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Weak Hom-bialgebras and weak Hom-Hopf algebras
by

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Abstract

The purpose of this paper is to introduce and study weak Hom-bialgebras and weak Hom-Hopf algebras. We provide relevant definitions, properties and constructions by twisting. Moreover, we extend to these classes of algebras, Kaplansky's procedure to construct weak Hom-bialgebras and weak Hom-Hopf algebras starting from any Hom-associative algebra and Hom-bialgebra.

Key Words: Hom-associative algebra, Weak Hom-bialgebra, weak Hom-Hopf algebra

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1 Introduction

Quantum deformations (or q-deformations) of Lie algebras have been investigated in various domains of Mathematics and Physics. They gave rise to a great interest in quantum groups and Hopf algebras. In physical contexts appeared q-deformations of infinite-dimensional Lie algebras, primarily the Heisenberg algebras (oscillator algebras) and the Virasoro algebra. It turns out that these algebras obey some deformed (twisted) version of Jacobi identity. This kind of algebras was called by Hartwig, Larsson and Silvestrov Hom-Lie algebras. The corresponding associative algebras, called Hom-associative algebras, were introduced by the second author and Silvestrov in [8]. Hom-bialgebras and Hom-Hopf algebras were introduced and studied in [10],[11], see also [16]. The weak Hom-bialgebras (resp. weak Hom-Hopf algebras) constitute a wider class which include weak bialgebras (resp. weak Hopf algebras). The categories of these algebras are monoidal categories. Recall that weak coproduct was also motivated by quantum symmetry and vector field algebras, see [7]. The weak bialgebras were introduced by Böhm et al. [1] in the context of operator algebras and quantum field theory.

The aim of this paper is to generalize weak bialgebras and weak Hopf algebras to Hom-setting and study their properties. We provide constructions of weak Hom-bialgebras and weak Hom-Hopf algebras using twisting principle and also Kaplansky's type constructions starting from any Hom-algebras, generalizing the results in [5, Theorem 1.3], and results in [13]. In Section 2, we provide the definitions and the main properties of weak Hom-bialgebras and weak Hom-Hopf algebras. In Section 3, we describe Twisting construction and show some examples, like deforming Sweedler-Taft Hopf algebra to a weak Hom-Hopf bialgebra. Section 4 is dedicated to study constructions of weak Hom-bialgebras starting from any Hom-algebra. These results generalize Kaplansky's constructions for bialgebras and Hopf algebras, which were extended to weak bialgebras and weak Hopf algebras in [3].

2 Preliminaries

Throughout this paper $\mathbb{K}$ is an algebraically closed field of characteristic 0. Let A be a finite-dimensional $\mathbb{K}$-vector space. In the sequel we use for a comultiplication $\Delta : A \to A \otimes A$, Sweedler's notation that is $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, $\forall x \in A$.

Recall that a *bialgebra* is a $\mathbb{K}$-vector space A, equipped with an algebra structure given by a multiplication μ, a unit η, and a coalgebra structure given by a comultiplication Δ and a counit ε, such that there is a compatibility condition between the two structures expressed by the fact that Δ and ε are algebra morphisms, that is for $x, y \in A$

$$\Delta(\mu(x \otimes y)) = \Delta(x) \cdot \Delta(y) \quad \text{and} \quad \varepsilon(\mu(x, y)) = \varepsilon(x)\varepsilon(y).$$

The multiplication $\cdot$ on $A \otimes A$ is the usual multiplication on tensor product,

$$(x \otimes y) \cdot (x' \otimes y') = \mu\left(x \otimes x'\right) \otimes \mu\left(y \otimes y'\right).$$

It is assumed also that the unit 1 is sent to the unit by the comultiplication and counit, $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. The unit η is completely determined by $\eta(1)$, which we denote by 1 and multiplication μ by a dot when there is no confusion. A bialgebra is said to be a *Hopf algebra* if the identity on A has an inverse for the convolution product defined by

$$f \star g := \mu \circ (f \otimes g) \circ \Delta. \tag{2.1}$$

For complete classical theory of Bialgebras and Hopf algebras, we refer to [2, 6]. Hom-bialgebras and Hom-Hopf algebras were studied in [10, 11, 16]. The associativity condition and the coassociativity condition are twisted by a homomorphism α. They write

$$\alpha(x)(yz) = (xy)\alpha(z) \text{ and } (\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta.$$

Definition 1. *A Hom-associative algebra is a triple (A, μ, α) consisting of a $\mathbb{K}$-vector space A, a bilinear map $\mu : A \times A \to A$ and a linear space homomorphism $\alpha : A \to A$ satisfying*

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)). \tag{2.2}$$

$$\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)). \tag{2.3}$$

The second condition is called multiplicativity and usually such a Hom-associative algebras are called multiplicative. Since we are dealing only with multiplicative Hom-associative algebras, we shall call them Hom-associative algebras for simplicity.

A Hom-associative algebra is said unital if there exists an element $u \in A$ such that $\alpha(u) = u$ and $\mu(x, u) = \mu(u, x) = \alpha(x) \quad \forall x \in A$.

In the following, we introduce weak Hom-bialgebras.

Definition 2. *A weak Hom-bialgebra is a sextuple $\mathcal{B} = (A, m, \eta, \Delta, \varepsilon, \alpha)$, where $m : A \otimes A \to A$ (multiplication), $\eta : \mathbb{K} \to A$ (unit), $\Delta : A \to A \otimes A$ (comultiplication), $\varepsilon : A \to K$ (counit) and $\alpha : A \to A$ are linear maps satisfying, for all $x, y, z \in A$ and by setting $\eta(1) = 1$:*

1. *the quadruple (A, m, η, α) is a unital multiplicative Hom-associative algebra, that is*

$$\alpha \circ m = m \circ (\alpha \otimes \alpha), \tag{2.4}$$

$$m \circ (m \otimes \alpha) = m \circ (\alpha \otimes m), \tag{2.5}$$

$$m(x \otimes 1) = m(1 \otimes x) = \alpha(x), \ \alpha(1) = 1, \tag{2.6}$$

2. *the quadruple* $(A, \Delta, \varepsilon, \alpha)$ *is a counital comultiplicative Hom-coassociative coalgebra, that is*

$$(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha, \tag{2.7}$$
$$(\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta, \tag{2.8}$$
$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = \alpha, \ \varepsilon \circ \alpha = \varepsilon, \tag{2.9}$$

3. *the compatibility condition is expressed by the following three identities:*

$$\Delta(m(x \otimes y)) = \sum m(x_{(1)} \otimes y_{(1)}) \otimes m(x_{(2)} \otimes y_{(2)}), \tag{2.10}$$
$$(\Delta \otimes \alpha) \circ \Delta(1) = (\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1)) = (1 \otimes \Delta(1)) \cdot (\Delta(1) \otimes 1), \tag{2.11}$$
$$\begin{aligned} \varepsilon(m(m(x \otimes y) \otimes \alpha(z)) &= \varepsilon(m(\alpha(x) \otimes y_{(1)}))\ \varepsilon(m(y_{(2)} \otimes \alpha(z)) \\ &= \varepsilon(m(\alpha(x) \otimes y_{(2)}))\ \varepsilon(m(y_{(1)} \otimes \alpha(z)). \end{aligned} \tag{2.12}$$

Notice that if $\alpha = id$, we recover usual definition of weak-bialgebra.

