Bull. Math. Soc. Sci. Math. Roumanie
Tome 61 (109) No. 1, 2018, 3–11

On some modules associated with Galois orbits

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Dedicated to the memory of Professor Nicolae Popescu

Abstract

Given a prime number p we consider $\mathbb{C}_p$, which is usually called the Tate field, the topological completion of the algebraic closure of the field of p-adic numbers. We introduce and study a class of modules associated with factor groups of profinite groups, especially of those which are the Galois groups of the normal closure of algebraic infinite extensions. In particular, we show that the module associated with a Galois orbit of an arbitrary element of $\mathbb{C}_p$ is a factor of the Iwasawa algebra of a normal element of $\mathbb{C}_p$ by an ideal which can be described.

Key words and phrases: Galois orbits, Iwasawa algebra, local fields, distributions.
2010 Mathematics Subject Classification: 11S99.

Introduction

For a prime number p, we consider $\mathbb{Q}_p$ the field of p-adic numbers and $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic valuation and $O_{\mathbb{C}_p}$ the valuation ring of $\mathbb{C}_p$. Let K be an infinite algebraic extension of $\mathbb{Q}_p$ and L the normal closure of K. Let A be a closed subring of $\mathbb{C}_p$. Denote $\mathcal{H} = Gal(L/K)$ and $G_L = Gal(L/\mathbb{Q}_p)$. We introduce and study a class of modules associated with factor groups $G_L/\mathcal{H}$ and respectively $G_L/\mathcal{H}^\sigma$, where $\mathcal{H}^\sigma = \sigma\mathcal{H}\sigma^{-1}$ with $\sigma \in G_L$. We associate with G_L a $\Lambda_A(G_L)$-module and relate it with $\Lambda_A(G_L/\mathcal{H})$ and respectively with a direct product of $\Lambda_A(G_L/\mathcal{H}^\sigma)$. This offers new ways to relate distributions in the sense of Mazur and Swinnerton-Dyer (see [6]), functionals, and linear maps defined on an infinite algebraic extension of $\mathbb{Q}_p$, with those defined on its normal closure. When x is an arbitrary element of $\mathbb{C}_p$, we consider $O(x)$, the Galois orbit with respect to the Galois group of all continuous automorphisms of $\mathbb{C}_p/\mathbb{Q}_p$. Here the module associated with $O(x)$ with scalars drawn from $\mathbb{Q}_p$, i.e. $\Lambda_{\mathbb{Q}_p}(O(x))$, which is isomorphic as $\mathbb{Q}_p$-module with the space of distributions in the sense of Mazur and Swinnerton-Dyer defined on $O(x)$ with values in $\mathbb{Q}_p$, is a factor of the Iwasawa algebra of a Galois orbit of a normal element of $\mathbb{C}_p$ with scalars drawn from $\mathbb{Q}_p$ by an ideal which can be described. A particular situation, when x is a normal element of $\mathbb{C}_p$, is studied in [1].

The present paper consists of four sections. The first one contains notation and some basic results. In the second section we introduce the module associated with a factor group of a profinite group with scalars drawn from a closed subring of $\mathbb{C}_p$, see Propositions 1 and 2. In section three we consider the special case of Galois orbits of elements of $\mathbb{C}_p$, which

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are not necessarily normal. We prove that the module associated with a Galois orbit of an arbitrary element of $\mathbb{C}_p$ with scalars drawn from $\mathbb{Q}_p$ is a factor of the Iwasawa algebra of a Galois orbit of a normal element of $\mathbb{C}_p$ with scalars drawn from $\mathbb{Q}_p$ by an ideal, which can be described, see Theorem 1. In the last section we embed the module associated with the Galois group of the normal closure of an algebraic infinite extension, with scalars drawn from a closed subring of $\mathbb{C}_p$, into a direct product of modules associated with some factor groups, see Theorem 2. These modules associated with factor groups of a profinite group, with scalars drawn from a closed subring of $\mathbb{C}_p$, enjoy some nice properties and deserve further study.

