

## Standard Bases with Special Generators of the Leading Ideal

by

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### Abstract

Let  $I \subseteq K[x]$  be an ideal,  $K$  a field,  $x = (x_1, \dots, x_n)$  and  $>$  a monomial ordering (not necessarily a well-ordering). Let  $L(I)$  be the leading ideal of  $I$  with respect to  $>$ . A standard basis (in case of a well-ordering a Gröbner basis)  $G = \{f_1, \dots, f_m\}$  is defined by the property that  $L(I)$  is generated by the leading monomials  $LM(f_1), \dots, LM(f_m)$  of  $G$ . Usually one considers a standard basis associated to the uniquely determined minimal system of monomials generating  $L(I)$ , a minimal standard basis. In case of border bases, Janet bases or Pommaret bases (usually only defined for well-orderings but we generalize the concept to any ordering) the underlying standard bases have as leading monomials special (non minimal) generators of  $L(I)$ . We describe an algorithm which computes these bases using a minimal standard bases and the corresponding special generators of  $L(I)$ . We have implemented these algorithms in SINGULAR (cf. [DGPS16]) including modular and parallel implementations and give timings to compare them (cf. [KP17]). We also discuss the verification of the modular algorithm to compute border bases and Janet bases.

**Key Words:** Gröbner basis, standard basis, Janet Basis, Border Basis, modular computation

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## 1 Introduction

Gröbner bases are in our days a very powerful tool for symbolic computations in mathematics and its applications. The ideas go back to Gordan (cf. [Go99]), Macaulay (cf. [M39]) and Gröbner (cf. [Gr39]). For ideals in a polynomial ring they are special generators with many good properties. 1965 Buchberger (cf. [Bu65]) gave his famous algorithm to compute them. For ideals in power series rings the corresponding theory goes back to Hironaka (cf. [H64]) and Grauert (cf. [Gr72]), called standard bases. Mora (cf. [Mo82], [Mo92]) gave the first algorithm to compute standard bases. Now we have a unified theory: Gröbner bases are standard bases in case of global orderings (all monomials are greater than 1) and the standard bases of Hironaka and Grauert correspond to local orderings (all monomials are smaller than 1), cf. [GP07]. Involutive bases or more general  $r$ -standard bases are Gröbner bases with additional properties. They were introduced by Gerdt and Blinkov (cf. [Ger05], [GB98], [GBY01]) influenced by ideas studying partial differential equations.

Let  $K$  be a field,  $x = (x_1, \dots, x_n)$  and  $I \subseteq K[x]$  an ideal, let  $>$  be a monomial ordering and  $L(I)$  the leading ideal of  $I$ . In [Ger05] V. Gerdt defined a restricted division  $r$  on the set of monomials  $Mon(x)$  to be a transitive relation

$$u|_r v, u, v \in Mon(x) \text{ such that } u|_r v \text{ implies } u|v.$$

Special choices of  $r$  lead to involutive division and involutive bases of  $I$  including Janet bases and Pommaret bases. The definition implies that an  $r$ -standard basis of  $I$  is a standard basis of  $I$ . An  $r$ -standard basis may be an infinite set. There exist always finite Janet bases and Pommaret bases (cf. [Ger05]).

**Example 1.** We give the following examples for restricted divisions.

1. The ordinary division of monomials is a restricted division.
2. The Pommeret division is defined as follows. Let  $u = x_1^{e_1} \dots x_n^{e_n}$  and  $v \in \text{Mon}(x)$  we define the restricted division  $u|_r v$  by the conditions  $u|v$  and  $v/u \in K[x_1, \dots, x_i]$  with  $i$  minimal such that  $e_i > 0$ .
3. More general assume that for any monomial  $u \in \text{Mon}(x)$  there is associated a subset  $x(u) \subset x$  of so-called multiplicative variables<sup>1</sup>. Let  $u = x_1^{e_1} \dots x_n^{e_n}$  and  $v \in \text{Mon}(x)$  we define the restricted division  $u|_r v$  by the conditions  $u|v$ . and  $v/u \in K[x(u)]$ .
4. The Janet division is defined as follows. Let  $U$  be a finite set of monomials<sup>2</sup>. If  $u \notin U$  we define  $x(u) = \emptyset$ . For  $u = x_1^{e_1} \dots x_n^{e_n} \in U$  we define  $[\emptyset] = U$  and for and  $1 \leq k \leq n$

$$[e_{k+1}, \dots, e_n] = \{x_1^{v_1} \dots x_n^{v_n} \in U | e_{k+1} = v_{k+1}, \dots, e_n = v_n\}.$$

