Oscillation of Impulsive Neutral Partial Differential Equations with Distributed Deviating Arguments

by

T. Kalaimani¹, T. Raja¹, V. Sadhasivam¹, S. H. Saker²

Abstract

In this paper, we will consider a class of even order nonlinear impulsive neutral partial functional differential equations with continuous distributed deviating arguments. Adequate conditions are obtained for the oscillation of solutions by using impulsive differential inequalities and averaging technique with two different boundary conditions. Examples are specified to illustrate the main results.

Key Words: Neutral partial differential equations, Oscillation, Impulse, Distributed deviating arguments

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1 Introduction

The oscillation theory of ordinary differential equations marks its commencement with the manuscript of Sturm [26] in 1836 and for partial differential equations by Hartman and Wintner [9] in 1955. To the best of the authors' knowledge, the work on impulsive delay differential equations was published and initiated in 1989 in [6]. Its consequences were integrated in the monograph [14]. On the other hand there are a few papers has been considered higher order partial differential equations with distributed deviating arguments, we refer the reader to the papers [7], [15], [16], [17]. The primary exertion for impulsive partial differential equations has been started in 1991 in [5]. In recent years the oscillation of parabolic and hyperbolic equations with or without impulse effect has been widely studied in the literature, we refer the reader to the papers [11], [12], [18], [21], [22], [23], [24], [27], [28] and the reference they are cited. In [31], population ecology, generic repression, control theory, climate models, coupled oscillators, viscoelastic materials, and structured population models studied with distributed delay and boundary conditions of the type Dirichlet, Neumann and Robin. From the essence of these mathematical models, we formulated this higher order problem. Distributed delay system models appear in logistics [4], traffic flow [25], microorganism growth [20], and hematopoiesis [1], and [2]. The wide interest on qualitative studies of impulsive ordinary and partial functional differential equations is returned to their varieties of applications in various fields of science and technology [3], [13], [32], and so it is desirable to study these equations scientifically.

In this paper, we will study the following even order nonlinear impulsive neutral partial
functional differential equation with continuous distributed deviating arguments

\[
\begin{gathered}
\frac{\partial^m}{\partial t^m} \left[ u(x, t) + \int_a^b g(t, \xi)u(x, \tau(t, \xi))d\eta(\xi) \right] + \int_a^b q(x, t, \xi)f(u(x, \sigma(t, \xi)))d\eta(\xi) = a(t)\Delta u(x, t) + \sum_{j=1}^n b_j(t)\Delta u(x, \rho_j(t)), \quad t \neq t_k, \quad (x, t) \in \Omega \times (0, +\infty) = G,
\end{gathered}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a piecewise smooth boundary \( \partial \Omega \) and \( \Delta \) is the Laplacian in the Euclidean space \( \mathbb{R}^N \).

Equation (1.1) is enhanced with one of the subsequent Dirichlet and Robin boundary conditions,

\[
\begin{gathered}
\frac{\partial u}{\partial \gamma} + \mu(x, t)u = 0, \quad (x, t) \in \partial \Omega \times (0, +\infty), \\
\text{ (1.2) }
\end{gathered}
\]

\[
\begin{gathered}
\frac{\partial u}{\partial \gamma} + \mu(x, t)u = 0, \quad (x, t) \in \partial \Omega \times (0, +\infty),
\end{gathered}
\]

where \( \gamma \) is the outer surface normal vector to \( \partial \Omega \) and \( \mu(x, t) \in C (\partial \Omega \times [0, +\infty), [0, +\infty)) \).

In the sequel, we assume that the following hypotheses \((H)\) hold:

\((H_1)\) \( a(t), b_j(t) \in PC ([0, +\infty), [0, +\infty]), j = 1, 2, ..., n, \) where \( PC \) represents the class of functions which are piecewise continuous in \( t \) with discontinuities of first kind only at \( t = t_k, k = 1, 2, ..., \) and left continuous at \( t = t_k, k = 1, 2, ..., a, b \) are non-positive constants with \( a < b \).

\((H_2)\) \( g(t, \xi) \in C^m ([0, +\infty) \times [a, b], [0, +\infty]), q(x, t, \xi) \in C(\bar{\Omega} \times [0, +\infty) \times [a, b], [0, +\infty]), \)

\( Q(t, \xi) = \min_{x \in \Omega} q(x, t, \xi), \rho_j(t) \in C([0, +\infty), \mathbb{R}), \lim_{t \to +\infty} \rho_j(t) = +\infty, j = 1, 2, ..., n, \)

\( f(u) \in C(\mathbb{R}, \mathbb{R}) \) is convex in \([0, +\infty), uf(u) > 0 \) and \( \frac{f(u)}{u} \geq c > 0 \) for \( u \neq 0 \).

\((H_3)\) \( \tau(t, \xi), \sigma(t, \xi) \in C([0, +\infty) \times [a, b], \mathbb{R}), \tau(t, \xi) \leq t, \sigma(t, \xi) \leq t \) for \( \xi \in [a, b], \)

\( \tau(t, \xi) \) and \( \sigma(t, \xi) \) are nondecreasing with respect to \( t \) and \( \xi \) respectively and \( \lim_{t \to +\infty} \tau(t, \xi) = +\infty, \xi \in [a, b] \),

\( \lim_{t \to +\infty} \sigma(t, \xi) = +\infty. \)

\((H_4)\) There exist a function \( \theta(t) \in C([0, +\infty), [0, +\infty]) \) satisfying \( \theta(t) \leq \sigma(t, a), \theta'(t) > 0 \) and \( \lim_{t \to +\infty} \theta(t) = +\infty, \eta(\xi) : [a, b] \to \mathbb{R} \) is nondecreasing and the integral is a Stieltjes integral in (1.1).