1 Notation and basic results

Let $\{X_n, \varphi_n\}_{n\geq 1}$ be a projective system, such that X_n is a finite set and $\varphi_n : X_{n+1} \to X_n$ is a surjective map, for all $n \geq 1$. Denote by $X = \varprojlim X_n$ the projective limit defined by this projective system. Consider a closed subring A of $\mathbb{C}_p$. The map φ_n induces a map of the corresponding A-modules $A[X_{n+1}] \to A[X_n]$. The projective limit $\varprojlim A[X_n]$ is a complete A-module which we denote by $A[[X]]$ or $\Lambda_A(X)$.

Definition 1. ([6]) *By a distribution on $\{X_n, \varphi_n\}_{n\geq 1}$ (or on X) with values in A we mean a set $\mu = \{\mu_n\}_{n\geq 1}$ of mappings: $\mu_n : X_n \to A$ such that the following compatibility relations*

$$\mu_n(x) = \sum_{y\in\varphi_n^{-1}(x)} \mu_{n+1}(y) \tag{1.1}$$

hold for all $n \geq 1$ and all $x \in X_n$.

Let $\mathcal{D}(X, A)$ be the set of all distributions $\mu = \{\mu_n\}_{n\geq 1}$ defined above. We have a canonical isomorphism of A-modules between $\mathcal{D}(X, A)$ and $\Lambda_A(X)$.

For any $\mu \in \mathcal{D}(X, A)$ denote

$$\|\mu\| = \sup_{n\geq 1}\{|\mu_n(x)| : x \in X_n\}, \tag{1.2}$$

the norm of μ. When $\|\mu\| < \infty$ we say that μ is a measure. Let $\mathcal{M}(X, A)$ be the set of all measures on X with values in A.

When X is a profinite group the space $\mathcal{D}(X, A)$ becomes an A-algebra, which is endowed with the convolution product of distributions.

The measures $\mathcal{M}(X, A) \subset \mathcal{D}(X, A)$ correspond, via the above isomorphism, to the sequences of $\Lambda_A(X)$ that are uniformly bounded.

Now, let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic valuation. Denote by G the p-adic absolute Galois group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. It is known that G is canonically isomorphic to $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, the group of all continuous automorphisms of $\mathbb{C}_p$. In what follows we shall identify these two groups.

For any subset X of $\mathbb{C}_p$ we denote by $\tilde{X}$ the topological closure of X in $\mathbb{C}_p$. It is clear that if X is a field then $\tilde{X}$ is a closed subfield of $\mathbb{C}_p$.

Let H be a closed subgroup of G. Denote $Fix(H) = \{x \in \mathbb{C}_p : \sigma(x) = x \text{ for all } \sigma \in H\}$. For an arbitrary element $x \in \mathbb{C}_p$ denote $H(x) = \{\sigma \in G : \sigma(x) = x\}$, which is a subgroup of G, not necessarily normal. One has $Fix(H(x)) = \widetilde{\mathbb{Q}_p[x]}$, the closure of the ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$. We say that x is a *topological generic element* of $\widetilde{\mathbb{Q}_p[x]}$. By [2], any closed subfield K of $\mathbb{C}_p$ has a topological generic element, i.e. there exists $x \in K$ such that $K = \widetilde{\mathbb{Q}_p[x]}$. It is proved in [3] that for any element y of $\mathbb{C}_p$, the ring $\mathbb{Q}_p[y]$ and the field $\mathbb{Q}_p(y)$ have the same topological closure in $\mathbb{C}_p$, that is, $\widetilde{\mathbb{Q}_p[y]} = \widetilde{\mathbb{Q}_p(y)}$. By this result, the topological closure in $\mathbb{C}_p$ of a ring of the form $\mathbb{Q}_p[y]$ is always a field. Now for an arbitrary element x of $\mathbb{C}_p$ let $O(x) = \{\sigma(x) : \sigma \in G\}$ be the orbit of x. It is clear that the map $\sigma \rightsquigarrow \sigma(x)$ from G to $O(x)$ is continuous, and it defines a homeomorphism from $G/H(x)$ to $O(x)$, see [2]. Then $O(x)$ is a compact and totally disconnected subspace of $\mathbb{C}_p$. Moreover, the group G acts continuously on $O(x)$: if $\sigma \in G$, $\tau(x) \in O(x)$ then $\sigma \star \tau(x) = (\sigma\tau)(x)$.