Definition 3. *Let* $(A, m, \eta, \Delta, \varepsilon, \alpha)$ *and* $(A', m', \eta', \Delta', \varepsilon', \alpha')$ *be two weak Hom-bialgebras, we say that a linear map* $f : A \to A'$ *is a weak Hom-bialgebra morphism if the following identities hold:*

$$f \circ m = m' \circ (f \otimes f), \tag{2.13}$$
$$(f \otimes f) \circ \Delta = \Delta' \circ f, \tag{2.14}$$
$$f \circ \eta = \eta', \tag{2.15}$$
$$\varepsilon' \circ f = \varepsilon, \tag{2.16}$$
$$f \circ \alpha = \alpha' \circ f. \tag{2.17}$$

Remark 1. *Condition* (2.10) *means that* Δ *is a Hom-algebra morphism, that is conditions* (2.13) *and* (2.15) *are satisfied. But condition* (2.11) *shows that* Δ *does not necessarily conserve the unit* 1.
If $\Delta(1) = 1 \otimes 1$ *then condition* (2.11) *is satisfied. Indeed*

$$(\Delta \otimes \alpha) \circ \Delta(1) = \Delta(1) \otimes \alpha(1) = 1 \otimes 1 \otimes 1,$$

and

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes 1 \otimes 1).(1 \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1.$$

Identity (2.12) *is a weak version of the fact that* ε *is a Hom-algebra morphism in the Hom-bialgebra case. When* $\Delta(1) = 1 \otimes 1$, *one can derive that the counit is a Hom-algebra morphism. Indeed*
$\varepsilon(m(x \otimes y)) = \varepsilon(\alpha(m(x \otimes y))) = \varepsilon(m((\alpha(x) \otimes \alpha(y)))) = \varepsilon(m(m(x \otimes 1) \otimes \alpha(y))) = \varepsilon(\alpha^2(x))\varepsilon(\alpha^2(y)) = \varepsilon(x)\varepsilon(y)$. *Thus, a weak Hom-bialgebra for which* $\Delta(1) = 1 \otimes 1$ *is a Hom-bialgebra. Conversely, any unital and counital Hom-bialgebra is a weak Hom-bialgebra.*

Definition 4. *A weak Hom-Hopf algebra is a 7-tuple* $\mathcal{H} = (A, m, \eta, \Delta, \varepsilon, S, \alpha)$, *where* $(A, m, \eta, \Delta, \varepsilon, \alpha)$ *is a weak Hom-bialgebra and* S *is an antipode, that is an endomorphism* *of A satisfying for all $x \in A$:*

$$m(id \otimes S)\Delta(x) = (\varepsilon \otimes \alpha)(\Delta(1)(x \otimes 1)), \tag{2.18}$$
$$m(S \otimes id)\Delta(x) = (\alpha \otimes \varepsilon)((1 \otimes x)\Delta(1)), \tag{2.19}$$
$$m(m \otimes \alpha)(S \otimes id \otimes S)(\Delta \otimes \alpha)\Delta = S \circ \alpha^4, \tag{2.20}$$
$$\alpha \circ S = S \circ \alpha. \tag{2.21}$$

Notice that the definition of weak Hom-Hopf algebras provided in [18] is different.

Example 1. *Let A be a 2-dimensional vector space generated by $\{e_1, e_2\}$. It carries a structure of weak Hom-bialgebra with the following operations,*

$$m(e_1, e_1) = e_1, m(e_1, e_2) = e_1 - e_2, m(e_2, e_1) = e_1 - e_2, m(e_2, e_2) = e_1 - e_2,$$

where $\eta(1) = e_1$,

$$\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2, \Delta(e_2) = (e_1 - e_2) \otimes (e_1 - e_2),$$

$$\varepsilon(e_1) = 2, \varepsilon(e_2) = 1,$$

$$\alpha(e_1) = e_1, \alpha(e_2) = e_1 - e_2.$$

In addition, it defines a weak Hom-Hopf algebra with the antipode $S = id$.

Example 2. *Let A be a 3-dimensional vector space generated by $\{e_1, e_2, e_3\}$. It carries a structure of weak Hom-bialgebra with the following operations,*

$$\begin{aligned}
&m(e_1, e_1) = e_1, m(e_1, e_2) = e_1 - e_3, m(e_2, e_1) = e_1 - e_3, m(e_2, e_2) = e_1 - e_3,\\
&m(e_1, e_3) = e_2 - e_3, m(e_3, e_1) = e_2 - e_3, m(e_3, e_2) = e_2 - e_3,\\
&m(e_2, e_3) = e_2 - e_3, m(e_3, e_3) = e_2 - e_3,
\end{aligned}$$

where $\eta(1) = e_1$,

$$\begin{aligned}
&\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + (e_2 - e_3) \otimes (e_2 - e_3) + e_3 \otimes e_3,\\
&\Delta(e_2) = (e_1 - e_2) \otimes (e_1 - e_2) + (e_2 - e_3) \otimes (e_2 - e_3),\\
&\Delta(e_3) = (e_2 - e_3) \otimes (e_2 - e_3),\\
&\varepsilon(e_1) = 3, \varepsilon(e_2) = 2, \varepsilon(e_3) = 1,\\
&\alpha(e_1) = e_1,\ \alpha(e_2) = e_1 - e_3,\ \alpha(e_3) = e_2 - e_3.
\end{aligned}$$

In addition, it defines a weak Hom-Hopf algebra with S=id.

3 Twisting constructions

We show in the following that a weak Hom-bialgebra and a weak Hom-bialgebra morphism give rise to a new weak Hom-bialgebra. In particular, one may construct weak Hom-bialgebras starting from a usual weak bialgebra and a bialgebra morphism.

Theorem 1. *Let $B = (A, m, \eta, \Delta, \varepsilon, \alpha)$ be a weak Hom-bialgebra and $\beta : A \to A$ be a weak Hom-bialgebra morphism. Then,*

$$B_\beta = (A, m_\beta = \beta \circ m, \eta, \Delta_\beta = \Delta \circ \beta, \varepsilon, \beta \circ \alpha)$$

is a weak Hom-bialgebra.

Proof. We check the identities (2.5)-(2.12):

1. *Hom-associativity*

$$\begin{aligned} m_\beta(\beta \circ \alpha(x) \otimes m_\beta(y \otimes z)) &= \beta \circ m(\beta \circ \alpha(x) \otimes \beta \circ m(y \otimes z)), \\ &= \beta^2 \circ m(\alpha(x) \otimes m(y \otimes z)) = \beta^2 \circ m(m(x \otimes y) \otimes \alpha(z)), \\ &= m_\beta(m_\beta(x \otimes y) \otimes \beta \circ \alpha(z)). \end{aligned}$$

Notice that we denote by β^2 the composition $\beta \circ \beta$.

2. *Unitality*

$$\eta \circ \beta \circ \alpha(1) = \eta \circ \beta(1) = \eta(1).$$

3. *Hom-coassociativity*

$$\begin{aligned} (\Delta_\beta \otimes \beta \circ \alpha)\Delta_\beta(x) &= (\beta \otimes \beta \otimes \beta)(\Delta \otimes \alpha)\Delta_\beta(x) = (\beta \otimes \beta \otimes \beta)(\alpha \otimes \Delta)\Delta_\beta(x), \\ &= ((\beta \circ \alpha) \otimes \Delta_\beta)\Delta_\beta(x). \end{aligned}$$

4. *Counitality*

$$\begin{aligned} (id \otimes \varepsilon)\Delta_\beta(x) &= \beta(x_1)\varepsilon(\beta(x_2)) = \beta(x_1)\varepsilon(x_2) = \beta(x_1)\varepsilon(\alpha(x_2)), \\ &= \beta(x_1\varepsilon(\alpha(x_2)) = \beta \circ \alpha(x). \end{aligned}$$

On the other hand,

$$(\varepsilon \otimes id)\Delta_\beta(x) = \beta(\varepsilon(\alpha(x_1))x_2) = \beta \circ \alpha(x).$$

5. *Compatibility condition*

(a)

$$\begin{aligned} \Delta_\beta(m_\beta(x \otimes y)) &= \Delta(\beta^2(x) \cdot \beta^2(y)) = (\beta^2 \otimes \beta^2)\Delta(x.y) = (\beta^2 \otimes \beta^2)(\Delta(x) \cdot \Delta(y)), \\ &= (\beta \otimes \beta)(\Delta_\beta(x) \cdot \Delta_\beta(y)) = \Delta_\beta(x) \cdot_\beta \Delta_\beta(y). \end{aligned}$$