The variable  $x_k$  is multiplicative for  $u$  (with respect to  $U$ ) if

$$e_k = \max\{v_k | x_1^{v_1} \dots x_n^{v_n} \in [e_{k+1}, \dots, e_n]\}.$$

5. As a special example for the Janet division consider  $U = \{x^5, x^2y^2, y^6\} \subset K[x, y]$ . The multiplicative variables for  $x^5$  and  $x^2y^2$  are  $\{x\}$  and for  $y^6$  are  $\{x, y\}$ .

Such a restricted division leads to a special system  $\{m_j\}_{j \in J}$  of monomials of  $L(I)$  with the following property<sup>3</sup>.

For every  $m \in L(I)$  there is a  $j \in J$  such that  $m_j|_r m$ .

The set  $\{m_j\}_{j \in J}$  is a special set of monomials generating  $L(I)$  with respect to the restricted division, so-called  $r$ -generators of  $L(I)$ .

**Definition 1.** A subset  $G \subseteq I$  is called an  $r$ -standard basis of  $I$  if the leading monomials of the elements of  $G$  form a set of  $r$ -generators of  $L(I)$ . If especially the restricted division is a Pommeret (resp. Janet) division the corresponding  $r$ -standard basis is called Pommeret (resp. Janet) basis.

Border bases (cf. [KK05],[KK06], [KKR05]) can also be interpreted as special  $r$ -standard bases.

**Definition 2.** Let  $O := \text{Mon}(x) \setminus \text{Mon}(x) \cap L(I)$  and  $\partial O := x_1 O \cup \dots \cup x_n O \setminus O$  the border of  $O$  then  $L(I) = \langle \partial O \rangle$ . A subset  $G \subset I$  is called border basis if  $\partial O$  is the set of leading monomials of  $G$ .

<sup>1</sup>In the previous example  $x(u) = \{x_1, \dots, x_i\}$ .

<sup>2</sup>In the applications  $U$  will be a finite set of generators of the leading ideal  $L(I)$  of an ideal  $I$  with respect to the fixed ordering.

<sup>3</sup>Note, that the restricted division may depend on the ideal  $L(I)$  as we will see for the Janet-division.

The elements of the border have the following property:

Let  $m, n, t$  be monomials,  $m, t \in \partial O$  and  $n \notin \partial O$ . If  $m|n$  then  $n \nmid t$ .

This implies that the following relation  $r$  is transitive:

If  $(m, n) \in \partial O \times \partial O$  then  $m \nmid_r n$ . If  $(m, n) \notin \partial O \times \partial O$  then  $m|_r n$  if and only if  $m|n$ .

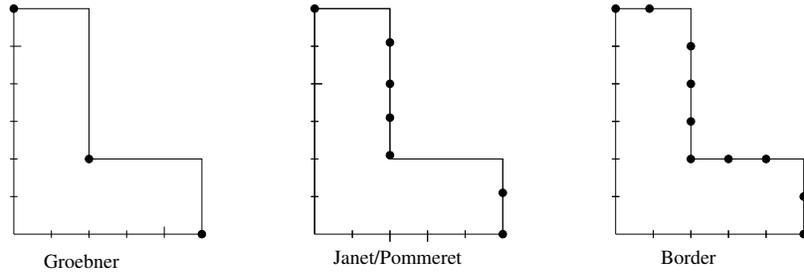
Using this restricted division a border basis is an  $r$ -standard basis. If  $\dim K[x]/I = 0$  then a border basis is finite.