Definition 2. ([1]) *An element $x \in \mathbb{C}_p$ is called* normal *if the extension $\widetilde{\mathbb{Q}_p(x)}/\mathbb{Q}_p$ is normal, i.e. $Gal_{cont}(\mathbb{C}_p/\widetilde{\mathbb{Q}_p(x)}) = H(x)$ is a normal subgroup of $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$.*

The notions of trace and trace series associated with an element x of $\mathbb{C}_p$ were introduced and investigated in [4,7,9]. Given an $x \in \mathbb{C}_p$, the trace $Tr(x)$ is defined by the equality

$$Tr(x) = \int_{O(x)} t d\pi_x(t), \tag{1.3}$$

provided that the integral with respect to the Haar distribution π_x on the right side of (1.3) is well defined. This is the case, for example, when π_x is bounded, i.e. when π_x is a measure. The integral is also well defined when x is a Lipschitz element, see [4, Theorem 4.4]. Precisely, an element $x \in \mathbb{C}_p$ is Lipschitz if one has

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{|N(x,\varepsilon)|} = 0,$$

where $N(x,\varepsilon)$ is the number of open balls of radius ε which cover $O(x)$ and $|\cdot|$ stands for the p-adic absolute value. An element $x \in \mathbb{C}_p$ is called p-bounded if there exists an $s \in \mathbb{N}$ such that p^s does not divide the number $N(x,\varepsilon)$, for any $\varepsilon > 0$. It is clear that a p-bounded element of $\mathbb{C}_p$ is also Lipschitz. The trace function $F(x,z)$ is defined by

$$F(x,z) = \int_{O(x)} \frac{1}{1-zt} d\pi_x(t),$$

for all those $z \in \mathbb{C}_p$ for which the integral is well defined. This is an analytic object that embodies a significant amount of algebraic data. The Taylor series expansion

$$F(x,z) = \sum_{n=0}^{\infty} Tr(x^n)\, z^n$$

is usually called the trace series associated with x.

2 Modules associated with factor groups

Let G be a profinite group, $G = \varprojlim G_n$ with G_n finite groups such that $G_n = G/H_n$, where $\{H_n\}_{n\geq 1}$ is a family of normal subgroup of G of finite index such that $H_{n+1} \subseteq H_n$, for any

$n \geq 1$ and $\cap_{n\geq 1} H_n = \{e\}$, the neutral element of G. Let H be a closed subgroup of G, which is not necessarily normal. Let $\overline{H}_n := H_n H$ be the subgroup of G that is generated by the product of H_n with H. We have that $\overline{H}_n$ is an open and closed subgroup of G of finite index and $\cap_{n\geq 1}\overline{H}_n = H$. For the sake of simplicity we denote by $G/H = \{\sigma \pmod{H} : \sigma \in G\}$ the left cosets of H in G and $\overline{G}_n = G/\overline{H}_n$ the left cosets of $\overline{H}_n$ in G. One has the following result, which is a generalization of a classical result for normal subgroups [8].