(b)

$$\begin{aligned} (\Delta_\beta \otimes \beta \circ \alpha)\Delta_\beta(1) &= (\beta^2 \otimes \beta^2 \otimes \beta^2)((\Delta \otimes \alpha)\Delta(1)) = (\beta^2 \otimes \beta^2 \otimes \beta^2)((\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1))) \\ &= (\Delta_\beta(1) \otimes 1) \cdot_\beta (1 \otimes \Delta_\beta(1)). \end{aligned}$$

and similarly

$$(\Delta_\beta \otimes \beta \circ \alpha)\Delta_\beta(1) = (1 \otimes \Delta_\beta(1)) \cdot_\beta (\Delta_\beta(1) \otimes 1).$$

(c) *ε weak condition*

$$\begin{aligned} &\varepsilon(m_\beta(m_\beta(x \otimes y) \otimes \beta \circ \alpha(z))) = \varepsilon \circ \beta^2 \circ (m(m(x \otimes y) \otimes \alpha(z))), \\ &= \varepsilon \circ (m(m(x \otimes y) \otimes \alpha(z))), \\ &= \varepsilon(\beta \circ m((\beta \circ \alpha(x) \otimes \beta(y_1))\varepsilon(\beta \circ m(\beta(y_2) \otimes \beta \circ \alpha(z))), \\ &= \varepsilon \circ \beta^2(m(\alpha(x) \otimes y_1))\varepsilon \circ \beta^2(m(y_2 \otimes \alpha(z))), \\ &= \varepsilon(m(\beta \circ \alpha(x) \otimes \beta(y_1)))\varepsilon(m(\beta(y_2) \otimes \beta \circ \alpha(z))). \end{aligned}$$

□

Corollary 1. *Let $B=(A,m,\eta,\Delta,\varepsilon)$ be a weak bialgebra and β a weak bialgebra morphism. Then*
$$B_\beta=(A,m_\beta=\beta\circ m,\eta,\Delta_\beta=\Delta\circ\beta,\varepsilon,\beta)$$
is a weak Hom-bialgebra.

Proof. We apply Theorem 1 to $B_{id}=(A,m,\eta,\Delta,\varepsilon,id)$. □

Corollary 2. *Let $B=(A,m,\eta,\Delta,\varepsilon,\alpha)$ be a weak Hom-bialgebra. Then for any natural number n,*
$$B_{\alpha^n}=(A,m_{\alpha^n}=\alpha^n\circ m,\eta,\Delta_{\alpha^n}=\Delta\circ\alpha^n,\varepsilon,\alpha^{n+1})$$
is a weak Hom-bialgebra.

Theorem 2. *Let $B=(A,m,\eta,\Delta,\varepsilon,S,\alpha)$ be a weak Hom-Hopf algebra and β be a weak Hom-Hopf morphism such that $\beta\circ S=S\circ\beta$. Then*
$$B_\beta=(A,m_\beta=\beta\circ m,\eta,\Delta_\beta=\Delta\circ\beta,\varepsilon,S,\beta\circ\alpha)$$
is a weak Hom-Hopf algebra.

Proof. We have to check antipode's conditions (2.18), (2.19), (2.20). Note first that $\Delta_\beta(1)=\Delta(1)$. We use Sweedler's notation and omit the summation symbol.
$$\begin{aligned}&m_\beta\circ(id\otimes S)\circ\Delta_\beta(x)=\beta\circ m\circ(id\otimes S)\circ\Delta\circ\beta(x)=\beta^2\circ m\circ(id\otimes S)\circ\Delta(x),\\&=\beta^2\circ(\varepsilon\otimes\alpha)\circ(m(1_{(1)}\otimes x)\otimes m(1_{(2)}\otimes 1)),\\&=(\varepsilon\otimes\beta\circ\alpha)\circ(\beta\otimes\beta)\circ(m(1_{(1)}\otimes x)\otimes m(1_{(2)}\otimes 1)),\\&=(\varepsilon\otimes\beta\circ\alpha)\circ(m_\beta(1_{(1)}\otimes x)\otimes m_\beta(1_{(2)}\otimes 1)).\end{aligned}$$

$$\begin{aligned}&m_\beta(S\otimes id)\Delta_\beta(x)=\beta\circ m(S\otimes Id)\Delta\circ\beta(x)=\beta^2\circ m(S\otimes id)\Delta(x),\\&=\beta^2\circ(\alpha\otimes\varepsilon)(m(1\otimes 1_{(1)})\otimes m(x\otimes 1_{(2)})),\\&=(\beta\circ\alpha\otimes\varepsilon)\circ(\beta\otimes\beta)\circ(m(1\otimes 1_{(1)})\otimes m(x\otimes 1_{(2)})),\\&=(\beta\circ\alpha\otimes\varepsilon)\circ(m_\beta(1\otimes 1_{(1)})\otimes m_\beta(x\otimes 1_{(2)})).\end{aligned}$$

$$\begin{aligned}&m_\beta(m_\beta\otimes\beta\circ\alpha)\circ(S\otimes id\otimes S)\circ(\Delta_\beta\otimes\beta\circ\alpha)\Delta_\beta(x),\\&=\beta\circ m\circ(\beta\circ m\otimes\beta\circ\alpha)\circ(S\otimes id\otimes S)\circ(\Delta\circ\beta\otimes\beta\circ\alpha)\circ\Delta\circ\beta(x),\\&=\beta\circ m\circ(\beta\otimes\beta)\circ(m\otimes\alpha)\circ(S\otimes id\otimes S)\circ(\beta^2\otimes\beta^2\otimes\beta^2)\circ(\Delta\otimes\alpha)\circ\Delta(x),\\&=\beta\circ m\circ(\beta\otimes\beta)\circ(m\otimes\alpha)\circ(\beta^2\otimes\beta^2\otimes\beta^2)\circ(S\otimes id\otimes S)\circ(\Delta\otimes\alpha)\circ\Delta(x),\\&=\beta\circ m\circ(\beta\otimes\beta)\circ(\beta^2\otimes\beta^2)\circ(m\otimes\alpha)\circ(S\otimes id\otimes S)\circ(\Delta\otimes\alpha)\circ\Delta(x),\\&=\beta^4\circ m\circ(m\otimes\alpha)\circ(S\otimes id\otimes S)\circ(\Delta\otimes\alpha)\circ\Delta(x)=\beta^4\circ S\circ\alpha^4(x),\\&=(\beta\circ\alpha)^4\circ S(x).\end{aligned}$$

□

Corollary 3. *Let $B = (A, m, \eta, \Delta, \varepsilon, S)$ be a weak Hopf algebra and β be a weak Hopf algebra morphism such that $\beta \circ S = S \circ \beta$. Then*

$$B_\beta = (A, m_\beta = \beta \circ m, \eta, \Delta_\beta = \Delta \circ \beta, \varepsilon, S, \beta)$$

is a weak Hom-Hopf algebra.

Corollary 4. *Let $B = (A, m, \eta, \Delta, \varepsilon, S, \alpha)$ be a weak Hom-Hopf algebra. Then, for any natural number n,*

$$B_{\alpha^n} = (A, m_{\alpha^n} = \alpha^n \circ m, \eta, \Delta_{\alpha^n} = \Delta \circ \alpha^n, \varepsilon, S, \alpha^{n+1})$$

is a weak Hom-Hopf algebra.

Example 3 (**Sweedler-Taft Weak Hom-Hopf algebras**). *Let $n \geq 2$ be an integer and λ be a primitive n^{th} root of unity. Consider the Taft's algebras $\mathcal{H}_{n^2}(\lambda)$, generalizing Sweedler's Hopf algebra, defined by the generators c, x and were e is the unit, with the relations: $c^n = e,\ x^n = 0,\ x \cdot c = \lambda\ c \cdot x$.*

Let $\mathcal{H}'$ be the algebra obtained by adjoining a new unit 1 to $\mathcal{H}_{n^2}(\lambda)$. It carries a weak Hopf algebra structure as in [3]. We define a linear map $\beta : \mathcal{H}' \to \mathcal{H}'$ as follows:

$$\beta(1) = 1, \quad \beta(e) = e, \quad \beta(c) = c, \quad \beta(x) = tx,$$

where t is a parameter in $\mathbb{K}$. It is a weak Hopf algebra morphism. Then, by applying Theorem 2, it leads to what we call a Sweedler-Taft weak Hom-Hopf algebra defined as follows.
The multiplication is defined by $m_\beta(x, y) = \beta(x \cdot y) = \beta(x) \cdot \beta(y)$. The Hom-coalgebra structure is defined by :

$$\begin{aligned}
&\Delta_\beta(1) = (1 - e) \otimes (1 - e) + e \otimes e, \\
&\Delta_\beta(e) = e \otimes e,\ \Delta_\beta(c) = c \otimes c,\ \Delta_\beta(x) = c \otimes x + x \otimes e, \\
&\varepsilon(1) = 2,\ \varepsilon(e) = 1,\ \varepsilon(c) = 1,\ \varepsilon(x) = 0.
\end{aligned}$$

Then $\mathcal{H}'$ becomes (n^2+1)-dimension weak Hom-bialgebra, having a basis $\{1, c^i x^j, 0 \leq i, j \leq n - 1\}$.