Let us consider the following example:

**Example 2.** Let  $I = \langle x^5, x^2y^2, y^6 \rangle \subset K[x, y]$  then

1.  $\{x^5, x^2y^2, y^6\}$  is a Gröbner basis of  $I$ .
2.  $\{x^5, x^2y^2, y^6, x^5y, x^2y^3, x^2y^4, x^2y^5\}$  is a Janet basis and also a Pommaret basis of  $I$  since  $x^5, x^2y^2, x^5y, x^2y^3, x^2y^4, x^2y^5$  have  $\{x\}$  as multiplicative variable and  $y^6$  has  $\{x, y\}$  as multiplicative variables.
3. A border basis of  $I$  is given by the set  $\{x^5, x^2y^2, y^6, x^5y, x^2y^3, x^2y^4, x^2y^5, xy^6, x^3y^2, x^4y^2\}$  since  $O = \{1, y, y^2, y^3, y^4, y^5, x, xy, xy^2, xy^3, xy^4, xy^5, x^2, x^2y, x^3, x^3y, x^4, x^4y\}$ .

The following diagrams visualize the bases in 1. to 3. in the example <sup>4</sup>.



Next we want to explain the idea of modular computations. We refer to [A03], [NY17], [BDFG15], [GY03] and [IPS11] for a detailed description and analysis of the modular approach. We describe here how to compute a standard basis using modular methods. We use the following notations.

**Definition 3.** Let  $K = \mathbb{Q}$  and  $S \subset \mathbb{Q}[x]$  be a set of polynomials, then

1.  $\text{LM}(S) := \{\text{LM}(f) \mid f \in S\}$  is the set of leading monomials of  $S$ .
2. If  $I = \langle f_1, \dots, f_r \rangle \subset \mathbb{Q}[x]$  and  $p$  is a prime number which does not divide any denominator of the coefficients of  $f_1, \dots, f_r$ . We write  $I_p := \langle f_1 \bmod p, \dots, f_r \bmod p \rangle \subset \mathbb{F}_p[x]$ .

The idea of the algorithm is as follows.

<sup>4</sup>The dots correspond to the exponents of the monomials in the corresponding basis.

1. We choose a set of prime numbers  $P$  and compute for  $p \in P$  a standard basis  $G_p$  of  $I_p \subset \mathbb{F}_p[x]$ .
2. Then we lift these modular standard bases to a set of polynomials  $G \subset \mathbb{Q}[x]$ .

The lifting process consists of the two steps. The set

$$GP = \{G_p \mid p \in P\} \text{ is lifted to } G_N \subset \mathbb{Z}/N\mathbb{Z}[x] \text{ with } N = \prod_{p \in P} p$$

using Chinese remainder theorem (here  $N$  should be larger than the moduli of all coefficients in the expected standard basis). We obtain

$$G \subset \mathbb{Q}[x] \text{ applying the Farey rational map to the coefficients in } G_N$$

(here we need that  $\sqrt{N/2}$  is larger than the moduli of all coefficients of  $G$ ). Since we do not have good bounds for  $N$  to guarantee the correct lifting, we enlarge the set of primes as long as the lifting stabilizes. Then the result is a standard basis of the ideal  $I$  with high probability (a so-called probabilistic standard basis). To guarantee a correct result, we need a final test in  $\mathbb{Q}[x]$ .

We test if  $I = \langle G \rangle$  and  $G$  is a standard basis of  $\langle G \rangle$ .

If the test fails we enlarge the set of primes and continue. The test is usually very time consuming. In many cases one can prove that it is enough to test that

$$I \subset \langle G \rangle \text{ and } G \text{ is a standard basis of } \langle G \rangle.$$

This is the so-called verification theorem (cf. [A03], [P07]). We will prove such a verification theorem for  $r$ -standard bases.

Finally, we recall the classical approach to rational reconstruction which is based on the lifting of modular to rational results by computing Farey preimages via Euclidean division with remainder ([BDFG15], [C94], [KG83]). Let

$$F_B = \left\{ \frac{a}{b} \in \mathbb{Q} \mid \gcd(a, b) = 1, 0 \leq a \leq B, 0 < |b| \leq B \right\}$$

and

$$\mathbb{Q}_N = \left\{ \frac{a}{b} \in \mathbb{Q} \mid \gcd(a, b) = \gcd(b, N) = 1 \right\}.$$

Let  $\varphi_N : \mathbb{Q}_N \rightarrow \mathbb{Z}/N$  defined by  $\varphi_N \left( \frac{a}{b} \right) = (a \bmod N)(b \bmod N)^{-1} := \left( \frac{a}{b} \right)_N$ .

