{[k]}$. Then it follows that since the 1-skeleton is connected then $\Delta^{[k]}$ is connected, too.

We assume without loss of generality that $F_0 = \{v_0, v_1, \ldots, v_k\} \in \Delta^{[k]}$ such that $v_0 = 0$, $v_1 = s \in S$. We define the set

$$F_j = \{v_{0,j}, v_{1,j}, \ldots, v_{k,j}\}$$

with $v_{i,j} = v_i + js \bmod n$. It is easy to observe that since F_0 is in $\Delta^{[k]}$ and G is circulant, F_j is in $\Delta^{[k]}$, too.

Moreover if we focus on the first two vertices of F_j we obtain that

$$v_{1,j} = v_{0,j-1} \text{ for all } j = 1, \ldots, n-1,$$

and $v_{0,n-1} = v_{1,0}$. Since

$$v_{0,j} = js \bmod n$$

the set $\{v_0, \ldots, v_{n-1}\}$, by the primality of n, is equal to V. Hence the cycle with vertices

$$v_{0,0}, v_{0,1}, \ldots, v_{0,n-1}$$

and edges

$$\{v_{0,0}, v_{0,1}\}, \ldots, \{v_{0,n-2}, v_{0,n-1}\}, \{v_{0,n-1}, v_{0,0}\}$$

is a Hamiltonian cycle and the assertion follows. $\square$

Recall that a finitely generated graded module M over a Noetherian graded K-algebra R is said to satisfy the Serre's condition S_r if

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min(r, \dim M_{\mathfrak{p}}),$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

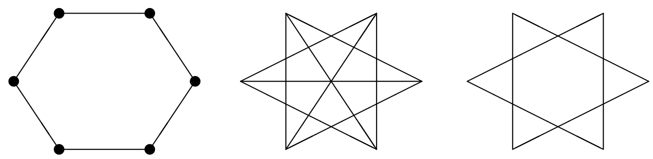

Figure 1: $G = C_6(\{1\}$, Δ and $\Delta^{[2]}$.

Definition 2. *Let M be a finitely generated $\mathbb{Z}$-graded module over a standard graded K-algebra R where K is a field. For a positive integer r we say that M is* sequentially S_r *if there exists a finite filtration of graded R-modules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} satisfies th S_r condition and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_t/M_{t-1}).$$

A nice characterization of sequentially S_2 simplicial complexes is the following:

Theorem 2 ([7])**.** *Let Δ be a simplicial complex with vertex set V. Then Δ is sequentially S_2 if and only if the following conditions hold:*

1. *$\Delta^{[i]}$ is connected for all $i \geq 1$;*
2. *$\operatorname{link}_\Delta(x)$ is sequentially S_2 for all $x \in V$.*

Example 1. *Let G be the circulant graph $C_6(\{1\})$. Then its simplicial complex Δ is connected, but $\Delta^{[2]}$ is not (see Figure 1).*

Sequentially Cohen-Macaulay cycles have been characterized in [6], that are in our notation are just $C_3(\{1\})$ and $C_5(\{1\})$. In [7] the authors proved that the only sequentially S_2 are the odd cycles. The following is related to these results.

Theorem 3. *Let G be the circulant graph $C_n(\{1,\ldots,s\})$ with n prime. Then G is sequentially S_2.*

Proof. By Lemma 2 the first condition of Theorem 2 is satisfied. To check the second condition of Theorem 2 we prove that $K[\operatorname{link}_\Delta(x_0)]$, is sequentially Cohen Macaulay. We observe that

$$K[\operatorname{link}_\Delta(x_0)] \cong (R/I(G))_{x_0} \cong K[x_0^{\pm 1}][x_1,\ldots,x_{n-1}]/I(G)'$$

where $I(G)'$ is obtained by the K-algebra homomorphism induced by the mapping $x_0 \to 1$. Since the vertices adjacent to 0 are $\{1,\ldots,s\} \cup \{n-s,\ldots,n-1\}$ we have that

$$I(G)' = I(G') + (x_1,\ldots,x_s) + (x_{n_s},\ldots,x_{n-1}).$$

with G' be the subgraph of G induced by the vertices $\{s+1,\dots,n-(s+1)\}$. That is

$$(R/I(G))_{x_0} \cong K[x_{s+1},\dots,x_{n-s-1}]/I(G').$$

We claim that $I(G')$ is chordal, hence it is sequentially Cohen-Macaulay by Theorem 3.2 of [6]. To prove the claim we observe that the labelling on the vertices of G'

$$s+1, s+2, \dots, n-s-1$$

induces a perfect elimination ordering, that is $N^+(i) = \{j : \{i,j\} \in E(G'), i < j\}$ is a clique. Let j, $k \in N^+(i)$. That is $\{i,j\}$ and $\{i,k\}$ are two edges with $i < j$ and $i < k$ and assume $j < k$. Then $|j-i|_n = j-i \leq s$ and $|k-i|_n = k-i \leq s$. Moreover

$$0 < j-i < k-i \leq s.$$

Hence it follows $|k-j| = k-j < s$. Therefore $\{j,k\} \in E(G')$ and $N^+(i)$ is a clique. □