It carries a structure of weak Hom-Hopf algebra with an antipode defined by :

$$S(1) = 1,\ S(e) = e,\ S(c) = c^{-1},\ S(x) = -c^{-1} \cdot x.$$

4 Hom-Type Kaplansky's Constructions

In this section, we provide constructions of weak Hom-bialgebras and weak Hom-Hopf algebras starting from any Hom-associative algebra. These constructions are inspired by Kaplansky's constructions for bialgebras (see [5]).

Theorem 3. *Let (A, m, α) be any multiplicative Hom-associative algebra (not necessarily unital) with basis $\mathfrak{b}$ and $\mathcal{B}$ be the result of adjoining to A two successive unit elements e and 1, that means $m(1 \otimes 1) = 1$, $m(e \otimes e) = e$, $m(1 \otimes e) = m(e \otimes 1) = e$, $m(1 \otimes a) = m(a \otimes 1) =$*

$m(e\otimes a)=m(a\otimes e)=\alpha(a)$, $\forall a\in\mathfrak{b}$ and $\alpha(1)=1$, $\alpha(e)=e$. We set, on the vector space $\mathcal{B}$ spanned by the vector space A together with the generators $\{e,\ 1\}$,

$$\begin{aligned}\Delta(1)&=(1-e)\otimes(1-e)+e\otimes e,\\ \Delta(e)&=e\otimes e,\\ \Delta(a)&=\alpha(a)\otimes\alpha(a),\quad\forall a\in\mathfrak{b},\\ \varepsilon(a)&=1,\quad\forall a\in\mathfrak{b},\\ \varepsilon(e)&=1,\\ \varepsilon(1)&=2,\end{aligned}$$

then extend Δ and ε by linearity for all element of $\mathcal{B}$. Assume moreover that for all $a,b,c\in\mathfrak{b}$ we have $m(a\otimes b)\in\mathfrak{b}$ and

$$\Delta(\alpha(a))=\alpha^2(a)\otimes\alpha^2(a),\ \varepsilon(\alpha(a))=\varepsilon(\alpha^2(a))=1,\ \varepsilon(m(m(a\otimes b)\otimes\alpha(c)))=1.\qquad(4.1)$$

Then $\mathcal{B}$ becomes a weak Hom-bialgebra.

Proof. We check that Δ is Hom-coassociative for a,b in $\mathfrak{b}$:

$$(\Delta\otimes\alpha)\Delta(a)=(\Delta\otimes\alpha)(\alpha(a)\otimes\alpha(a))=\Delta(\alpha(a))\otimes\alpha^2(a)=\alpha^2(a)\otimes\alpha^2(a)\otimes\alpha^2(a),$$

$$(\alpha\otimes\Delta)\Delta(a)=(\alpha\otimes\Delta)(\alpha(a)\otimes\alpha(a))=\alpha^2(a)\otimes\alpha^2(a)\otimes\alpha^2(a),\quad\forall a\in\mathfrak{b}.$$

We have

$$\begin{aligned}&(\Delta\otimes\alpha)\Delta(1)=\Delta(1)\otimes 1-\Delta(1)\otimes e-\Delta(e)\otimes 1+2\Delta(e)\otimes e\\ &=1\otimes 1\otimes 1-1\otimes e\otimes 1-e\otimes 1\otimes 1+2e\otimes e\otimes 1-1\otimes 1\otimes e+1\otimes e\otimes e\\ &+e\otimes 1\otimes e-2e\otimes e\otimes e-e\otimes e\otimes 1+2e\otimes e\otimes e\\ &=1\otimes 1\otimes 1-1\otimes e\otimes 1-e\otimes 1\otimes 1+e\otimes e\otimes 1-1\otimes 1\otimes e+1\otimes e\otimes e+e\otimes 1\otimes e\\ &=(1-e)\otimes(1-e)\otimes(1-e)+e\otimes e\otimes e.\end{aligned}$$

On the other hand

$$\begin{aligned}&(\alpha\otimes\Delta)\Delta(1)=1\otimes\Delta(1)-1\otimes\Delta(e)-e\otimes\Delta(1)+2e\otimes\Delta(e)\\ &=1\otimes 1\otimes 1-1\otimes e\otimes 1-e\otimes 1\otimes 1+e\otimes e\otimes 1-1\otimes 1\otimes e+1\otimes e\otimes e+e\otimes 1\otimes e\\ &=(1-e)\otimes(1-e)\otimes(1-e)+e\otimes e\otimes e.\end{aligned}$$

Then

$$(\Delta\otimes\alpha)\Delta(1)=(\alpha\otimes\Delta)\Delta(1).$$

We check the compatibility condition (2.11). For simplicity, we denote in the sequel the multiplication by a central dot (even for tensor product).

$$\begin{aligned}(1\otimes\Delta(1))\cdot(\Delta(1)\otimes 1)&=(\Delta(1)\otimes 1)\cdot(1\otimes\Delta(1)),\\ &=(1-e)\otimes(1-e)\otimes(1-e)+e\otimes e\otimes e.\end{aligned}$$

Using previous calculation, we derive that (2.11) is satisfied.

Now, we check the remaining identities for all $a \in \mathfrak{b}$:

$$\begin{array}{rcl}(\varepsilon\otimes id)\Delta(1) & = & \varepsilon(1)1-\varepsilon(1)e-\varepsilon(e)1+2\varepsilon(e)e=\alpha(1),\\ (id\otimes\varepsilon)\Delta(1) & = & \varepsilon(1)1-\varepsilon(e)1-\varepsilon(1)e+2\varepsilon(e)e=\alpha(1),\\ (\varepsilon\otimes id)\Delta(e) & = & \varepsilon(e)e=e=\alpha(e),\\ (id\otimes\varepsilon)\Delta(e) & = & \varepsilon(e)e=e=\alpha(e),\\ (\varepsilon\otimes id)\Delta(a) & = & \varepsilon(\alpha(a))\alpha(a)=\alpha(a),\\ (id\otimes\varepsilon)\Delta(a) & = & \varepsilon(\alpha(a))\alpha(a)=\alpha(a).\end{array}$$

The comultiplication Δ is in fact an algebra homomorphism. Indeed, for $a \in \mathfrak{b}$, we have $\Delta(a)\cdot\Delta(1) = (\alpha(a)\otimes\alpha(a))\cdot(1\otimes 1 - 1\otimes e - e\otimes 1 + 2e\otimes e) = \alpha^2(a)\otimes\alpha^2(a) - \alpha^2(a)\otimes\alpha^2(a) - \alpha^2(a)\otimes\alpha^2(a) + 2\alpha^2(a)\otimes\alpha^2(a) = \alpha^2(a)\otimes\alpha^2(a) = \Delta(\alpha(a)) = \Delta(a\cdot 1)$.

$\Delta(a)\cdot\Delta(e) = (\alpha(a)\otimes\alpha(a))\cdot(e\otimes e) = \alpha(a)\cdot e\otimes\alpha(a)\cdot e = \alpha^2(a)\otimes\alpha^2(a) = \Delta(a\cdot e)$.

Let $a_1, a_2 \in \mathfrak{b}$, $\Delta(a_1)\cdot\Delta(a_2) = (\alpha(a_1)\otimes\alpha(a_1))\cdot(\alpha(a_2)\otimes\alpha(a_2)) = \alpha(a_1)\cdot\alpha(a_2)\otimes\alpha(a_1)\cdot\alpha(a_2) = \alpha(a_1\cdot a_2)\otimes\alpha(a_1\cdot a_2) = \Delta(a_1\cdot a_2)$.