If  $2B^2 < N$  then the Farey map  $\varphi_{B,N} : F_B \cap \mathbb{Q}_N \rightarrow \mathbb{Z}/N$  is injective. The following algorithm computes the preimage of the Farey map.

**Algorithm 1** FAREY( $N$ )**Require:** Integers  $N \geq 2$  and  $0 \leq r \leq N - 1$ .**Ensure:** **false** or a rational  $a/b$  with  $\gcd(a, b) = \gcd(b, N) = 1$ ,  $a/b \equiv r \pmod{N}$ ,  $0 \leq a, |b| \leq \sqrt{(N-1)/2}$ , $(a_0, b_0) := (N, 0)$ ,  $(a_1, b_1) := (r, 1)$ ,  $i := -1$ **while**  $2a_{i+2}^2 \geq N - 1$  **do** $i := i + 1$ divide  $a_i$  by  $a_{i+1}$  to find  $q_i, a_{i+2}, b_{i+2}$  such that

$$(a_i, b_i) = q_i(a_{i+1}, b_{i+1}) + (a_{i+2}, b_{i+2})$$

and  $0 \leq a_{i+2} < a_{i+1}$ **if**  $2b_{i+2}^2 < N$  and  $\gcd(a_{i+2}, b_{i+2}) = 1$  **then****return**  $\frac{a_{i+2}}{b_{i+2}}$ **return false**

The aim of this paper is to define and compute border basis and Janet bases also for non-well orderings in a general setting, analyze the modular and parallel algorithms to compute them and compare the results with other implementations. We also compare our implementation of Janet bases with the implementation in SINGULAR based on the algorithm of V. Gerdt (cf. [Ger05]).

## 2 $r$ -Standard Bases

To explain our approach to compute  $r$ -standard bases let us recall the properties of a weak normal form (cf. [GP07]).

Let  $G = \{g_1, \dots, g_m\}$  be a standard basis of  $I$  with respect to  $>$ . For any  $f \in K[x]$  there exist polynomials  $u, a_1, \dots, a_m \in K[x]$  such that

$$uf = \sum_{i=1}^m a_i g_i + h.$$

satisfying

1. If  $h \neq 0$  then  $LM(h)$  is not divisible by  $LM(g_i)$ ,  $i = 1, \dots, m$ .
2.  $LM(\sum_{i=1}^m a_i g_i) \geq LM(a_i g_i)$  for all  $i$  with  $a_i g_i \neq 0$ .
3.  $LM(u) = 1$ , if  $>$  is a global ordering (i.e. a well-ordering), then  $u$  can be chosen to be 1.

The polynomial  $h$  is called weak normal form of  $f$  with respect to  $G$ . There exist algorithms to compute a weak normal form. We fix now such an algorithm computing uniquely (unique by using the algorithm)  $u, h$  such that  $uf - h \in I$  and write  $(h, u) = NF(f | G)$  i.e.,  $h = NF(f | G)[1]$  and  $u = NF(f | G)[2]$ .

Using the normal form we can construct an  $r$ -standard basis as described in the following proposition.

**Proposition 1.** *Let  $M = \{m_1, \dots, m_k\}$  be a set of  $r$ -generators of  $L(I)$  and  $G = \{g_1, \dots, g_m\}$  a standard basis of  $I$ . Let  $(h_i, u_i) = \text{NF}(m_i \mid G)$  and  $f_i := u_i m_i - h_i, i = 1, \dots, k$ . Then  $\{f_1, \dots, f_k\}$  is an  $r$ -standard basis of  $I$ .*

*Proof.*  $\text{LM}(f_i) = \text{LM}(u_i m_i)$  using property (2) of the weak normal form,  
 $\text{LM}(u_i m_i) = m_i$  since  $\text{LM}(u_i) = 1$ . □

On the basis of Proposition 1 we have implemented algorithms to compute Janet bases and border bases. They can be used to compute this bases using modular and parallel methods.