\((H_5)\) \( \frac{\partial^{(i)} u(x, t)}{\partial t^{(i)}} \) are piecewise continuous in \( t \) with discontinuities of first kind only at

\( t = t_k, k = 1, 2, ..., \) and left continuous at \( t = t_k, k = 1, 2, ..., i = 0, 1, 2, ..., m - 1. \)

\((H_6)\) \( I_k^{(i)} \left(x, t_k, \frac{\partial^{(i)} u(x, t_k)}{\partial t^{(i)}} \right) \in PC(\bar{\Omega} \times [0, +\infty) \times \mathbb{R}, \mathbb{R}), k = 1, 2, \ldots, i = 0, 1, 2, ..., m - 1, \)

and there exist positive constants \( a_k^{(i)}, b_k^{(i)} \) with \( b_k^{(m-1)} \leq a_k^{(0)} \) such that for
$i = 0, 1, 2, \ldots, m - 1, \; k = 1, 2, \ldots,$

\[ a_k^{(i)} \leq I_k^{(i)} \left( x, t_k, \frac{\partial^{(i)} u(x, t_k)}{\partial t^{(i)}} \right) \leq b_k^{(i)}. \]

This paper is organized as follows: In Section 2, we present the definitions and notations that will be needed. In Section 3, we discuss the oscillation of the problem (1.1) and (1.2). In Section 4, we discuss the oscillation of the problem (1.1) and (1.3). In Section 5, we present some examples to illustrate the main results. The results in this paper expand and improve numerous findings in the earlier publications not including impulsive effects. We anticipate that this work acquire the concentration of numerous researchers functioning on the even order impulsive partial functional differential equations.

2 Preliminaries

In this section, we begin with definitions and well known results which are required throughout this paper.

**Definition 1.** A solution $u$ of the problem (1.1) is a function $u \in C^m(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (1.1), where

\[
\begin{align*}
t_{-1} := \min \left\{ 0, \min_{\xi \in [a, b]} \left\{ \inf_{t \geq 0} \tau(t, \xi) \right\} \right\}, \\
\hat{t}_{-1} := \min \left\{ 0, \min_{1 \leq j \leq n} \left\{ \inf_{t \geq 0} \rho_j(t) \right\} \right\}.
\end{align*}
\]

**Definition 2.** The solution $u$ of the problem (1.1), (1.2) [(1.1), (1.3)] is said to be oscillatory in the domain $G$ if for any positive number $\ell$ there exists a point $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ such that $u(x_0, t_0) = 0$ holds.

**Definition 3.** A function $V(t)$ is said to be eventually positive (negative) if there exists a $t_1 \geq t_0$ such that $V(t) > 0$ ($< 0$) holds for all $t \geq t_1$.

It is identified that [29] the least eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

\[ \Delta \omega(x) + \lambda \omega(x) = 0, \quad in \; \Omega, \quad \omega(x) = 0, \quad on \; \partial \Omega, \]

and the consequent eigenfunction $\Phi(x) > 0$ in $\Omega$.

For each positive solution $u(x, t)$ of the problem (1.1), (1.2) [(1.1), (1.3)] we combine the functions $V(t)$ and $\tilde{V}(t)$ defined by

\[
V(t) = K_\Phi \int_{\Omega} u(x, t) \Phi(x) dx, \quad \tilde{V}(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx,
\]

\[
F(t) = M(\theta(t))^{m-2} \theta'(t), \quad and \quad G(t) = c_0 \int_{a}^{b} Q(t, \xi) d\eta(\xi),
\]

where $a_k^{(i)}$ and $b_k^{(i)}$ are the lower and upper bounds of the integrals.

For the oscillation of the problem (1.1) and (1.2) we show that

\[
\int_{t_k}^{t_{k+1}} \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx dt > 0,
\]

where $\int_{t_k}^{t_{k+1}}$ is the integral over the interval $[t_k, t_{k+1}]$. This is achieved by using the properties of the eigenfunction $\Phi(x)$ and the solution $u(x, t)$.

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where

\[ K_\Phi = \left( \int_{\Omega} \Phi(x)dx \right)^{-1}, \quad |\Omega| = \int_{\Omega} dx, \quad \text{and} \quad g_0 = 1 - \int_a^b g(t, \xi, \eta) d\eta(\xi). \]

**Lemma 1.** [10] Let \( y(t) \) be a positive and \( n \) times differentiable function on \([0, +\infty)\). If \( y^{(n)}(t) \) is constant sign and not identically zero on any ray \([t_1, +\infty)\) for \( t_1 > 0 \), then there exists a \( t_g \geq t_1 \) and integer \( l \) (\( 0 \leq l \leq n \)), with \( n + l \) even for \( y(t)y^{(n)}(t) \geq 0 \) or \( n + l \) odd for \( y(t)y^{(n)}(t) \leq 0 \); and for \( t \geq t_g \), \( y(t)y^{(k)}(t) > 0 \), \( 0 \leq k \leq l \); \((-1)^{k-l}y(t)y^{(k)}(t) > 0 \), \( l \leq k \leq n \).

**Lemma 2.** [19] Suppose that the conditions of Lemma 1 is satisfied, and \( y^{(n-1)}(t)y^{(n)}(t) \leq 0 \), for \( t \geq t_g \). Then there exist constant \( \alpha \in (0, 1) \) and \( M > 0 \) such that \( |y'(at)| \geq M^{n-2}|y^{(n-1)}(t)| \) for sufficiently large \( t \).

**Lemma 3.** [8] If \( X \) and \( Y \) are nonnegative, then

\[
X^\alpha - \alpha XY^{\alpha-1} + (\alpha - 1)Y^\alpha \geq 0, \quad \alpha > 1,
\]

\[
X^\alpha - \alpha XY^{\alpha-1} - (1 - \alpha)Y^\alpha \leq 0, \quad 0 < \alpha < 1,
\]

where the equality holds if and only if \( X = Y \).

### 3 Oscillation of the problem (1.1), (1.2)

In this section, we establish some sufficient conditions for the oscillation of all solutions of the problem (1.1), (1.2).

**Lemma 4.** If the functional impulse differential inequality

\[
\begin{align*}
Z^{(m)}(t) + G(t)Z(\theta(t)) &\leq 0, \quad t \neq t_k \\
\frac{\partial^{(i)} Z(t_k^+)}{\partial t^{(i)}} - k = 1, 2, \ldots, &\leq b_k^{(i)} \quad i = 1, 2, \ldots, m - 1,
\end{align*}
\]

has no eventually positive solution, then every solution of the boundary value problem defined by (1.1), (1.2) is oscillatory in \( G \).