Also straightforward computation shows that ε satisfies the weak condition (2.12).

□

Remark 2. *A class of examples of Hom-associative algebras satisfying the hypotheses of the previous theorem may be obtained as follows. Take (S, m, α) a multiplicative Hom-semigroup as defined in [4] and define A to be the semigroup Hom-algebra of S.*

Proposition 1. *The weak Hom-bialgebra defined in Theorem 3 cannot be endowed with a weak Hom-Hopf algebra structure.*

Proof. Indeed, Condition (2.19) cannot be satisfied. Since

$$m\circ(id\otimes S)\Delta(x) = m\circ(id\otimes S)(\alpha(x)\otimes\alpha(x)) = m(\alpha(x)\otimes S(\alpha(x)))$$

and

$$\begin{array}{l}(\varepsilon\otimes\alpha)(\Delta(1)(x\otimes 1)) = (\varepsilon\otimes\alpha)((1-e)\otimes(1-e)\cdot(x\otimes 1) + (e\otimes e)\cdot(x\otimes 1)),\\ = (\varepsilon\otimes\alpha)(\alpha(x)\otimes e) = \varepsilon(\alpha(x))e = e.\end{array}$$

Then, we should have

$$\alpha(x)\cdot S(\alpha(x)) = e.$$

This is impossible because $\alpha(x)\cdot S(\alpha(x))$ belongs to A and cannot be equal to e.

□

Corollary 5. *Let (A, m, α) be a multiplicative unital Hom-associative algebra where 1 is the unit and let $\mathfrak{b}$ be a basis of A. We assume that for all $a, b \in \mathfrak{b}$ we have $m(a\otimes b) \in \mathfrak{b}$*

and there exists an element e in A such that: $m(e\otimes a)=m(a\otimes e)=\alpha(a)$, $\forall a\in\mathfrak{b}$, and $\alpha(e)=e$. We set

$$\begin{aligned}\Delta(1)&=(1-e)\otimes(1-e)+e\otimes e,\\ \Delta(a)&=\alpha(a)\otimes\alpha(a),\forall a\in\mathfrak{b}\\ \varepsilon(a)&=1,\quad\forall a\in\mathfrak{b},a\neq 1,\\ \varepsilon(1)&=2,\end{aligned}$$

and moreover, for all $a,b,c\in\mathfrak{b}$ we have

$$\Delta(\alpha(a))=\alpha^2(a)\otimes\alpha^2(a),\ \varepsilon(\alpha(a))=\varepsilon(\alpha^2(a))=1,\ \varepsilon(m(m(a\otimes b)\otimes\alpha(c)))=1.\qquad(4.2)$$

Then, $(A,m,1,\Delta,\varepsilon,\alpha)$ is a weak Hom-bialgebra.

Proof. The map Δ is a Hom-algebra morphism according to Theorem 3. Furthermore
$\Delta(1)\cdot\Delta(1)=\Delta(1\cdot 1)=\Delta(1)$,
$\Delta(1)\cdot\Delta(a)=((1-e)\otimes(1-e)+e\otimes e)\cdot(\alpha(a)\otimes\alpha(a))=\Delta(1\cdot a)=\Delta(\alpha(a))$, $\forall a\in\mathfrak{b}$
$\Delta(a_1)\cdot\Delta(a_2)=\Delta(a_1\cdot a_2)=\alpha(a_1)\cdot\alpha(a_2)\otimes\alpha(a_1)\cdot\alpha(a_2)$, $\forall a_1,a_2\in\mathfrak{b}$. □

Corollary 6. *Let (A,m,α) be a n-dimensional multiplicative unital Hom-associative algebra where 1 is the unit. We assume that there is a basis $\mathfrak{b}=\{e_i\}_{1\leq i\leq n}$ of A where $e_1=1$, $\alpha(e_i)\in\mathfrak{b}$ and the elements $\{e_i\}_{2\leq i\leq n}$ are nilpotent of order 2 and generate a hyperplane with an orthogonal basis. Then, there exists a weak Hom-bialgebra structure on A given by setting, for a fixed integer $k\in\{2,\cdots,n\}$, on the orthogonal basis $\{e_i\}_{2\leq i\leq n}$ of the hyperplane,*

$$\begin{aligned}\Delta(1)&=(1-\alpha(e_k))\otimes(1-\alpha(e_k))+\alpha(e_k)\otimes\alpha(e_k),\\ \Delta(e_i)&=\alpha(e_i)\otimes\alpha(e_i)\ \textit{and}\ \Delta(\alpha(e_i))=\alpha^2(e_i)\otimes\alpha^2(e_i),\quad i=2,\cdots,n,\\ \varepsilon(1)&=2,\\ \varepsilon(e_i)&=\varepsilon(\alpha(e_i))=\varepsilon(\alpha^2(e_i))=1,\quad i=2,\cdots,n.\end{aligned}\qquad(4.3)$$

Proof. Straightforward calculations. □

We have the following more general result.

Theorem 4. *Consider (A,m,α) to be a n-dimensional multiplicative unital Hom-associative algebra with unit $e_1=1$ defined as follows. Fix an integer p such that $p\leq n$ and assume that there is a basis $\mathfrak{b}=\mathfrak{b}_1\cup\mathfrak{b}_2$ of A where $\mathfrak{b}_1=\{e_i\}_{i=1,\cdots,p}$ and $\mathfrak{b}_2=\{e_i\}_{i=p+1,\cdots,n}$. Set*

$$\begin{aligned}&m(e_i\otimes e_j)=e_{max(i,j)},\quad i,j=2,\cdots,p,\\ &m(e_i\otimes g)=m(g\otimes e_i)=\alpha(g),\quad\forall g\in\mathfrak{b}_2.\\ &\alpha(e_i)=e_i,\ i=1,\cdots,p,\\ &\alpha(g)\in\mathfrak{b}_2,\quad\forall g\in\mathfrak{b}_2.\end{aligned}\qquad(4.4)$$

The comultiplication Δ *and the counit* ε *defined by*

$$\begin{aligned}
&\Delta(e_p) = e_p \otimes e_p \\
&\Delta(e_i) = (e_i - e_{i+1}) \otimes (e_i - e_{i+1}) + \Delta(e_{i+1}), \quad i = 1, \cdots, p-1, \\
&\Delta(g) = \alpha(g) \otimes \alpha(g), \quad \forall g \in \mathfrak{b}_2, \\
&\varepsilon(e_i) = p - i + 1, \quad i = 1, \cdots, p, \\
&\varepsilon(g) = 1, \quad \forall g \in \mathfrak{b}_2,
\end{aligned}$$

endow A with a weak Hom-bialgebra structure.

Proof. The sub-Hom-bialgebra generated by $\mathfrak{b}_1$ is a weak bialgebra see [3, Theorem 2.7]. We check the conditions involving $g \in \mathfrak{b}_2$. Notice that $\Delta(\alpha(g)) = \alpha^2(g) \otimes \alpha^2(g)$ $\forall g \in \mathfrak{b}_2$. We show that Δ is compatible with the multiplication. Indeed

$\Delta(e_{p-i})\cdot\Delta(g) = [(e_{p-i}-e_{p-i+1})\otimes(e_{p-i}-e_{p-i+1})+(e_{p-i+1}-e_{p-i+2})\otimes(e_{p-i+1}-e_{p-i+2})+ \ldots+(e_{p-i+j}-e_{p-i+j+1})\otimes(e_{p-i+j}-e_{p-i+j+1})+\ldots+(e_{p-1}-e_p)\otimes(e_{p-1}-e_p)+e_p\otimes e_p].(\alpha(g)\otimes \alpha(g)) = e_p \cdot \alpha(g) \otimes e_p \cdot \alpha(g) = \alpha^2(g) \otimes \alpha^2(g) = \Delta(e_{p-i} \cdot g), \forall g \in \mathfrak{b}_2, \forall i = 0, \ldots, p-1.$