### 3 Modular Computations of $r$ -Standard Bases

In this chapter we fix a monomial ordering  $>$  and a restricted division  $r$  on  $\text{Mon}(x)$ . We also assume that we have an algorithm `RMON` which associates to every monomial ideal  $I$  in a unique way a finite set of  $r$ -generators of  $I$  (`RMON` returns the empty set if there is no finite set of  $r$ -generators). Examples for `RMON` are algorithms to compute a border basis for a zero-dimensional monomial ideal or to compute a Janet basis (more general an involutive basis) for a monomial ideal.

We describe now an algorithm to compute an  $r$ -standard basis using modular methods. We use the notations used in the introduction.

The idea of the algorithm is as follows. We choose a set of prime numbers  $P$  and compute for  $p \in P$  a standard basis  $G_p$  of  $I_p \subset \mathbb{F}_p[x]$ . Then we lift these modular standard bases  $GP = \{G_p \mid p \in P\}$  to a set of polynomials  $G \subset \mathbb{Q}[x]$ . We obtain a probabilistic standard basis of  $I$  or after a verification process a standard basis of  $I$ . Now we apply the algorithm `RMON` to the leading ideal  $L(G)$  of  $G$  and consider the monomials  $J := \text{RMON}(L(G)) \setminus \text{LM}(G)$ , the “extra” generators coming from the restricted division. Using the normal form we obtain for every monomial  $m \in J$  the normal form  $(u_p, h_p) = \text{NF}(m \mid G_p)$ . For  $p \in P$  let  $R_p = \{u_p m - h_p \mid m \in J\}$ . As before we lift  $RP = \{R_p \mid p \in P\}$  to  $R \subset \mathbb{Q}[x]$ .  $G \cup R$  is the  $r$ -standard basis we wanted to compute.

We want to use only primes  $p$  such that  $\text{LM}(G) = \text{LM}(G_p)$ , so-called lucky primes. The set of unlucky primes is finite. The following procedure tries to avoid the unlucky primes.

`DELETEUNLUCKYPRIMESSB`: *We define an equivalence relation on  $(GP, P)$  by  $(G_p, p) \sim (G_q, q) :\iff \text{LM}(G_p) = \text{LM}(G_q)$ . Then the equivalence class of largest cardinality is stored in  $(GP, P)$ , the others are deleted.*

Since we do not know if the number of lucky primes we have chosen is big enough, we test from time to time whether the lifting gives the correct result, i.e.  $G$  is a standard basis and  $I = \langle G \rangle$ . This test is very expensive. In case of a local ordering or a homogeneous ideal in case of a global ordering Theorem 1 simplifies the test. In all cases the following test in

positive characteristic accelerates the algorithm very much.

**PTESTSB:** We randomly choose a prime number  $p \notin P$  such that  $p$  does not divide the numerator and denominator of any polynomial of the input. The test is positive if and only if  $(G \bmod p)$  is a standard basis of  $I_p$ .

For the verification of the  $r$ -standard basis we use the following theorem.

**Theorem 1.** Let  $I \subset \mathbb{Q}[x]$  be an ideal,  $>$  a monomial ordering, we assume that either  $>$  is global and  $I$  is homogeneous<sup>5</sup> or  $>$  is a local ordering<sup>6</sup>. Let  $R \subset \mathbb{Q}[x]$  be a set of polynomials such that  $\text{LM}(R) = \text{LM}(R_p)$  where  $R_p$  is an  $r$ -standard basis of  $I_p$  for some prime  $p$ . If  $R$  is a standard basis of  $\langle R \rangle$  and  $I \subset \langle R \rangle$  then  $I = \langle R \rangle$  and  $R$  is an  $r$ -standard basis of  $I$ .

*Proof.* Since  $R_p$  is a standard basis of  $I_p$  it follows from [A03] resp. [P07] that  $I = \langle R \rangle$ . Since  $R_p$  is a standard basis of  $I_p$  and  $\text{LM}(R) = \text{LM}(R_p)$  it follows that  $R$  is an  $r$ -standard basis.  $\square$

We obtain the following algorithms (pseudo-code without the optimization used in the implementation in SINGULAR).