**Proof.** Assume that there exist a nonoscillatory solution \( u(x, t) \) of the boundary value problem (1.1), (1.2) and \( u(x, t) > 0 \). By the hypothesis \((H_3)\), that there exists a \( t_1 > t_0 > 0 \) such that \( \tau(t, \xi) \geq t_0 \), \( \sigma(t, \xi) \geq t_0 \) for \((t, \xi) \in [t_1, +\infty) \times [a, b] \) and \( \rho_j(t) > t_0 \), \( j = 1, 2, \ldots, n \) for \( t \geq t_1 \), then for \( t \geq t_1 \), \( t \neq t_k \), \( k = 1, 2, \ldots, n \) we get that

\[
\begin{align*}
u(x, \tau(t, \xi)) &> 0, \quad \text{for} \quad (x, t, \xi) \in \Omega \times [t_1, +\infty) \times [a, b], \\
u(x, \sigma(t, \xi)) &> 0, \quad \text{for} \quad (x, t, \xi) \in \Omega \times [t_1, +\infty) \times [a, b], \\
u(x, \rho_j(t)) &> 0, \quad \text{for} \quad (x, t) \in \Omega \times [t_1, +\infty), \quad j = 1, 2, \ldots, n.
\end{align*}
\]
Combining (3.2)-(3.5), we get that

\[ \text{Set } Z \text{ over the domain } \Omega, \text{ we obtain} \]

\[ \frac{d^m}{dt^m} \left\{ \int_{\Omega} u(x,t)K_\Phi(x)dx + \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u}{\partial \gamma} - u \frac{\partial \Phi(x)}{\partial \gamma} \right] dS + \int_{\Omega} u(x,t)\Delta \Phi(x)dx \right\} \]

\[ = a(t) \int_{\Omega} \Delta u(x,t)K_\Phi(x)dx + \sum_{j=1}^{n} b_j(t) \int_{\Omega} \Delta u(x,\rho_j(t))K_\Phi(x)dx. \]

From Green’s formula and the boundary condition (1.2), we see that

\[ K_\Phi \int_{\Omega} \Delta u(x,t)\Phi(x)dx = K_\Phi \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u}{\partial \gamma} - u \frac{\partial \Phi(x)}{\partial \gamma} \right] dS + K_\Phi \int_{\Omega} u(x,t)\Delta \Phi(x)dx \]

\[ = -\lambda_0 V(t) \leq 0, \quad (3.3) \]

and for \( j = 1, 2, \ldots, n \), we have

\[ K_\Phi \int_{\Omega} \Delta u(x,\rho_j(t))\Phi(x)dx = K_\Phi \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u(x,\rho_j(t))}{\partial \gamma} - u(x,\rho_j(t)) \frac{\partial \Phi(x)}{\partial \gamma} \right] dS \]

\[ + K_\Phi \int_{\Omega} u(x,\rho_j(t))\Delta \Phi(x)dx \]

\[ = -\lambda_0 V(\rho_j(t)) \leq 0, \quad (3.4) \]

where \( dS \) is surface component on \( \partial \Omega \). Furthermore applying Jensen’s inequality for convex functions and using the assumptions on \((H_2)\), we get that

\[ \int_{\Omega} \int_{a}^{b} q(x,t,\xi)f(u(x,\sigma(t,\xi)))K_\Phi(x)d\eta(\xi)dx \]

\[ \geq \int_{a}^{b} Q(t,\xi) \int_{\Omega} f(u(x,\sigma(t,\xi)))K_\Phi(x)dxd\eta(\xi) \]

\[ \geq c \int_{a}^{b} Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi). \quad (3.5) \]

Combining (3.2)-(3.5), we get that

\[ \frac{d^m}{dt^m} \left[ V(t) + \int_{a}^{b} g(t,\xi)V(\tau(t,\xi))d\eta(\xi) \right] + c \int_{a}^{b} Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \leq 0. \quad (3.6) \]

Set \( Z(t) = V(t) + \int_{a}^{b} g(t,\xi)V(\tau(t,\xi))d\eta(\xi) \). Equation (3.6), can be written as

\[ Z^{(m)}(t) + c \int_{a}^{b} Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \leq 0, \quad t \neq t_k. \quad (3.7) \]

From the assumption of \( \int_{a}^{b} g(t,\xi)d\eta(\xi) = Q(t,\xi) \), we have \( Z(t) \geq V(t) > 0 \) and \( Z^{(m)}(t) \leq 0 \). Simultaneously, we can further prove \( Z^{(m-1)}(t) \geq 0, \ t \geq t_2 \). In addition, from Lemma 1, there exists a \( t_3 \geq t_2 \) and an odd number \( l, \ 0 \leq l \leq m - 1 \), and for \( t \geq t_3 \), we have

\[ Z^{(i)}(t) > 0, \ 0 \leq i \leq l, \ (-1)^{(i-1)}Z^{(i)}(t) > 0, \ \text{for} \ l \leq i \leq m - 1. \]
By choosing $i = 1$, we have $Z'(t) > 0$, since $Z(t) \geq x(t) > 0$, $Z'(t) \geq 0$, we have

$$Z(\sigma(t, \xi)) \geq Z(\sigma(t, \xi) - \tau(t, \xi)) \geq x(\sigma(t, \xi) - \tau(t, \xi)),$$

and thus

$$Z^{(m)}(t) + c \int_a^b Q(t, \xi) Z(\sigma(t, \xi)) \left(1 - \int_a^b g(\sigma(t, \xi), \xi) \, d\eta(\xi)\right) d\eta(\xi) \leq 0. \quad (3.8)$$

From equation (3.7), we get

$$Z^{(m)}(t) + G(t) Z(\sigma(t, \xi)) \leq 0.$$

From $(H_3)$ and $(H_4)$, we have

$$Z(\sigma(t, \xi)) \geq Z(\sigma(t, a)) > 0, \quad \xi \in [a, b] \quad \text{and} \quad \theta(t) \leq \sigma(t, \xi) \leq t.$$