We assume $i < k$,
$\Delta(e_{p-i}) \cdot \Delta(e_{p-k}) = [(e_{p-i} - e_{p-i+1}) \otimes (e_{p-i} - e_{p-i+1}) + (e_{p-i+1} - e_{p-i+2}) \otimes (e_{p-i+1} - e_{p-i+2}) + \ldots + (e_{p-i+j} - e_{p-i+j+1}) \otimes (e_{p-i+j} - e_{p-i+j+1}) + \ldots + (e_{p-1} - e_p) \otimes (e_{p-1} - e_p) + e_p \otimes e_p] \cdot [(e_{p-k} - e_{p-k+1}) \otimes (e_{p-k} - e_{p-k+1}) + (e_{p-k+1} - e_{p-k+2}) \otimes (e_{p-k+1} - e_{p-k+2}) + \ldots + (e_{p-k+t} - e_{p-k+t+1}) \otimes (e_{p-k+t} - e_{p-k+t+1}) + \ldots + (e_{p-1} - e_p) \otimes (e_{p-1} - e_p) + e_p \otimes e_p = (e_{p-i} - e_{p-i+1}) \otimes (e_{p-i} - e_{p-i+1}) + (e_{p-i+1} - e_{p-i+2}) \otimes (e_{p-i+1} - e_{p-i+2}) + \ldots + (e_{p-i+j} - e_{p-i+j+1}) \otimes (e_{p-i+j} - e_{p-i+j+1}) + \ldots + (e_{p-1} - e_p) \otimes (e_{p-1} - e_p) + e_p \otimes e_p$
$= \Delta(e_{p-i}) = \Delta(e_{p-i} \cdot e_{p-k}), \quad \forall i, k = 0, \ldots, p-1.$
The compatibility is straightforward for elements in $\mathfrak{b}_2$ by using to the multiplicativity.

We show that ε satisfies condition (2.12). First, we consider the case $j \leq k$. Since for $i, j, k \leq p$, we have

$$\varepsilon((e_i \cdot e_j) \cdot \alpha(e_k)) = \varepsilon(\alpha(e_i) \cdot (e_j \cdot e_k)) = \varepsilon(e_i \cdot (e_j \cdot e_k)) = \varepsilon(e_i \cdot e_k) = p - \max(i,k) + 1.$$

In the other hand side,

$\varepsilon(e_i\cdot(e_j)_{(1)})\varepsilon((e_j)_{(2)}\cdot e_k) = \varepsilon(e_i\cdot(e_j-e_{j+1}))\varepsilon((e_j-e_{j+1})\cdot e_k))+\varepsilon(e_i\cdot(e_{j+1}-e_{j+2}))\varepsilon((e_{j+1}- e_{j+2})\cdot e_k)+\varepsilon(e_i\cdot(e_{j+2}-e_{j+3}))\varepsilon((e_{j+2}-e_{j+3})\cdot e_k))+\ldots+\varepsilon(e_i.\cdot(e_k-e_{k+1}))\varepsilon((e_k-e_{k+1})\cdot e_k)+ \ldots+\varepsilon(e_i.\cdot(e_{p-1}-e_p))\varepsilon((e_{p-1}-e_p)\cdot e_k)+\varepsilon(e_i\cdot e_p)\varepsilon(e_p\cdot e_k) = \varepsilon(e_i\cdot(e_k-e_{k+1}))\varepsilon(e_k-e_{k+1})+ \varepsilon(e_i \cdot (e_{k+1} - e_{k+2}))\varepsilon(e_{k+1} - e_{k+2}) + \ldots + \varepsilon(e_i \cdot (e_{p-1} - e_p))\varepsilon(e_{p-1} - e_p) + \varepsilon(e_i \cdot e_p)\varepsilon(e_p) = \varepsilon(e_i \cdot (e_k - e_{k+1})) + \varepsilon(e_i \cdot (e_{k+1} - e_{k+2})) + \ldots + \varepsilon(e_i \cdot (e_{p-1} - e_p) + \varepsilon(e_i \cdot e_p) = \varepsilon(e_i \cdot e_k) - \varepsilon(e_i \cdot e_{k+1})+\varepsilon(e_i\cdot e_{k+1})-\varepsilon(e_i\cdot e_{k+2})+\varepsilon(e_i\cdot e_{k+2})+\ldots-\varepsilon(e_i\cdot e_{p-1})+\varepsilon(e_i\cdot e_{p-1})-\varepsilon(e_i\cdot e_p)+\varepsilon(e_i\cdot e_p) = \varepsilon(e_i \cdot e_k) = p - \max(i,k) + 1.$

If $j > k$, then $\varepsilon((e_i \cdot e_j) \cdot \alpha(e_k)) = \varepsilon(\alpha(e_i) \cdot (e_j \cdot e_k)) = \varepsilon(e_i \cdot (e_j \cdot e_k)) = \varepsilon(e_i \cdot e_j) = p - \max(i,j) + 1$. In the other side
$\varepsilon(e_i\cdot(e_j)_{(1)})\varepsilon((e_j)_{(2)}\cdot e_k) = \varepsilon(e_i\cdot(e_j-e_{j+1}))\varepsilon((e_j-e_{j+1})\cdot e_k)+\varepsilon(e_i\cdot(e_{j+1}-e_{j+2}))\varepsilon((e_{j+1}- e_{j+2}) \cdot e_k) + \varepsilon(e_i \cdot (e_{j+2} - e_{j+3}))\varepsilon((e_{j+2} - e_{j+3}) \cdot e_k) + \ldots + \varepsilon(e_i \cdot (e_{p-1} - e_p)\varepsilon((e_{p-1} - e_p) \cdot e_k) + \varepsilon(e_i \cdot e_p)\varepsilon(e_p \cdot e_k) = \varepsilon(e_i \cdot (e_j - e_{j+1}))\varepsilon(e_j - e_{j+1}) + \varepsilon(e_i \cdot (e_{j+1} - e_{j+2}))\varepsilon(e_{j+1} - e_{j+2}) +$

$... + \varepsilon(e_i \cdot (e_{p-1} - e_p))\varepsilon(e_{p-1} - e_p) + \varepsilon(e_i \cdot e_p)\varepsilon(e_p) = \varepsilon(e_i \cdot (e_j - e_{j+1})) + \varepsilon(e_i \cdot (e_{j+1} - e_{j+2})) + ... + \varepsilon(e_i \cdot (e_{p-1} - e_p)) + \varepsilon(e_i \cdot e_p) = \varepsilon(e_i \cdot e_j) - \varepsilon(e_i \cdot e_{j+1}) + \varepsilon(e_i \cdot e_{j+1}) - \varepsilon(e_i \cdot e_{j+2}) + \varepsilon(e_i \cdot e_{j+2}) + ... - \varepsilon(e_i \cdot e_{p-1}) + \varepsilon(e_i \cdot e_{p-1}) - \varepsilon(e_i \cdot e_p) + \varepsilon(e_i \cdot e_p) = \varepsilon(e_i \cdot e_j) = p - \max(i, j) + 1$.
Other conditions are obtained by straightforward calculations. □

Theorem 5. *Let (A, m) be a unital algebra with unit e_2 and α be an algebra map defined on A such that the multiplication on a basis $\{e_i\}_{i=2,\cdots,n}$ of A is given by $m(e_i \otimes e_j) = \alpha(e_{max(i,j)})$, $i, j = 2, \cdots, n$, and $\alpha(e_2) = e_2$. Let $\mathcal{B}$ be the vector space resulting on adjoining a new unit element $e_1 = 1$ to A with $\alpha(e_1) = e_1$. The comultiplication and the counit defined as*

$$\begin{aligned} \Delta(e_n) &= \alpha(e_n) \otimes \alpha(e_n), \\ \Delta(e_i) &= (\alpha(e_i) - \alpha(e_{i+1})) \otimes (\alpha(e_i) - \alpha(e_{i+1})) + \Delta(e_{i+1}), \quad i = 1, \cdots, n-1, \\ \varepsilon(e_i) &= \varepsilon(\alpha(e_i)) = n - i + 1, \quad i = 1, \cdots, n, \end{aligned}$$

provide $\mathcal{B}$ with a weak Hom-bialgebra structure.