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**Algorithm 2** RSTANDARD( $G, J$ )

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**Require:**  $G$  a standard basis,  $J \subseteq \text{LM}(G)$  a finite set.

**Ensure:**  $R$  a set of polynomials of  $G$  such that  $\text{LM}(R) = J$ .

```

 $R = \emptyset;$ 
for  $m \in J$  do
   $(u, h) = \text{NF}(m \mid G);$ 
   $R = R \cup \{um - h\};$ 
return  $R;$ 

```

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The main algorithm will be Algorithm 3.

**Remark 1.** Algorithm 3 can easily be parallelized as follows:

1. The standard bases  $G_p$  can be computed in parallel.
2. The verification can be parallelized:
  - $R \subset \langle G \rangle$  can be checked in parallel for every generator of  $R$ .
  - The check for  $G$  being a standard basis can be done reducing every  $s$ -polynomial with respect to  $G$  in parallel.

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<sup>5</sup>If the ideal is not homogeneous and the ordering is not local the theorem is wrong. K. Yokoyama(cf. [Y12]) gave an example that Theorem 2.4 in [IPS11] fails. The correct statement can be found in [NY17].

<sup>6</sup>For the definition of a local ordering cf. [GP07].

**Algorithm 3** RSTANDARDBASIS**Require:**  $I \subseteq \mathbb{Q}[x]$  an ideal given by generators,  $>$  a monomial ordering,  $s$  an integer.**Ensure:**  $R \subseteq \mathbb{Q}[x]$  *r*-standard basis of  $I$  (with verification if  $s = 1$  and  $>$  is local or  $I$  is homogeneous).**if**  $s = 1$  and  $>$  is not local and  $I$  is not homogeneous **then** $s = 0;$ choose  $P$ , a list of random primes; $GP = \emptyset, RP = \emptyset;$ **loop****for**  $p \in P$  **do**  compute a standard basis  $G_p$  of  $I_p$ ;   $GP = GP \cup \{G_p\};$    $(GP, P) = \text{DELETEUNLUCKYPRIEMESSB}(GP, P);$   lift  $(GP, P)$  to  $G \subseteq \mathbb{Q}[x]$  by applying Chinese remainder and Farey fraction rational map;**if**  $\text{PTTESTSB}(I, G, P)$  **then**  **if**  $s = 1$  **then**    **if**  $I \subseteq \langle G \rangle$  **then**      **if**  $G$  is a standard basis of  $\langle G \rangle$  **then**        compute  $J := \text{RMON}(L(G)) \setminus \text{LM}(G);$         **for**  $p \in P$  **do**           $R_p = \text{RSTANDARD}(G_p, J);$            $RP = RP \cup \{R_p\};$           lift  $(RP, P)$  to  $R \subseteq \mathbb{Q}[x]$  by applying Chinese remainder and Farey fraction rational map;          **if**  $R \subset \langle G \rangle$  **then**             $R = R \cup G;$             **return**  $R;$     **else**      compute  $J := \text{RMON}(L(G)) \setminus \text{LM}(G);$       **for**  $p \in P$  **do**         $R_p = \text{RSTANDARD}(G_p, J);$          $RP = RP \cup \{R_p\};$         lift  $(RP, P)$  to  $R \subseteq \mathbb{Q}[x]$  by applying Chinese remainder and Farey fraction rational map;        **if**  $R \subset \langle G \rangle$  **then**           $R = R \cup G;$           **return**  $R;$   enlarge  $P;$ 

## 4 Examples, timings and conclusions

In this chapter we study examples computing border bases and Janet bases.

We use the following examples 1-19 which can be found in the SymbolicData project of H.-G. Gräbe (cf.[Gr16]), using the link <http://symbolicdata.org/XMLResources/IntPS/>.