Thus $Z(\theta(t)) \leq Z(\sigma(t, a))$ for $t > t_2$. Then (3.8) can be written as

$$Z^{(m)}(t) + G(t) Z(\theta(t)) \leq 0.$$

Multiplying both sides of the equation (1.1) by $K_\Phi \Phi(x) > 0$, integrating with respect to $x$ over the domain $\Omega$, and from $(H_6)$, we obtain

$$a_k^{(i)} \leq \frac{\partial^{(i)} u(x, t_k^+)}{\partial t^{(i)} u(x, t_k^+)} \leq b_k^{(i)}.$$

According to $V(t) = K_\Phi \int_\Omega u(x, t) \Phi(x) \, dx$, we have

$$a_k^{(i)} \leq \frac{\partial^{(i)} V(x, t_k^+)}{\partial t^{(i)} V(x, t_k^+)} \leq b_k^{(i)}.$$

Since $Z(t) = V(t) + \int_a^b g(t, \xi) V(\tau(t, \xi)) \, d\eta(\xi)$, we obtain

$$a_k^{(i)} \leq \frac{\partial^{(i)} Z(x, t_k^+)}{\partial t^{(i)} Z(x, t_k^+)} \leq b_k^{(i)}.$$

Therefore $Z(t)$ is an eventually positive solution of (3.1). This disagree with the hypothesis. \qed

**Theorem 1.** If there exists a function $\varphi(t) \in C'([0, +\infty), (0, +\infty))$ which is nondecreasing with respect to $t$, such that

$$\int_{t_1}^{+\infty} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[\varphi(s) G(s) - \frac{(\varphi'(s))^2}{4 F(s) \varphi(s)}\right] \, ds = \infty, \quad (3.9)$$

then every solution of the boundary value problem (1.1), (1.2) is oscillatory in $G$.  

Proof. Assume that there exist a non-oscillatory solution \( u(x,t) \) of the boundary value problem (1.1), (1.2) and \( u(x,t) > 0 \). Proceeding as in the proof of Lemma 4 to get that \( Z^{(m)}(t) + G(t)Z(\theta(t)) \leq 0 \), where \( Z(t) = V(t) + \int_{a}^{b} g(t,\xi) V(\tau(t,\xi)) d\eta(\xi) \) and satisfies \( Z^{(m)}(t) \leq 0 \), \( Z^{(m-1)}(t) \geq 0 \) and an odd number \( l \), \( 0 \leq l \leq m-1 \), such that \( Z^{(i)}(t) > 0 \), \( 0 \leq i \leq l \), \( (-1)^{(l+1)}Z^{(i)}(t) > 0 \), for \( l \leq i \leq m-1 \). Define

\[
W(t) := \frac{\varphi(t)Z^{(m-1)}(t)}{Z(\theta(t))}, \quad \text{for } t \geq t_0,
\]

then \( W(t) \geq 0 \) for \( t \geq t_1 \), and

\[
W'(t) \leq \frac{\varphi'(t)}{\varphi(t)} W(t) + \frac{\varphi(t)Z^{(m)}(t)}{Z(\theta(t))} - \frac{\varphi(t)Z^{(m-1)}(t)Z'(\theta(t))\varphi'(t)}{Z(\theta(t))^2}.
\]

From \( Z^{(m)}(t) \leq 0 \), according to Lemma 2, we obtain

\[
Z'(\theta(t)) \geq M(\theta(t))^{m-2}Z^{(m-1)}(t).
\]

Thus

\[
W'(t) \leq \frac{\varphi'(t)}{\varphi(t)} W(t) - G(t)\varphi(t) - \frac{F(t)}{\varphi(t)} W^2(t), \quad W(t^+_k) \leq \frac{b_k^{(m-1)}}{a_k^{(0)}} W(t_k).
\]

Define

\[
U(t) = \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t).
\]

In fact, \( W(t) \) is continuous on each interval \( (t_k, t_{k+1}) \), and in consideration of \( W(t^+_k) \leq (b_k^{(m-1)}/a_k^{(0)}) W(t_k) \). It follows for \( t \geq t_0 \) that

\[
U(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k^+) \leq \prod_{t_0 \leq t_j < t_k} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k) = U(t_k)
\]

and for all \( t \geq t_0 \), we have

\[
U(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k^-) \leq \prod_{t_0 \leq t_j < t_k} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k) = U(t_k),
\]
which implies that $U(t)$ is continuous on $[t_0, +\infty)$ and satisfies

$$U'(t) + \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right) U^2(t) F(t) + \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} G(t) \phi(t) - \frac{\phi'(t) U(t)}{\phi(t)}$$

$$= \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W'(t) + \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-2} F(t) \phi(t) W^2(t)$$

$$+ \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} G(t) \phi(t) - \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{\phi'(t) U(t)}{\phi(t)} W(t)$$

$$= \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[ W'(t) + W^2(t) \frac{F(t)}{\phi(t)} - W(t) \frac{\phi'(t)}{\phi(t)} + G(t) \phi(t) \right] \leq 0.$$ 

That is

$$U'(t) \leq - \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{F(t)}{\phi(t)} U^2(t) + \frac{\phi'(t)}{\phi(t)} U(t) - \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} G(t) \phi(t).$$

Applying Lemma 3 with

$$X = \sqrt{\prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{F(t)}{\phi(t)} U(t)}, \quad Y = \frac{\phi'(t)}{2} \sqrt{\prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{1}{F(t) \phi(t)}},$$

we have

$$\frac{\phi'(t)}{\phi(t)} U(t) - \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{F(t)}{\phi(t)} U^2(t) \leq \frac{\phi'(t)}{4F(t) \phi(t)} \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1}.$$

Thus

$$U'(t) \leq - \prod_{t_0 \leq t_k < t} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[ G(t) \phi(t) - \frac{(\phi'(t))^2}{4F(t) \phi(t)} \right].$$

Integrating both sides from $t_1$ to $t$, we have

$$U(t) \leq U(t_1) - \int_{t_1}^{t} \prod_{t_0 \leq t_k < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[ G(s) \phi(s) - \frac{(\phi'(s))^2}{4F(s) \phi(s)} \right] ds.$$

Letting $t \to +\infty$, from (3.9), we have $\lim_{t \to +\infty} U(t) = -\infty$, which leads to a contradiction with $U(t) \geq 0$. The proof is complete. 