Proof. It easy to check that α is an algebra morphism for $\mathcal{B}$. We show now that m is Hom-associative. Indeed
$m(e_i \otimes e_1) = m(e_1 \otimes e_i) = \alpha(e_{max(i,1)}) = \alpha(e_i)$,
$m(\alpha(e_i) \otimes m(e_j \otimes e_k)) = m(\alpha(e_i) \otimes \alpha(e_{max(j,k)})) = \alpha \circ m(e_i \otimes e_{max(j,k)}) = \alpha^2(e_{max(i,j,k)})$.
In the other hand,
$m(m(e_i \otimes e_j) \otimes \alpha(e_k)) = m(\alpha(e_{max(i,j)}) \otimes \alpha(e_k)) = \alpha \circ m(e_{max(i,j)} \otimes e_k) = \alpha^2(e_{max(i,j,k)})$.
Hence, m is Hom-associative with respect to α.
Now, we check the Hom-coalgebra structure by induction,
$(\Delta \otimes \alpha)\Delta(e_n) = (\Delta \otimes \alpha)(\alpha(e_n) \otimes \alpha(e_n)) = \alpha^2(e_n) \otimes \alpha^2(e_n) \otimes \alpha^2(e_n)$,
We suppose that the property is true for $i + 1$ and prove it for i.

$$\begin{aligned} &(\alpha \otimes \Delta)\Delta(e_i) = (\alpha \otimes \Delta)(\alpha(e_i) - \alpha(e_{i+1})) \otimes (\alpha(e_i) - \alpha(e_{i+1})) + (\alpha \otimes \Delta)\Delta(e_{i+1}), \\ &= (\alpha^2(e_i) - \alpha^2(e_{i+1})) \otimes (\Delta(\alpha(e_i)) - \Delta(\alpha(e_{i+1}))) + (\Delta \otimes \alpha)\Delta(e_{i+1}), \\ &= (\alpha^2(e_i) - \alpha^2(e_{i+1})) \otimes (\alpha^2(e_i) - \alpha^2(e_{i+1})) \otimes (\alpha^2(e_i) - \alpha^2(e_{i+1})) + (\Delta \otimes \alpha)\Delta(e_{i+1}), \end{aligned}$$

In the other hand:

$$\begin{aligned} &(\Delta \otimes \alpha)\Delta(e_i) = (\Delta \otimes \alpha)(\alpha(e_i) - \alpha(e_{i+1})) \otimes (\alpha(e_i) - \alpha(e_{i+1})) + (\Delta \otimes \alpha)\Delta(e_{i+1}), \\ &= (\Delta(\alpha(e_i)) \otimes (\alpha^2(e_i) - \alpha^2(e_{i+1})) - \Delta(\alpha(e_{i+1}))) + (\Delta \otimes \alpha)\Delta(e_{i+1}), \\ &= (\alpha^2(e_i) - \alpha^2(e_{i+1})) \otimes (\alpha^2(e_i) - \alpha^2(e_{i+1})) \otimes (\alpha^2(e_i) - \alpha^2(e_{i+1})) + (\Delta \otimes \alpha)\Delta(e_{i+1}), \\ &= (\alpha \otimes \Delta)\Delta(e_i), \end{aligned}$$

which achieves the proof.
Now, we show that the multiplication m is compatible with the Hom-coalgebra structure.
We denote in the sequel the multiplication by " $\cdot$ " .
We assume $i < k$,
$\Delta(e_{p-i}) \cdot \Delta(e_{p-k}) = [(\alpha(e_{p-i}) - \alpha(e_{p-i+1})) \otimes (\alpha(e_{p-i}) - \alpha(e_{p-i+1})) + (\alpha(e_{p-i+1}) - \alpha(e_{p-i+2})) \otimes (\alpha(e_{p-i+1}) - \alpha(e_{p-i+2})) + ... + (\alpha(e_{p-i+j}) - \alpha(e_{p-i+j+1})) \otimes (\alpha(e_{p-i+j}) - \alpha(e_{p-i+j+1})) + ... + (\alpha(e_{p-1}) - \alpha(e_p)) \otimes (\alpha(e_{p-1}) - \alpha(e_p)) + \alpha(e_p) \otimes \alpha(e_p)] \cdot [(\alpha(e_{p-k}) - \alpha(e_{p-k+1})) \otimes (\alpha(e_{p-k}) -$

$\alpha(e_{p-k+1})) + (\alpha(e_{p-k+1}) - \alpha(e_{p-k+2})) \otimes (\alpha(e_{p-k+1}) - \alpha(e_{p-k+2})) + ... + (\alpha(e_{p-k+t}) - \alpha(e_{p-k+t+1})) \otimes (\alpha(e_{p-k+t}) - \alpha(e_{p-k+t+1})) + ... + (\alpha(e_{p-1}) - \alpha(e_p)) \otimes (\alpha(e_{p-1}) - \alpha(e_p)) + \alpha(e_p) \otimes \alpha(e_p)] = (\alpha(e_{p-i}) - \alpha(e_{p-i+1})) \otimes (\alpha(e_{p-i}) - \alpha(e_{p-i+1})) + (\alpha(e_{p-i+1}) - \alpha(e_{p-i+2})) \otimes (\alpha(e_{p-i+1}) - \alpha(e_{p-i+2})) + ... + (\alpha(e_{p-i+j}) - \alpha(e_{p-i+j+1})) \otimes (\alpha(e_{p-i+j}) - \alpha(e_{p-i+j+1})) + ... + (\alpha(e_{p-1}) - \alpha(e_p)) \otimes (\alpha(e_{p-1}) - \alpha(e_p)) + \alpha(e_p) \otimes \alpha(e_p) = \Delta(e_{p-i}) = \Delta(e_{p-i} \cdot e_{p-k}), \quad \forall i, k = 0, \dots, p-1.$

Now, we check the compatibility with the counit :
$(id \otimes \varepsilon)\Delta(e_n) = (id \otimes \varepsilon)(\alpha(e_n) \otimes \alpha(e_n)) = \varepsilon(e_n)\alpha(e_n) = \alpha(e_n)$. Again by induction, we suppose that the property is true for $i+1$ and prove it for i. Indeed

$$(id \otimes \varepsilon)\Delta(e_i) = (id \otimes \varepsilon)[(\alpha(e_i) - \alpha(e_{i+1}) \otimes \alpha(e_i - \alpha(e_{i+1}))] + (id \otimes \varepsilon)\Delta(e_{i+1}) = (\alpha(e_i) - \alpha(e_{i+1}))(\varepsilon(e_i) - \varepsilon(e_{i+1})) + \alpha(e_{i+1})) = \alpha(e_i).$$

$$(\varepsilon \otimes id)\Delta(e_i) = (\varepsilon \otimes id)[(\alpha(e_i) - \alpha(e_{i+1}) \otimes \alpha(e_i - \alpha(e_{i+1}))] + (\varepsilon \otimes id)\Delta(e_{i+1}) = (\varepsilon(e_i) - \varepsilon(e_{i+1}))(\alpha(e_i) - \alpha(e_{i+1})) + \alpha(e_{i+1})) = \alpha(e_i).$$

The proof for the weak condition of the counit is similar to the proof of Theorem 4.

∎

Proposition 2. *Under the assumptions of Theorem 5, there exists $n-1$ weak Hom-bialgebras associated to A which are pairwise non isomorphic.*
Pick j such that $j = 2, \cdots, n$. The Hom-coalgebra structures are defined by
$\Delta(e_1) = (\alpha(e_1) - \alpha(e_2)) \otimes (\alpha(e_1) - \alpha(e_2)) + (\alpha(e_2) - \alpha(e_3)) \otimes (\alpha(e_2) - \alpha(e_3)) + ... + (\alpha(e_{j-1}) - \alpha(e_j)) \otimes (\alpha(e_{j-1}) - \alpha(e_j)) + \alpha(e_j) \otimes \alpha(e_j)$,
If $i < j$,
$\Delta(e_i) = (\alpha(e_i) - \alpha(e_{i+1})) \otimes (\alpha(e_i) - \alpha(e_{i+1})) + ... + (\alpha(e_{j-1}) - \alpha(e_j)) \otimes (\alpha(e_{j-1}) - \alpha(e_j)) + \alpha(e_j) \otimes \alpha(e_j)$,
If i=j,
$\Delta(e_j) = \alpha(e_j) \otimes \alpha(e_j)$,
If $n \geq i > j$,
$\Delta(e_i) = (\alpha(e_i) - \alpha(e_{i+1})) \otimes (\alpha(e_i) - \alpha(e_{i+1}))$,

and $\varepsilon(e_i) = \varepsilon(\alpha(e_i)) = j - i + 1$ *for* $i = 1, \cdots, j-1$;
$\varepsilon(e_j) = \varepsilon(\alpha(e_j)) = 1$; $\varepsilon(e_i) = \varepsilon(\alpha(e_i)) = 0$ *for* $i = j+1, \cdots, n$.