Proposition 1. *Let $G = \varprojlim G/H_n$ be a profinite group where $\{H_n\}_{n\geq 1}$ is a family of normal subgroups of G of finite index such that $H_{n+1} \subseteq H_n$ for any $n \geq 1$, and $\cap_{n\geq 1} H_n = \{e\}$, the neutral element of G. Let H be a closed subgroup of G, which is not necessarily normal, and $\overline{H}_n := H_n H$ the subgroup of G that is generated by the product of H_n with H. There is a homeomorphism between G/H and $\varprojlim G/\overline{H}_n$.*

Proof. Because $H \subset \overline{H}_{n+1} \subset \overline{H}_n \subset G$ we have the following commutative diagram

$$\begin{array}{ccc} \varprojlim \overline{G}_n & \xrightarrow{\pi_n} & \overline{G}_n \\ {\scriptstyle f}\big\uparrow & & {\scriptstyle \varphi_n}\big\uparrow \\ G/H & \xrightarrow{f_{n+1}} & \overline{G}_{n+1} \end{array}$$

where $\pi_n, \varphi_n, f_{n+1}$ are canonical maps and f is the unique map obtained via the universal property of projective limit such that $\pi_n \circ f = \varphi_n \circ f_{n+1} = f_n$, for any $n \geq 1$. The map f is a bijection. Indeed, let us suppose that $f(\sigma \pmod{H}) = f(\tau \pmod{H})$. By this one has $f_n(\sigma \pmod{H}) = f_n(\tau \pmod{H})$, which means that $\sigma \pmod{\overline{H}_n} = \tau \pmod{\overline{H}_n}$, for any $n \geq 1$. Because $\cap_{n\geq 1}\overline{H}_n = H$, clear $\sigma \pmod{H} = \tau \pmod{H}$, which means that f is an injective map. Now, let $x = (x_1, x_2, \ldots, x_n, \ldots) \in \varprojlim \overline{G}_n$. There is $\sigma_n \in G$ such that $f_n(\sigma_n \pmod{H}) = \sigma_n \pmod{\overline{H}_n} = x_n$, for any $n \geq 1$. By using the commutativity of the above diagram we have that

$$\sigma_{n+k} \pmod{\overline{H}_n} = \sigma_n \pmod{\overline{H}_n}, \tag{2.1}$$

for any $n, k \geq 1$. From hypothesis G is profinite, so compact, and by this there exists a subsequence $\{\sigma_{n_m}\}_{m\geq 1}$ of the sequence $\{\sigma_n\}_{n\geq 1}$ that converges to an element $\sigma \in G$. Because $\overline{H}_n$ is a closed subgroup of G with respect to Krull's topology, by using (2.1), one has

$$\sigma \pmod{\overline{H}_n} = \sigma_n \pmod{\overline{H}_n}, \tag{2.2}$$

for any $n \geq 1$, and by (2.2) one obtains $f(\sigma \pmod{H}) = x$, which means that f is a surjective map. The continuity of f follows from [8, Proposition 1.1.1] and the proof is done. ∎

We have a natural action of G_n on $A[\overline{G}_n]$. Indeed, let $\hat{\tau} \in G_n$, $\tau \in G$, and $\overline{\sigma} \in \overline{G}_n$, which is the left coset of $\sigma \in G$ in $\overline{G}_n$. By defining $\hat{\tau} \cdot \overline{\sigma} = \overline{\tau\sigma}$ one has that $A[\overline{G}_n]$ is an $A[G_n]$-finite generated module, which is free over A. The canonical morphisms $A[\overline{G}_{n+1}] \to A[\overline{G}_n]$ verify the compatibility relations of a projective system. By definition, one considers that

$$\Lambda_A(G/H) = A[[G/H]] = \varprojlim A[\overline{G}_n]$$

is the module of G/H with scalars drawn from A. We can see that $\Lambda_A(G/H) = \varprojlim_U A[G/UH]$, where the inverse limit is taken over all open subgroups U of G, so this module does not depend on the choice of $\{H_n\}_{n\geq 1}$ and it becomes an $\Lambda_A(G)$-module.