The following theorem is of Philos type [19] and can be obtained by applying the inequality (3.10) and the Philos technique. The details will be left to the interested reader. To formulate the results we assume that there exist two functions $H(t, s), h(t, s) \in C^1(D, \mathbb{R})$, in which $D = \{(t, s) | t \geq s \geq t_0 > 0\}$, such that
(H₇). \( H(t,t) = 0, \ t \geq t₀; \ H(t,s) > 0, \ t > s \geq t₀, \)

(H₅). \( H'(t,s) \geq 0, \ H'(s,t) \leq 0, \)

(H₆). \( -\frac{∂}{∂s}[H(t,s)\rho(s)] - H(t,s)\rho(s)\varphi'(s) = h(t,s). \)

**Theorem 2.** Assume that there exist functions \( \varphi(t) \) and \( \rho(s) \in C^1([0, +\infty), (0, +\infty)) \) such that \( \varphi(t) \) is non-decreasing. If there exist two functions \( H(t,s), h(t,s) \in C¹(D, \mathbb{R}) \) satisfy \( (H₇) - (H₉) \) and

\[
\limsup_{t \to +\infty} \frac{1}{H(t,t₀)} \int_{t₀}^{t} \prod_{t₀ \leq t_k < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \Pi(s)ds = \infty, \tag{3.11}
\]

where
\[
\Pi(s) = G(s)\varphi(s)H(t,s)\rho(s) - \frac{1}{4} \frac{|h(t,s)|^2}{F(s)H(t,s)\rho(s)},
\]

then every solution of the boundary value problem (1.1), (1.2) is oscillatory in \( G. \)

In Theorem 2, by choosing \( \rho(s) = \varphi(s) = 1, \) we have the following corollary.

**Corollary 1.** Assume that the conditions of Theorem 2 hold, and

\[
\limsup_{t \to +\infty} \frac{1}{H(t,t₀)} \int_{t₀}^{t} \prod_{t₀ \leq t_k < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \Gamma(s)ds = \infty,
\]

where
\[
\Gamma(s) = G(s)H(t,s) - \frac{1}{4} \frac{|h(t,s)|^2}{F(s)H(t,s)},
\]

then every solution of the boundary value problem (1.1), (1.2) is oscillatory in \( G. \)

**Remark 1.** From Theorem 2 and Corollary 1, we can attain various oscillatory criteria by different choices of the weighted function \( H(t,s). \) For example, choosing \( H(t,s) = (t-s)^{n-1}, \ t \geq s \geq t₀, \) in which \( n > 2 \) is an integer, then \( h(t,s) = (n-1)(t-s)^{(n-3)/2}, \ t \geq s \geq t₀. \) From Corollary 1, we have the following Kamenev type result.

**Corollary 2.** If there exists an integer \( n > 2 \) such that

\[
\limsup_{t \to +\infty} \frac{1}{(t-t₀)^{n-1}} \int_{t₀}^{t} \prod_{s \geq t₀<s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[ G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2F(s)} \right] ds = +\infty,
\]

then every solution of the boundary value problem (1.1), (1.2) is oscillatory in \( G. \)

Also by applying the method of Philos [19] one can obtain the following new oscillation theorem.
**Corollary 3.** Let the functions $H(t, s)$, $h(t, s)$, $\varphi(s)$ and $\rho(s)$ be as defined in Theorem 2. Additionally, suppose that $0 < \inf_{s \geq t_0} \{ \liminf_{t \to +\infty} H(t, t_0) \} \leq +\infty$, and
\[
\limsup_{t \to +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} |h(t, s)|^2 \frac{\varphi(s)}{F(s)H(t, s)\rho(s)} ds < +\infty.
\]
If there exists a function $A(t) \in C([t_0, +\infty), \mathbb{R})$ such that
\[
\limsup_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} \frac{F(s)(A_+(s))^2}{\rho(s)\varphi(s)} ds = +\infty,
\]
and for every $T \geq t_0$
\[
\limsup_{t \to +\infty} \frac{1}{H(t, T)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} \left[ G(s)H(t, s)\varphi(s)\rho(s) - \frac{1}{4} \frac{|h(t, s)|^2 \varphi(s)}{F(s)H(t, s)\rho(s)} \right] ds \geq A(T),
\]
where $A_+(s) = \max\{A(s), 0\}$, then every solution of the boundary value problem (1.1), (1.2) is oscillatory in $G$.

In Theorem 3, by choosing $\rho(s) = \varphi(s) \equiv 1$, we get the following corollary.

**Corollary 3.** Assume that the conditions of Theorem 3 hold and assume that $\rho(s) = \varphi(s) \equiv 1$. If
\[
\limsup_{t \to +\infty} \frac{1}{H(t, T)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} \left[ G(s)H(t, s) - \frac{1}{4} \frac{|h(t, s)|^2 \varphi(s)}{F(s)H(t, s)\rho(s)} \right] ds \geq A(T),
\]
for every $T \geq t_0$, where $A_+(s) = \max\{A(s), 0\}$, then every solution of the boundary value problem (1.1), (1.2) is oscillatory in $G$.

Similar to Corollary 2, we can obtain the following corollary from Corollary 3.

**Corollary 4.** Assume that the conditions of Theorem 3 hold, and
\[
\limsup_{t \to +\infty} \frac{1}{(t - t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} \frac{(n-1)^2}{(t-s)^2F(s)} ds < \infty.
\]
If there exists an integer $n > 2$ and function $A(t) \in C([0, +\infty), \mathbb{R})$ such that
\[
\limsup_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} \frac{F(s)(A_+(s))^2}{\varphi(s)} ds = \infty,
\]
and for every $T \geq t_0$
\[
\limsup_{t \to +\infty} \frac{1}{(t - t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{b_k^{(m-1)}}{a_k^{(0)}} \left[ G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2F(s)} \right] ds \geq A(T),
\]
where $A_+(s) = \max\{A(s), 0\}$, then every solution of the boundary value problem (1.1), (1.2) is oscillatory in $G$. 