Proof. For any j such that $j = 2, \cdots, n$, we consider the structure of weak Hom-bialgebra defined above. The verification of the axioms is similar to Theorem 5. ∎

Corollary 7. *There exists a structure of weak Hom-bialgebra over any Hom-associative algebra generated by n orthogonal nilpotent elements of order two.*

Theorem 6. *Let $(A, m, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra and set $e_2 = \eta(1)$ be its unit. We consider the set $\mathcal{B}$ as a result of adjoining a unit e_1 to A with respect to the multiplication and α is extended linearly in $\mathcal{B}$. Assume,*

$$\begin{aligned} &\alpha(e_1) = e_1, \alpha(e_2) = e_2,\\ &m(e_1 \otimes e_1) = e_1, m(e_1 \otimes e_2) = m(e_2 \otimes e_1) = m(e_2 \otimes e_2) = e_2,\\ &m(e_1 \otimes x) = m(x \otimes e_1) = \alpha(x), \forall x \in A,\\ &\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2,\\ &\varepsilon(e_1) = 2, \end{aligned} \tag{4.5}$$

Then $\mathcal{B}$ is a weak Hom-bialgebra.

Proof. Condition (1.7) is satisfied, see Theorem 4. Also we have
$(\varepsilon \otimes id)\Delta(e_1) = \varepsilon(e_1)e_1 - \varepsilon(e_1)e_2 - \varepsilon(e_2)e_1 + \varepsilon(e_2)e_2 = e_1,$
$(id \otimes \varepsilon)\Delta(e_1) = \varepsilon(e_1)e_1 - \varepsilon(e_1)e_2 - \varepsilon(e_2)e_1 + \varepsilon(e_2)e_2 = e_1,$

$\varepsilon((e_1 \cdot e_1) \cdot \alpha(e_1)) = \varepsilon((e_1 \cdot e_1) \cdot e_1) = \varepsilon(e_1 \cdot e_1)\varepsilon(e_1 \cdot e_1) - \varepsilon(e_1 \cdot e_1)\varepsilon(e_2 \cdot e_1) - \varepsilon(e_1 \cdot e_2)\varepsilon(e_1 \cdot e_1) + \varepsilon(e_1 \cdot e_2)\varepsilon(e_2 \cdot e_1) = \varepsilon(e_1)\varepsilon(e_1) - \varepsilon(e_1)\varepsilon(e_2) - \varepsilon(e_2)\varepsilon(e_1) + 2\varepsilon(e_2)\varepsilon(e_2) = 4 - 2 - 2 + 2 = 2.$

Other conditions are obtained similarly by straightforward calculation.

□

Remark 3. *The counit of $\mathcal{B}$ is not an algebra homomorphism and $dim\mathcal{B} = dimA + 1$.*

Corollary 8. *Let $(A, m, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra and set $u = \eta(1)$ be its unit. Let $\mathcal{B}$ be a result of adjoining to A two successive unit elements e and 1 with respect to the multiplication and α extended linearly in $\mathcal{B}$ such that:*

$$\begin{aligned} &\alpha(1) = 1, \alpha(e) = e, \alpha(u) = u,\\ &m(1 \otimes 1) = 1, m(e \otimes 1) = m(1 \otimes e) = m(e \otimes e) = e,\\ &m(1 \otimes u) = m(u \otimes 1) = m(u \otimes e) = m(e \otimes u) = m(u \otimes u) = u,\\ &\Delta(1) = 1 \otimes (e - u) + u \otimes (1 - 2e + 2u),\\ &\Delta(e) = e \otimes (e - u) + u \otimes (2u - e),\\ &\varepsilon(1) = 2,\ \varepsilon(e) = 2. \end{aligned}$$

Then $\mathcal{B}$ is weak Hom-bialgebra.

The counit of $\mathcal{B}$ obtained in the corollary is not an algebra homomorphism.

Corollary 9. *Let $(A, m, \eta, \Delta, \varepsilon, \alpha)$ be a Hom-bialgebra and set $u = \eta(1)$ be its unit. Let $\mathcal{B}$ be a result of adjoining to A two successive unit elements e and 1 with respect to its multiplication and α extended linearly in $\mathcal{B}$ such that:*

$$\begin{aligned} &\alpha(1) = 1, \alpha(e) = e, \alpha(u) = u,\\ &m(1 \otimes 1) = 1, m(e \otimes 1) = m(1 \otimes e) = m(e \otimes e) = e,\\ &m(1 \otimes u) = m(u \otimes 1) = m(u \otimes e) = m(e \otimes u) = m(u \otimes u) = u,\\ &\Delta(1) = (1 - e) \otimes (1 - e) + (e - u) \otimes (e - u) + u \otimes u,\\ &\Delta(e) = (e - u) \otimes (e - u) + u \otimes u,\\ &\varepsilon(1) = 3,\ \varepsilon(e) = 2. \end{aligned}$$

Then $\mathcal{B}$ is a weak Hom-bialgebra.

The counit of the weak bialgebra $\mathcal{B}$ is not an algebra homomorphism.

Theorem 7. *Let $\mathcal{B}$ be a bialgebra and $\mathcal{B}' = span\{\mathcal{B}, e, 1\}$ be the weak bialgebra obtained by Kaplansky's Construction Theorem. Applying a Twisting construction to $\mathcal{B}$ and then a Kaplansky's Hom-type Construction is equivalent to apply the Twisting construction to the weak bialgebra $\mathcal{B}'$.*

Proof. If $\mathcal{B}$ is a bialgebra then the weak bialgebra $\mathcal{B}' = span\{\mathcal{B}, e, 1\}$ is obtained by setting $\Delta(1) = (1-e)\otimes(1-e) + e\otimes e, \Delta(e) = e\otimes e, \varepsilon(1) = 2, \varepsilon(e) = 1$.
Now, we construct a Hom-bialgebra structure using a bialgebra morphism β. We set

$$\beta(1) = 1, \beta(e) = e, m_\beta(x, y) = \beta(x)\cdot\beta(y), \Delta_\beta(x) = \Delta\circ\beta(x), \forall x \in \mathcal{B},$$

which leads to the Hom-bialgebra $(\mathcal{B}', m_\beta, \eta, \varepsilon, \Delta_\beta)$.
In another hand, we take $\mathcal{B}$ and apply the twisting construction with β. We obtain the Hom-bialgebra $(\mathcal{B}, m_\beta, \eta, \varepsilon, \Delta_\beta, \beta)$, with $m_\beta(x\cdot y) = \beta(x)\cdot\beta(y)$ and $\Delta_\beta = \Delta\circ\beta$, $\varepsilon\circ\beta = \varepsilon$.
For the latest structure we carry out the Kaplansky's construction, see Theorem 3. Hence

$$\begin{gathered}
\mathcal{B}' = span\{\mathcal{B}, e, 1\}, \beta(1) = 1, \beta(e) = e,\\
\Delta_\beta(1) = \Delta\circ\beta(1) = \Delta(1) = (1-e)\otimes(1-e) + e\otimes e,\\
\Delta_\beta(e) = \Delta\circ\beta(e) = \Delta(e) = e\otimes e,\\
\varepsilon\circ\beta = \varepsilon.
\end{gathered}$$

It turns out that it is the same as the weak Hom-bialgebra given above. □

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