Now, let A be a closed subring of $\mathbb{C}_p$ which is compact or is not contained in $O_{\mathbb{C}_p}$. In the second situation A is a field which contains $\mathbb{Q}_p$, see [3, Theorem 7] and [3, Lemma 6]. We have the following commutative diagramm

$$\begin{array}{ccccc} A[G_n] & \longrightarrow & A[\overline{G}_n] & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ A[G_{n+1}] & \longrightarrow & A[\overline{G}_{n+1}] & \longrightarrow & 0 \end{array}$$

and, by [8, Lemma 1.1.5] and [5, Proposition 9.1], one has the following exact sequence

$$\Lambda_A(G) \to \Lambda_A(G/H) \to 0.$$

The first arrow in the above sequence is continuous, see also [8]. We collect the above results in the following proposition.

Proposition 2. *Let A be a closed subring of $\mathbb{C}_p$, G a profinite group and H a closed subgroup of G, which is not necessarily normal. If A is compact or is not contained in $O_{\mathbb{C}_p}$ then $\Lambda_A(G/H)$ is a factor of $\Lambda_A(G)$.*

When H is a normal divisor of G then $\Lambda_A(G/H)$ becomes an algebra. In particular, when G/H is a compact p-adic Lie group and $A \simeq \mathbb{Z}_p$, the module $\Lambda_A(G/H)$ turns out to be the Iwasawa algebra of G/H.

In the next section we will focus on the following important example: G is the p-adic absolute Galois group and $H = H(x) = \{\sigma \in G : \sigma(x) = x\}$ is the stabilizer of an arbitrary element x of $\mathbb{C}_p$, which is a subgroup of G, not necessarily normal.

3 The case of Galois orbits

It is known that the p-adic absolute Galois group G is profinite so we have a tower of fields $\mathbb{Q}_p = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \mathbb{L}_2 \subset \cdots \subset \mathbb{L}_n \subset \cdots \subset \overline{\mathbb{Q}}_p$ such that $\mathbb{L}_n/\mathbb{Q}_p$ is finite and normal with Galois group G_n. If we denote $H_n = Gal(\overline{\mathbb{Q}}_p/\mathbb{L}_n)$ then $G/H_n \simeq G_n$.

Let x be an arbitrary element of $\mathbb{C}_p$, which is not necessarily normal. By denoting $H = Gal(\overline{\mathbb{Q}}_p/\widetilde{\mathbb{Q}_p[x]} \cap \overline{\mathbb{Q}}_p)$, which is not necessarily a normal subgroup of G, one has that $H \simeq H(x) = \{\sigma \in Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p) : \sigma(x) = x\}$, via [1, Proposition 1]. Let $\overline{H}_n = H_n H$ be the subgroup of G that is generated by the product of H_n with H and $\overline{G}_n = G/\overline{H}_n$ the left cosets of $\overline{H}_n$ in G. Let $K_n = Fix(\overline{H}_n)$ be the field fixed by $\overline{H}_n$ and K'_n the normal closure of K_n. Denote $G'_n = Gal(K'_n/\mathbb{Q}_p)$, $H'_n = Gal(\overline{\mathbb{Q}}_p/K'_n)$ and $K_x = \cup_{n\geq 1} K'_n$. In fact K_x is the Galois closure of the subfield of $\overline{\mathbb{Q}}_p$ fixed by H. Denote $G_{K_x} = Gal(K_x/\mathbb{Q}_p)$. For any $n \geq 1$, we have a morphism $\Phi_n : \mathbb{Q}_p[G'_n] \to \mathbb{Q}_p[\overline{G}_n]$, which is defined canonically, such that the following diagram of $\mathbb{Q}_p$-vector spaces

$$\begin{array}{ccccc}
\mathbb{Q}_p[G'_n] & \xrightarrow{\Phi_n} & \mathbb{Q}_p[\overline{G}_n] & \longrightarrow & 0 \\
{\scriptstyle \pi'_n}\big\uparrow & & {\scriptstyle \pi_n}\big\uparrow & & \\
\mathbb{Q}_p[G'_{n+1}] & \xrightarrow{\Phi_{n+1}} & \mathbb{Q}_p[\overline{G}_{n+1}] & \longrightarrow & 0
\end{array}$$