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4 Oscillation of the problem (1.1), (1.3)

In this section, we establish sufficient conditions for the oscillation of all solutions of the problem (1.1), (1.3).

**Lemma 5.** If the functional impulsive differential inequality

$$
\begin{align*}
\hat{Z}^{(m)}(t) + G(t)\hat{Z}(\theta(t)) &\leq 0, \quad t \neq t_k, \\
\frac{\partial^{(i)}\hat{Z}(t)}{\partial t^{(i)}} &\leq b_k^{(i)}, \quad k = 1, 2, \ldots, \quad i = 0, 1, 2, \ldots, m - 1,
\end{align*}
$$

has no eventually positive solution, then every solution of the boundary value problem defined by (1.1), (1.3) is oscillatory in G.

**Proof.** Assume that there exist a nonoscillatory solution \( u(x, t) \) of the boundary value problem (1.1), (1.3) and \( u(x, t) > 0 \). By the assumption \((H_3)\), that there exists a \( t_1 > t_0 > 0 \) such that \( \tau(t, \xi) \geq t_0, \) \( \sigma(t, \xi) \geq t_0 \) for \( (t, \xi) \in [t_1, +\infty) \times [a, b] \) and \( \rho_j(t) \geq t_0, \)

\( j = 1, 2, \ldots, n \), then

$$
\begin{align*}
u(x, \tau(t, \xi)) > 0, &\quad \text{ for } (x, t, \xi) \in \Omega \times [t_1, +\infty) \times [a, b], \\
u(x, \sigma(t, \xi)) > 0, &\quad \text{ for } (x, t, \xi) \in \Omega \times [t_1, +\infty) \times [a, b], \\
u(x, \rho_j(t)) > 0, &\quad \text{ for } (x, t) \in \Omega \times [t_1, +\infty), \quad j = 1, 2, \ldots, n.
\end{align*}
$$

Multiplying both sides of equation (1.1) by \( 1/|\Omega| \) and integrating with respect to \( x \) over the domain \( \Omega \), we obtain

$$
\begin{align*}
d^{m}_{\cdot} \left[ \int_{\Omega} u(x, t)dx + \frac{1}{|\Omega|} \int_{\Omega} \int_{a}^{b} g(t, \xi)u(x, \tau(t, \xi))d\eta(\xi)dx \right] \\
+ \frac{1}{|\Omega|} \int_{\Omega} \int_{a}^{b} g(x, t, \xi)f(u(x, \sigma(t, \xi)))d\eta(\xi)dx \\
= a(t) \frac{1}{|\Omega|} \int_{\Omega} \Delta u(x, t)dx + \sum_{j=1}^{n} b_j(t) \frac{1}{|\Omega|} \int_{\Omega} \Delta u(x, \rho_j(t))dx.
\end{align*}
$$

By Green’s formula and boundary condition (1.3),

$$
\int_{\Omega} \Delta u(x, t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial \gamma}dS = - \int_{\partial\Omega} \mu(x, t)u(x, t)dS \leq 0,
$$

and for \( j = 1, 2, \ldots, n, \)

$$
\int_{\Omega} \Delta u(x, \rho_j(t))dx = \int_{\partial\Omega} \frac{\partial u(x, \rho_j(t))}{\partial \gamma}dS = - \int_{\partial\Omega} \mu(x, \rho_j(t))u(x, \rho_j(t))dS \leq 0
$$
where $dS$ is surface element on $\partial \Omega$. Also from $(H_2)$ and Jensen’s inequality, we have
\[
\frac{1}{|\Omega|} \int_{\Omega} \int_{a}^{b} q(x, t, \xi) f(u(x, \sigma(t, \xi)))d\eta(\xi)dx \\
\geq \int_{a}^{b} Q(t, \xi) \frac{1}{|\Omega|} \int_{\Omega} f(u(x, \sigma(t, \xi)))dx d\eta(\xi) \\
\geq c \int_{a}^{b} Q(t, \xi) \hat{V}(\sigma(t, \xi))d\eta(\xi).
\] (4.5)

In view of (4.2)-(4.5), yield
\[
\frac{d^n}{dt^n} \left[ \hat{V}(t) + \int_{a}^{b} g(t, \xi) \hat{V}(\tau(t, \xi))d\eta(\xi) \right] + c \int_{a}^{b} Q(t, \xi) \hat{V}(\sigma(t, \xi))d\eta(\xi) \leq 0.
\] (4.6)

Set $\hat{Z}(t) = \hat{V}(t) + \int_{a}^{b} g(t, \xi) \hat{V}(\tau(t, \xi))d\eta(\xi)$. Equation (4.6), can be written as
\[
\hat{Z}^{(m)}(t) + c \int_{a}^{b} Q(t, \xi) \hat{V}(\sigma(t, \xi))d\eta(\xi) \leq 0, \quad t \neq t_k.
\]

The rest of the proof is similar to the proof of Lemma 4, and hence the details are omitted.

As in the proofs of the results in Section 3, we can also obtain the following results for (1.1), (1.3).

**Theorem 4.** If there exists a function $\hat{\varphi}(t) \in C^1([0, +\infty), (0, +\infty))$ which is nondecreasing with respect to $t$, such that
\[
\int_{t_1}^{+\infty} \prod_{t_0 \leq t_k < s} \left( \frac{b_k^{(m-1)}}{a_k(0)} \right)^{-1} \left[ \hat{\varphi}(s)G(s) - \frac{(\hat{\varphi}'(s))^2}{4F(s)\hat{\varphi}(s)} \right] ds = \infty,
\]
then every solution of the boundary value problem (1.1), (1.3) is oscillatory in $G$.