Example 2. *If a ring is Cohen-Macaulay it is pure and sequentially S_n for all n. The circulant graph of prime order with minimum number of vertices that is Cohen-Macaulay and has Krull dimension greater than 2 is $C_{13}(\{1,5\})$ (see [4]).*

4 Cohen-Macaulay circulant graphs of dimension 2 and their Castelnuovo-Mumford regularity

We start this section by the following

Theorem 4. *Let G be the circulant graph $C_n(S)$ with $S \subset \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$. The following conditions are equivalent:*

1. *G is Cohen-Macaulay of dimension 2;*
2. *$\Delta(G)$ is connected of dimension 1;*
3. *$\gcd(n, \bar{S}) = 1$ and $\forall a, b \in \bar{S}$ we have $b-a \notin \bar{S}$ and $n-(b+a) \notin \bar{S}$.*

Proof. (1) ⇔ (2). Known fact. See also [9] Corollary 4.54.

(2) ⇒ (3). If $\Delta(G)$ is connected then there is a path in $\bar{G} \cong \Delta(G)$ connecting the vertices 0 and 1 (see Remark 1) whose vertices are

$$0 = v_0, v_1, \dots, v_r = 1$$

and edges

$$\{0, s_1\}, \{s_1, s_1+s_2\}, \dots, \{\sum_{i=1}^{r-1} s_i, \sum_{i=1}^{r} s_i \equiv 1 \bmod n\}$$

with $s_i \in \bar{S}$. Hence there exists a relation

$$\sum a_i s_i \equiv 1 \bmod n, \text{ with } a_i \in \mathbb{N}, s_i \in \bar{S}.$$

By the Euclidean algorithm we have that $\gcd(n, \bar{S}) = 1$. Suppose there exist $a, b \in \bar{S}$ with $b - a \in \bar{S}$. This implies $a \neq b$. We observe that $\{0, a, b\}$ is a clique in $\Delta(G)$, that is $\dim \Delta(G) \geq 2$. In fact since $\bar{G}$ is circulant $\{0, a\}$, $\{0, b\}$ and $\{a, a + (b - a) = b\}$ are edges in $\bar{G}$. Now suppose that $n - (b + a) \in \bar{S}$. We observe that $\{0, a, a + b\}$ is a clique in $\Delta(G)$. In fact since $\bar{G}$ is circulant $\{0, a\}$, $\{a, a + b\}$ and $\{a + b, a + b + n - (a + b) \equiv 0\}$ are edges in $\bar{G}$. The implication (3) $\Rightarrow$ (2) follows by similar arguments.

Theorem 5. *Let G be a Cohen-Macaulay circulant graph $C_n(S)$ of dimension 2. Then* $\operatorname{reg} R/I(G) = 2$.

Proof. It is sufficient to prove that $h_2 \neq 0$ (see Remark 2 and the proof of Corollary 1). We need to compute $h_2 = f_1 - f_0 + f_{-1}$. We observe that f_1 is the number of edges of $\bar{G}$. By Remark 1 one of the two cases to study is

$$\binom{n}{2} - ns,$$

with $h_2 = \binom{n}{2} - n(s + 1) + 1$. The only roots $n \in \mathbb{N}$ of the quadratic equation

$$\binom{n}{2} - n(s + 1) + 1 = 0$$

are 1 and 2 with $s = 0$. Absurd. The other case follows by the same argument. $\square$

Theorem 6. *Let G be a Cohen-Macaulay circulant graph $C_n(S)$ of dimension 2. Then its Cohen-Macaulay type is*

$$h_2 = \begin{cases} \binom{n}{2} - n(s + \frac{1}{2}) + 1 & \textit{if } n \textit{ is even and } \frac{n}{2} \in S \\ \binom{n}{2} - n(s + 1) + 1 & \textit{otherwise.} \end{cases}$$