is commutative, where π_n and π'_n are canonical projections. By [5, Proposition 9.1] one obtains the following exact sequence

$$\Lambda_{\mathbb{Q}_p}(G_{K_x}) \to \Lambda_{\mathbb{Q}_p}(O(x)) \to 0, \tag{3.1}$$

where the first arrow is continuous, see [8]. From the proof of [1, Theorem 2] one has $Hom(K'_n, \mathbb{Q}_p) \simeq \mathbb{Q}_p[G'_n]$, which is an isomorphism of $\mathbb{Q}_p$-vector spaces, so $\Lambda_{\mathbb{Q}_p}(G_{K_x}) \simeq Hom(K_x, \mathbb{Q}_p)$. By (3.1) we have the following exact sequence

$$Hom(K_x, \mathbb{Q}_p) \xrightarrow{\Phi} \Lambda_{\mathbb{Q}_p}(O(x)) \longrightarrow 0, \tag{3.2}$$

where $\Phi = \varprojlim \Phi_n$, via the above isomorphism. Denote

$$I = ker\ \Phi \simeq \varprojlim ker\ \Phi_n, \tag{3.3}$$

where

$$ker\ \Phi_n = \left\{ \sum_{\hat{\sigma} \in G'_n} a_{\hat{\sigma}} \hat{\sigma} : \sum_{\hat{\sigma} \in G'_n,\ \overline{\sigma} = \overline{\tau}} a_{\hat{\sigma}} = 0 \text{ for any } \overline{\tau} \in \overline{G}_n \right\}.$$

From [1, Theorem 2] it follows that there is a normal element y of $\widetilde{K_x}$ such that $\Lambda_{\mathbb{Q}_p}(O_{K_x}(y)) \simeq Hom(K_x, \mathbb{Q}_p)$. Summing up, one has the following result.

Theorem 1. *For any element x of $\mathbb{C}_p$ there is a minimal Galois extension K_x of $\mathbb{Q}_p$, which is the Galois closure of the subfield of $\overline{\mathbb{Q}}_p$ fixed by the stabilizer of x, such that the following sequence*

$$Hom(K_x, \mathbb{Q}_p) \xrightarrow{\Phi} \Lambda_{\mathbb{Q}_p}(O(x)) \longrightarrow 0$$

is exact, where Φ is defined in (3.2)*. Moreover, there is a normal element y of $\widetilde{K}_x$ such that*

$$\Lambda_{\mathbb{Q}_p}(O(x)) \simeq \frac{\Lambda_{\mathbb{Q}_p}(O_{K_x}(y))}{I}$$

where $I = ker\ \Phi$ is the ideal described in (3.3)*.*

Remark 1. In other words, the above result says that the module associated with a Galois orbit $O(x)$ of an arbitrary element x of $\mathbb{C}_p$ with scalars drawn from $\mathbb{Q}_p$, which is isomorphic as $\mathbb{Q}_p$-module with the space of distributions in the sense of Mazur and Swinnerton-Dyer defined on $O(x)$ with values in $\mathbb{Q}_p$, is a factor of the Iwasawa algebra of a Galois orbit of a normal element of $\mathbb{C}_p$ with scalars drawn from $\mathbb{Q}_p$ by an ideal which can be described. Also, a finite field extension of $\mathbb{Q}_p$ can be used instead of the base field in Theorem 1.

4 A complementary result

We preserve the same notation and definitions as in the previous sections. Now, let $\mathbb{Q}_p \subset K$ be an infinite algebraic extension and $\mathbb{Q}_p \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset K$ a tower of fields such that $\mathbb{Q}_p \subset K_n$ is a finite extension for any $n \geq 1$ and, moreover, $K = \cup_{n\geq 1} K_n$. We denote by L the normal closure of K and by L_n the normal closure of K_n. One has $\mathbb{Q}_p \subset L_1 \subset L_2 \subset \cdots \subset L_n \subset \cdots \subset L$ and $L = \cup_{n\geq 1} L_n$.