**Theorem 5.** Assume that there exist functions $\hat{\varphi}(t)$ and $\hat{\rho}(s) \in C^1([0, +\infty), (0, +\infty))$ such that $\hat{\varphi}(t)$ is nondecreasing. Assume that there exist two functions $H(t, s)$, $h(t, s) \in C^1(D, \mathbb{R})$, in which $D = \{(t, s) | t \geq s \geq t_0 > 0\}$, such that $(H_7) - (H_9)$ hold. If
\[
\limsup_{t \to +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left( \frac{b_k^{(m-1)}}{a_k(0)} \right)^{-1} \hat{\Pi}(s)ds = \infty,
\]
where
\[
\hat{\Pi}(s) = G(s)\hat{\varphi}(s)H(t, s)\hat{\rho}(s) - \frac{1}{4} \frac{|h(t, s)|^2}{F(s)H(t, s)\hat{\rho}(s)},
\]
then every solution of the boundary value problem (1.1), (1.3) is oscillatory in $G$.
By choosing $\tilde{\rho}(s) = \hat{\varphi}(s) \equiv 1$, we have the following corollary.

**Corollary 5.** Assume that the conditions $(H_7) - (H_9)$ hold, and

$$\limsup_{t \to +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t < s} \left( \frac{b^{(m-1)}_k}{a_k} \right)^{-1} \Gamma(s) ds = \infty,$$

then every solution of the boundary value problem (1.1), (1.3) is oscillatory in $G$.

**Remark 2.** From Theorem 5 and Corollary 5, we can attain various oscillatory criteria by different choices of the weighted function $H(t, s)$. For example, choosing $H(t, s) = (t - s)^{n-1}$, $t \geq s \geq t_0$, in which $n > 2$ is an integer, then $h(t, s) = (n-1)(t-s)^{(n-3)/2}$, $t \geq s \geq t_0$. From Corollary 5, we get the following result.

**Corollary 6.** If there exists an integer $n > 2$ such that

$$\limsup_{t \to +\infty} \frac{1}{(t-t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \leq t < s} \left( \frac{b^{(m-1)}_k}{a_k} \right)^{-1} \left[ G(s)(t - s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2 F(s)} \right] ds = +\infty,$$

then every solution of the boundary value problem (1.1), (1.3) is oscillatory in $G$.

**Theorem 6.** Let the functions $H(t, s)$, $h(t, s)$, $\hat{\varphi}(s)$ and $\tilde{\rho}(s)$ be as defined in Theorem 5. Additionally, suppose that $0 < \inf_{s \geq t_0} \left\{ \liminf_{t \to +\infty} \frac{H(t, s)}{H(t, t_0)} \right\} \leq +\infty$, and

$$\limsup_{t \to +\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t < s} \left( \frac{b^{(m-1)}_k}{a_k} \right)^{-1} \frac{|h(t, s)|^2}{F(s) H(t, s) \tilde{\rho}(s)} ds < +\infty.$$

If there exists a function $\hat{A}(t) \in C([t_0, +\infty), \mathbb{R})$ such that

$$\limsup_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 \leq t < s} \left( \frac{b^{(m-1)}_k}{a_k} \right)^{-1} F(s) \hat{A}(s)^2 ds = +\infty,$$

and for every $T \geq t_0$

$$\limsup_{t \to +\infty} \frac{1}{H(t, T)} \int_{T}^{t} \prod_{t_0 \leq t < s} \left( \frac{b^{(m-1)}_k}{a_k} \right)^{-1} \left[ G(s) H(t, s) \hat{\varphi}(s) \tilde{\rho}(s) - \frac{1}{4} \frac{|h(t, s)|^2}{F(s) H(t, s) \tilde{\rho}(s)} \right] ds \geq \hat{A}(T),$$

where $\hat{A}(s) = \max\{\hat{A}(s), 0\}$, then every solution of the boundary value problem (1.1), (1.3) is oscillatory in $G$.

By choosing $\tilde{\rho}(s) = \hat{\varphi}(s) \equiv 1$, we attain the following corollary.
Corollary 7. Assume that the conditions of Theorem 6 hold and assume that \( \rho(s) = \hat{\phi}(s) \equiv 1 \). If

\[
\limsup_{t \to +\infty} \frac{1}{H(t, T)} \int_{T}^{t} \prod_{t_0 \leq t_0 < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[ G(s)H(t, s) - \frac{1}{4} |h(t, s)|^2 \right] ds \geq \hat{A}(T),
\]

for every \( T \geq t_0 \), then every solution of the boundary value problem (1.1), (1.3) is oscillatory in \( G \).

Similar to Corollary 6, we can obtain the following corollary from Corollary 7.

Corollary 8. Assume that the conditions of Theorem 6 hold, and

\[
\limsup_{t \to +\infty} \frac{1}{(t - t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \leq t_0 < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left( \frac{n-1}{(t-s)^2} \right) ds < \infty.
\]

If there exists an integer \( n > 2 \) and function \( \hat{A}(t) \in C([0, +\infty), \mathbb{R}) \) such that

\[
\limsup_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_0 < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right) F(s) (\hat{A}(s))^2 ds = \infty,
\]

and for every \( T \geq t_0 \)

\[
\limsup_{t \to +\infty} \frac{1}{(t - t_0)^{n-1}} \int_{T}^{t} \prod_{t_0 \leq t_0 < s} \left( \frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[ G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2} \right] ds \geq \hat{A}(T),
\]

then every solution of the boundary value problem (1.1), (1.3) is oscillatory in \( G \).

5 Examples

In this section, we present couple of examples to point up our results established in Section 3 and Section 4.