1. Cyclic\_7.xml
2. Milnor1.xml
3. Tjurina1.xml
4. random1.xml
5. random2.xml
6. Singular.schwarz\_6.xml (dehomogenized: h=1)
7. Twomat3.xml
8. Klein.xml
9. Meintjes.xml
10. Wilfred.xml
11. Behnke.xml
12. Paris.ilias13.xml
13. Pfister\_2.xml
14. Singular.gerhard\_3.xml
15. Singular.gerhard\_1.xml (dehomogenized w=1)
16. Milnor3.xml
17. Tjurina1.xml
18. Milnor4.xml
19. Steidel\_1.xml

In Table 1 (resp. Table 2) we compare timings to compute the border bases (resp. Janet bases) with Algorithm 1 using as `RMON` an algorithm to compute the border of the order ideal associated to the ordering ( `degrevlex` in examples 1-13 and local `degrevlex` in examples 14-19) (resp. a Janet basis of the leading ideal). `borderbasisP`, `rJanetP` (resp. `borderBasis0`, `rJanet0`) are the timings in characteristic 32003 (resp. 0). `modBorder1`, `modJanet1` (resp. `modBorder0`, `modJanet0`) give the timings for the parallel and modular computation with (resp. without) verification of the result. In the second table `JanetP` (resp. `Janet0`) are the timings using the `SINGULAR` implementation of V. Gerdt's algorithm for computing Janet bases in characteristic 32003 (resp. 0) which is implemented only for well-orderings.

| Example | borderBasisP | borderBasis0 | modBorder1 | modBorder0 |
|---------|--------------|--------------|------------|------------|
| 1       | 148          | –            | 1899       | 1224       |
| 3       | 57           | –            | 647        | 416        |
| 6       | 31           | 18           | 414        | 148        |
| 11      | 1038         | 73891        | 83021      | 27758      |
| 12      | 37           | 10041        | 1997       | 1216       |
| 13      | 7            | 3617         | 99         | 66         |
| 16      | 782          | –            | –          | –          |
| 17      | 243          | –            | 7838       | 4695       |
| 18      | 1            | 6283         | 10         | 4          |
| 19      | 2            | –            | 17         | 6          |

Table 1: Total running times for computing a border basis of the considered examples.

| Example | JanetP | Janet0 | rJanetP | rJanet0 | modJanet1 | modJanet0 |
|---------|--------|--------|---------|---------|-----------|-----------|
| 1       | 4      | 29     | 1       | –       | 71        | 46        |
| 3       | 126    | –      | 13      | –       | 563       | 373       |
| 4       | 10     | 17456  | 1       | –       | 56        | 19        |
| 5       | 60     | –      | 3       | –       | 154       | 54        |
| 6       | –      | –      | 0       | 0       | 0         | 0         |
| 7       | 74555  | 77496  | 41      | 38      | 5854      | 2760      |
| 8       | 31     | 8885   | 3       | –       | –         | 382673    |
| 9       | –      | 65982  | 807     | 3921    | 72639     | 71591     |
| 10      | –      | –      | 68452   | 83420   | –         | –         |
| 12      | 114    | 1635   | 4       | 6554    | 602       | 348       |
| 13      | 48     | 325    | 1       | 1368    | 30        | 20        |
| 14      | 61     | 87     | 31      | 239     | 3403      | 2500      |
| 15      |        |        | 0       | –       | 5         | 2         |
| 16      |        |        | 91      | –       | –         | 277878    |
| 17      |        |        | 34      | –       | 4015      | 1734      |
| 18      |        |        | 0       | 3022    | 2         | 1         |
| 19      |        |        | 0       | –       | 11        | 4         |

Table 2: Total running times for computing a Janet basis of the considered examples.

The timings are given in seconds, “–” means that the computation was stopped after one day or the memory exceeded 50 GB. The timings show that our implementation of the algorithm to compute Janet basis is faster than the implementation of V. Gredt’s algorithm. They also show that the modular and parallel approach is much more powerful than the direct computation in characteristic 0 and the verification step in the modular approach is very time consuming for difficult examples.

The computations are done using the 64-bit version of SINGULAR 4-1-0 on an Dell PowerEdge R720 2x Intel Xeon E5-2690 2.9 -3.8 GHz 20 MB Cache, 16 Cores, 32 Threads, 192 GB RAM under the Gentoo Linux operating system.

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