We use the following notation: $G_L = Gal(L/\mathbb{Q}_p) \simeq Gal_{cont}(\tilde{L}/\mathbb{Q}_p)$, $\mathcal{H} = Gal(L/K)$, $\mathcal{H}_n = Gal(L/K_n)$, $U_n = Gal(L/L_n)$. We denote by $N_{G_L}(\mathcal{H})$ the normalizer of $\mathcal{H}$ in G_L and by $S_{\mathcal{H}}$ a system of representatives for the left cosets of $N_{G_L}(\mathcal{H})$ in G_L. For any $\sigma \in S_{\mathcal{H}}$ we define $\mathcal{H}^\sigma = \sigma\mathcal{H}\sigma^{-1}$. Also, we denote by $N_{G_L}(\mathcal{H}_n)$ the normalizer of $\mathcal{H}_n$ in G_L and by $S_{\mathcal{H}_n}$ a system of representatives for the left cosets of $N_{G_L}(\mathcal{H}_n)$ in G_L. One has $U_n = \cap_{\sigma \in S_{\mathcal{H}_n}} \mathcal{H}_n^\sigma$.

The map

$$F : G_L \to \prod_{\sigma \in S_{\mathcal{H}}} (G_L/\mathcal{H}^\sigma), \; F(g) = (g \pmod{\mathcal{H}^\sigma})_{\sigma \in S_{\mathcal{H}}}, \; g \in G_L, \tag{4.1}$$

is injective. Indeed, let $g_1, g_2 \in G_L$ be such that $F(g_1) = F(g_2)$. Then $(g_1 \pmod{\mathcal{H}^\sigma})_{\sigma \in S_{\mathcal{H}}} = (g_2 \pmod{\mathcal{H}^\sigma})_{\sigma \in S_{\mathcal{H}}}$, so $g_2^{-1} g_1 \in \cap_{\sigma \in S_{\mathcal{H}}} \mathcal{H}^\sigma = \{e\}$, the neutral element of G_L. This is happening because L is the normal closure of K, which is the composition of the fields $\sigma(K)$, $\sigma \in S_{\mathcal{H}}$.

Also, the following diagram

$$\begin{array}{ccc} G_L & \xrightarrow{F} & \prod_{\sigma \in S_{\mathcal{H}}} (G_L/\mathcal{H}^\sigma) \\ {\scriptstyle \pi_\sigma} \downarrow & & \downarrow {\scriptstyle \overline{\pi}_\sigma} \\ G_L/\mathcal{H}^\sigma & \xrightarrow{Id} & G_L/\mathcal{H}^\sigma \end{array} \tag{4.2}$$

is commutative, where π_σ is the canonical map, $\overline{\pi}_\sigma$ is the projection onto the σ component of the direct product, and Id stands for the identity map.

In the same way as in (4.1) and (4.2) we have the following map

$$F_n : G_L/U_n \to \prod_{\sigma \in S_{\mathcal{H}}} G_L/(U_n\mathcal{H})^\sigma, \; F_n(g \pmod{U_n}) = (g \pmod{(U_n\mathcal{H})^\sigma})_{\sigma \in S_{\mathcal{H}}}, \; g \in G_L, \tag{4.3}$$

which is injective because $\cap_{\sigma \in S_{\mathcal{H}}} (U_n\mathcal{H})^\sigma = U_n$. Moreover, the following diagram

$$\begin{array}{ccc} G_L/U_n & \xrightarrow{F_n} & \prod_{\sigma \in S_{\mathcal{H}}} G_L/(U_n\mathcal{H})^\sigma \\ {\scriptstyle \pi_{\sigma,n}} \downarrow & & \downarrow {\scriptstyle \overline{\pi}_{\sigma,n}} \\ G_L/(U_n\mathcal{H})^\sigma & \xrightarrow{Id} & G_L/(U_n\mathcal{H})^\sigma, \end{array} \tag{4.4}$$

is commutative, where $\pi_{\sigma,n}$ is the canonical map, $\overline{\pi}_{\sigma,n}$ is the projection onto the σ component of the direct product, and Id stands for the identity map.