Proof. By Auslander-Buchsbaum Theorem (Theorem 1.3.3, [3]) and since the depth $R/I(G) = 2$ we need to compute the Betti number in position $\beta_{i,j}$ when $i = n - 2$. By Theorem 5 and the definition of Castelnuovo-Mumford regularity, the Betti numbers that are not trivially 0 are $\beta_{n-2,j}$ in the degrees $j \in \{n - 1, n\}$. We recall the Hochster's formula (see [10], Corollary 5.1.2)

$$\beta_{i,\sigma}(R/I_\Delta) = \dim_K \widetilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma}; K)$$

where $\widetilde{H}(\cdot)$ is the simplicial homology and $\sigma \in \Delta$ is interpreted as squarefree degree in the minimal free resolution and it induces a restriction in Δ defined by

$$\Delta_{|\sigma} = \{F \in \Delta : F \subseteq \sigma\}.$$

We observe that in the squarefree degree σ having total degree $n - 1$

$$\beta_{i,\sigma} = \dim_K \widetilde{H}_0(\Delta_{|\sigma}; K) = 0.$$

In fact $\Delta \cong \bar{G}$ is connected and the same happens removing one of the vertices of the circulant graph $\bar{G}$ since circulant graphs are biconnected. Now, if we consider the squarefree degree σ having total degree n, again, by Hochster formula, we obtain

$$\beta_{i,\sigma} = \dim_K \widetilde{H}_1(\Delta_{|\sigma}; K).$$

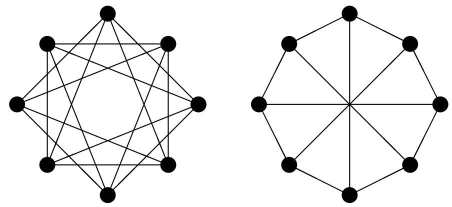

Figure 2: $G = C_8(\{2,3\}$ and $C_8(\{1,4\}) \cong \Delta(G)$.

In this case $\Delta_{|\sigma} \cong \Delta \cong \bar{G}$ and the chain complex of Δ

$$\mathcal{C} : 0 \to C_1 \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} C_{-1} \to 0,$$

has the two homologies $\widetilde{H}_0 = \widetilde{H}_{-1} = 0$. Therefore

$$\dim_K \widetilde{H}_1(\bar{G}; K) = \beta_{i,\sigma} = f_1 - f_0 + f_{-1}$$

and the assertion follows by Remark 1. $\square$

Example 3. *Let $G = C_8(\{2,3\})$ that is $\bar{S} = \{1,4\}$ (see Figure 2). We observe that it satisfies conditions (3) of Theorem 4. Its Cohen-Macaulay type by Theorem 6 is*

$$\binom{8}{2} - 8(2+1) + 1 = 5.$$

Remark 4. *We observe that the rings satisfying Theorem 6 are level. For a description of level algebras see Chapter 5.4 and 5.7 of [3].*

Corollary 2. *Let G be the circulant graph $C_n(S)$ with $S \subset \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and $s = |S|$. The following conditions are equivalent:*

1. *G is Gorenstein of dimension 2;*
2. *$S = \{1, \dots, \hat{i}, \dots, n\}$ and $\gcd(n,i) = 1$ with $n \geq 4$;*
3. *$\Delta(G) \cong \bar{G}$ is a n-gon with $n \geq 4$.*

Proof. (1) $\Rightarrow$ (2). G is Gorenstein if and only if G is Cohen-Macaulay of type 1. Hence by Theorem 4 $\Delta(G)$ is connected that is $\gcd(n, \bar{S}) = 1$. Moreover by Theorem 6 $h_2 = 1$ and solving the two quadratic equations

$$\binom{n}{2} - n(s + \frac{1}{2}) + 1 = 1, \ \binom{n}{2} - n(s+1) + 1 = 1,$$

we obtain respectively

$$n = 2s + 2 \text{ and } n = 2s + 3.$$

In both cases $s = \lfloor \frac{n}{2} \rfloor - 1$. Hence $\bar{S} = i$ with $\gcd(i,n) = 1$ and the assertion follows.

(2) $\Rightarrow$ (3). Let $\bar{S} = \{i\}$ with $\gcd(n, i) = 1$. We easily observe that the vertices

$$0, i, \ldots, (n-1)i \bmod n$$

and edges

$$\{0, i\}, \{i, 2i\}, \ldots, \{(n-1)i, (n)i \equiv 0 \bmod n\}$$

define a Hamiltonian cycle that is $\bar{G}$ itself.

(3) $\Rightarrow$ (1). Since $\Delta(G)$ is a simplicial 1-sphere is Gorenstein of Krull dimension 2 (see Corollary 5.6.5 of [3]). $\square$

We observe that in Theorem 4.1 of [4] the Cohen-Macaulayness of the graphs described in Corollary 2 has been studied by a different point of view.

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Received: 06.06.2017
Accepted: 12.10.2017

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