Example 1. Consider the following equation

\[
\begin{align*}
\frac{\partial^4}{\partial t^4} \left( u(x, t) + \frac{2}{3} \int_{-\pi/2}^{\pi/2} u(x, t + 2\xi) d\xi \right) + \frac{5}{3} \int_{-\pi/2}^{\pi/2} u(x, t + 2\xi) d\xi \\
= \frac{1}{6} \Delta u(x, t) + \frac{7}{6} \Delta u(x, t - \frac{3\pi}{2}) , \quad t > 1, \ t \neq t_k, \ k = 1, 2, ..., \\
u(x, (t_k)^+) = \frac{k + 1}{k} u(x, t_k), \\
\frac{\partial^{(i)}}{\partial t^{(i)}} u(x, (t_k)^+) = \frac{\partial^{(i)}}{\partial t^{(i)}} u(x, t_k), \quad i = 1, 2, 3, \quad k = 1, 2, ...,
\end{align*}
\]

(5.1)

for \((x, t) \in (0, \pi) \times [0, +\infty) \), with the boundary condition

\[
u(0, t) = u(\pi, t) = 0, \quad t \neq t_k.
\]

(5.2)
Here $\Omega = (0, \pi), m = 4, n = 1, a_k^{(0)} = b_k^{(0)} = \frac{k + 1}{k}, a_k^{(i)} = b_k^{(i)} = 1, i = 1, 2, 3,$

$$g(t, \xi) = \frac{2}{3}, \quad Q(t, \xi) = \frac{5}{3}, \quad f(u) = u, \quad \sigma(t, \xi) = \tau(t, \xi) = t + 2\xi, \quad a(t) = \frac{1}{6},$$

$$b_1(t) = \frac{7}{6}, \quad \rho_1(t) = t - \frac{3\pi}{2}, \quad \eta(\xi) = \xi, \quad \theta(t) = t, \quad \theta'(t) = 1, \quad c = 1.$$

Also $G(s) = \frac{5\pi}{12} - \frac{5\pi^2}{72}, \quad F(s) = s^2.$ Since $t_0 = 1, \quad t_k^{(1)} = 2^k, \quad g_0 = 1 - \frac{\pi}{6},$ we see from the above assumption that the hypotheses (H1) – (H6) hold, moreover

$$\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < s < t_k} a_k^{(0)} \frac{k}{k + 1} ds = \int_{1}^{+\infty} \prod_{1 < s < t} k \frac{k}{k + 1} ds + \cdots$$

$$= 1 + \frac{1}{2} \times 2 + \frac{1}{3} \times 2^2 + \frac{1}{3} \times 3 \times 2^3 + \cdots$$

$$= \sum_{n = 0}^{+\infty} \frac{2^n}{n + 1} = +\infty.$$

Now, the condition (3.12) reads,

$$\lim_{t \to +\infty} \left\{ \frac{1}{t - 1} \int_{1}^{t} \prod_{1 < s < t} k \frac{k}{k + 1} \left[ \left( \frac{5\pi}{12} - \frac{5\pi^2}{72} \right) (t - s)^2 - \frac{1}{\sigma^2(t - s)^2} \right] ds \right\} = +\infty.$$

Therefore all the conditions of the Corollary 2 are satisfied. Therefore, every solution of equation (5.1)–(5.2) is oscillatory in $G.$ In fact $u(x, t) = \sin x \cos t$ is such a solution.

Example 2. Consider the following equation of the form

$$\frac{\partial^2}{\partial t^2} \left( u(x, t) + \frac{1}{2(t + 1)} \int_{-\pi}^{0} u(x, t + \xi) d\xi \right) + \frac{1}{2(t + 1)} \int_{-\pi}^{0} u(x, t + \xi) d\xi$$

$$= \left( 1 + \frac{1}{(t + 1)^2} \right) \Delta u(x, t) + \frac{2}{(t + 1)^3} \Delta u(x, t - \frac{7\pi}{2}), \quad t > 1, \quad t \neq t_k,$$

$$u(x, (t_k)^+) = \frac{k + 1}{k} u(x, t_k), \quad k = 1, 2, \ldots,$$

$$\frac{\partial}{\partial t} u(x, (t_k)^+) = \frac{k}{k} \frac{\partial}{\partial t} u(x, t_k), \quad k = 1, 2, \ldots,$$

for $(x, t) \in (0, \pi) \times [0, +\infty),$ with the boundary condition

$$u_x(0, t) = u_x(\pi, t) = 0, \quad t \neq t_k.$$

Here $\Omega = (0, \pi), m = 2, \quad n = 1, \quad \mu(x, t) = 1, \quad a_k^{(0)} = b_k^{(0)} = \frac{k + 1}{k}, \quad a_k^{(i)} = b_k^{(i)} = 1, \quad i = 1,$

$$g(t, \xi) = \frac{1}{2(t + 1)}, \quad Q(t, \xi) = \frac{1}{2(t + 1)}, \quad f(u) = u, \quad F(s) = 2s,$$
\( \tau(t, \xi) = \sigma(t, \xi) = t + \xi, \ a(t) = 1 + \frac{2}{(t + 1)^2}, \ b_1(t) = \frac{2}{(t + 1)^3}, \)

and \( \rho_1(t) = t - \frac{7\pi}{2}, \ \eta(\xi) = \xi, \ \theta(t) = t^2, \ \theta'(t) = 2t, \ c = 1. \ \) Since \( t_0 = 1, \ t_k = 2^k, \)

\[
g_0 = 1 - \frac{1}{2} \log \left( \frac{t + 1}{t + 1 - \pi} \right), \quad G(s) = \frac{\pi}{2(t + 1)} \left( 1 - \frac{1}{2} \log \left( \frac{t + 1}{t + 1 - \pi} \right) \right).
\]

From the above assumptions it is easy to see that the hypotheses \((H_1) - (H_6)\) hold. Still to show that the condition \((4.7)\) is satisfied. In fact this condition reads

\[
\limsup_{t \to +\infty} \frac{1}{(t - 1)^2} \left\{ \int_1^t \prod_{1 < s < t, s < s_k} \frac{k}{k + 1} \left[ \frac{\pi}{2(t + 1)} \left( 1 - \frac{1}{2} \log \left( \frac{t + 1}{t + 1 - \pi} \right) \right) \right] (t - s)^2 - \frac{1}{2s(t - s)^2} \right\} ds \right\} = +\infty.
\]

Therefore all the conditions of the Corollary 6 are satisfied. Therefore, every solution of equation \((5.3)-(5.4)\) is oscillatory in \(G\). In fact \(u(x, t) = \cos x \sin t\) is such a solution.

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1 PG and Research Department of Mathematics, Thiruvalluvar Government Arts College, (Affiliated with Periyar University, Salem - 636 011) Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, India. E-mail: kalaimaths4@gmail.com, trmaths19@gmail.com ovsadha@gmail.com

and

2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, e-mail: shsaker@mans.edu.eg