By considering a closed subring A of $\mathbb{C}_p$, we obtain from (4.3), in a canonical way, an injective map

$$\overline{F}_n : A[G_L/U_n] \to \prod_{\sigma \in S_{\mathcal{H}}} A[G_L/(U_n\mathcal{H})^\sigma]. \tag{4.5}$$

By passing to inverse limit, one has the following result.

Theorem 2. *Let K be an infinite algebraic extension of $\mathbb{Q}_p$ and A a closed subring of $\mathbb{C}_p$. Let L be the normal closure of K. Denote $\mathcal{H} = Gal(L/K)$ and $G_L = Gal(L/\mathbb{Q}_p)$. Also, denote by $N_{G_L}(\mathcal{H})$ the normalizer of $\mathcal{H}$ in G_L and let $S_{\mathcal{H}}$ be a system of representatives for the left cosets of $N_{G_L}(\mathcal{H})$ in G_L. For any $\sigma \in S_{\mathcal{H}}$ we define $\mathcal{H}^\sigma = \sigma\mathcal{H}\sigma^{-1}$. Then, there exists a monomorphism $\overline{F} : \Lambda_A(G_L) \to \prod_{\sigma \in S_{\mathcal{H}}} \Lambda_A(G_L/\mathcal{H}^\sigma)$ of $\Lambda_A(G_L)$-modules, such that the following diagram*

$$\begin{array}{ccc} \Lambda_A(G_L) & \xrightarrow{\overline{F}} & \prod_{\sigma \in S_{\mathcal{H}}} \Lambda_A(G_L/\mathcal{H}^\sigma) \\ {\scriptstyle\lambda_\sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle\overline{\lambda}_\sigma} \\ \Lambda_A(G_L/\mathcal{H}^\sigma) & \xrightarrow{Id} & \Lambda_A(G_L/\mathcal{H}^\sigma), \end{array} \tag{4.6}$$

is commutative, where λ_σ is the canonical map, $\overline{\lambda}_\sigma$ is the projection onto the σ component of the direct product, and Id stands for the identity map.

Remark 2. Theorem 2, Theorem 1, and Proposition 2 provide a way to relate distributions in the sense of Mazur and Swinnerton-Dyer, functionals, and linear maps defined on an infinite algebraic extension K of $\mathbb{Q}_p$ with those defined on the normal closure of K.

Questions. We end this paper with a few questions which arise naturally, and which we pose to interested readers.

Let $\mathbb{Q}_p \subseteq K$ be an infinite algebraic extension and let L be the normal closure of K. Denote $G_L = Gal(L/\mathbb{Q}_p)$.

1) Let φ and ψ be two linear maps on L whose restrictions to $\sigma(K)$ coincide, for each $\sigma \in G_L$. Does it follow that $\varphi = \psi$?

2) If the restriction of φ to any $\sigma(K)$ is continuous, does it follow that φ is continuous ?

3) An interesting problem would be to study cases when $\varphi(\sigma x) = \varphi(x)$, for any $\sigma \in G$, where x is a generic element of $\tilde{K}$. A special example is provided by the trace function, which satisfies this property.

4) Can any $\mathbb{Q}_p$-functional on K be extended to a $\mathbb{Q}_p$-functional on L ?

5) If $Tr_{K/\mathbb{Q}_p}$ is continuous, does it follow that $Tr_{L/\mathbb{Q}_p}$ is continuous ?

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Received: 11.10.2016
Accepted: 12.02.2017

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