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Correctors for a class of transmission problems in chemical reactive flows

by

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Abstract

In this paper, we are concerned with some chemical reactive flows of a fluid through periodically perforated granular materials. The fluid penetrates the grains where the chemical reactions take place. Using the periodic unfolding method, we derive the corrector results which complete the previous homogenization results.

Key Words: homogenization, correctors, chemical reactive flow, periodic unfolding method

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Correctors for transmission problems Short title for running head (top of right hand page)

1 Introduction

In this paper, we are concerned with some chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid grains. The situation we will treat is that the reactive fluid penetrates the grains. Assume that there is an internal reaction inside the grains, instead just on their boundaries. Therefore, this is a transmission problem with an unknown flux on the boundary of each grain. For a presentation of the chemical aspects associated to this model, we refer the reader to Hornung [18], Norman [22] and the references therein.

Denote by ε a small parameter related to the characteristic size of the reactive grains. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with an ε-periodic structure, consisting of two parts: a fluid phase $\Omega_{1\varepsilon}$ and a set of ε-periodic reactive grains $\Omega_{2\varepsilon}$. We are interested in studying the stationary flow of a fluid confined in $\Omega_{1\varepsilon}$, of concentration $u_{1\varepsilon}$. Let $u_{2\varepsilon}$ be the concentration inside the grains. This nonlinear problem is stated as follows:

$$\begin{cases} -D_f \,\triangle u_{1\varepsilon} = f & \text{in } \Omega_{1\varepsilon}, \\ -D_p \,\triangle u_{2\varepsilon} + a g(u_{2\varepsilon}) = 0 & \text{in } \Omega_{2\varepsilon}, \\ -D_f \dfrac{\partial u_{1\varepsilon}}{\partial \nu} = D_p \dfrac{\partial u_{2\varepsilon}}{\partial \nu} & \text{on } \Gamma^\varepsilon, \\ u_{1\varepsilon} = u_{2\varepsilon} & \text{on } \Gamma^\varepsilon, \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a > 0$, $f \in L^2(\Omega)$, $\Gamma^\varepsilon = \partial\Omega_{2\varepsilon}$ and ν is the exterior unit normal to $\partial\Omega_{1\varepsilon}$. Assume that the reactive fluid is homogeneous and isotropic. Here D_f and D_p are two positive constant diffusion coefficients characterizing the fluid and the granular material filling the reactive grains, respectively. For the function g, we suppose that

(H_1) g is continuous, monotone, non-decreasing and $g(0) = 0$.

This general case is well illustrated by the two important practical examples: Langmuir kinetics and Freundlich kinetics (see for instance [6]-[8]). Throughout this paper, we let $n \geq 3$. We further suppose that

(H_2) There exist a positive constant C and an exponent ρ, with $1 \leq \rho < \frac{n}{n-2}$, such that

$$|g(x)| \leq C(1 + |x|^{\rho}).$$

In [6], Conca, Díaz, Liñán and Timofte carried out a study of the homogenization of problem (1.1). The proof is based on the oscillating test functions method due to Tartar (see [23]). A related work was given by Hornung, Jäger and Mikelić [19] where the two-scale convergence is used. Subsequently, using the oscillating test functions method, Conca, Díaz and Timofte [8] studied the homogenization of a similar problem with a nonlinear term associated to $\Omega_{1\varepsilon}$. Problem (1.1) is also related to a linear transmission problem (see for instance [20] and [11]) in a domain with the same structure as Ω. The conditions prescribed on the interface between the two components are the continuity of the conormal derivatives and a jump of the solution proportional to the conormal derivatives via a function of order ε^{γ}. For this linear transmission problem, the homogenization and corrector results were achieved in [20], [10] and [11]. For more investigations on the related problems, we refer to [1, 12, 15-17, 24-26] and the references therein.

In the present paper, we are devoted to the study of the corrector results for problem (1.1). The proof mainly depends on the periodic unfolding method, which was first introduced by Cioranescu, Damlamian and Griso in [3] for the case of fixed domains (see [4] for more details) and extended to the two-component domains which are separated by a periodic interface (see [11]). For further developments and various applications of the unfolding method, we refer the interested readers to the work in [2, 5, 11, 12, 25, 28] and the references therein.

This paper consists of two parts. In the first part, we shall give a new proof of the homogenization results achieved in [6]. The proof follows from an unfolded formulation of the homogenization results (See Theorem 3). This unfolded formulation is also crucial to the proof of our corrector results. Moreover, we derive the precise convergence of flux. Now we state the (standard) homogenization results where we will use some notations to be defined in the next section.

Theorem 1. *Let $(u_{1\varepsilon}, u_{2\varepsilon})$ be the solution of problem (1.1). Then there exists $u_1 \in H_0^1(\Omega)$ such that*

$$\widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u_1 \quad \textit{weakly in } L^2(\Omega).$$

Also, u_1 is the unique solution of the homogenized problem

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_1) + a\theta_2 g(u_1) = \theta_1 f & \textit{in } \Omega, \\ u_1 = 0 & \textit{on } \partial\Omega. \end{cases} \tag{1.2}$$

The homogenized matrix $A^0 = (a^0_{ij})_{1\le i,j\le n}$ *is defined by*

$$a^0_{ij} = \mathcal{M}_Y\big(a_{ij} + \sum_{k=1}^{n} a_{ik}\frac{\partial \chi_j}{\partial y_k}\big), \tag{1.3}$$

where a_{ij} *is the entry of the following matrix* A*:*

$$A = (a_{ij})_{1\le i,j\le n} := \begin{cases} D_f Id, & in\ Y_1, \\ D_p Id, & in\ Y_2, \end{cases}$$

and $\chi_j (j = 1, \cdots, n)$ *is the solution of the following cell problem:*

$$\begin{cases} -\mathrm{div}\big(A\nabla(\chi_j + y_j)\big) = 0 & in\ Y, \\ \mathcal{M}_Y(\chi_j) = 0, \quad \chi_j\ is\ Y\text{-}periodic. \end{cases} \tag{1.4}$$

Moreover, we have the following convergences:

$$\begin{aligned} D_f \widetilde{\nabla u_{1\varepsilon}} &\rightharpoonup A^1\nabla u_1 \quad weakly\ in\ L^2(\Omega), \\ D_p \widetilde{\nabla u_{2\varepsilon}} &\rightharpoonup A^2\nabla u_1 \quad weakly\ in\ L^2(\Omega), \end{aligned} \tag{1.5}$$

where $A^l = (a^l_{ij})_{n\times n}$ $(l = 1, 2)$ *is defined by*

$$a^l_{ij} = \theta_l \mathcal{M}_{Y_l}\big(a_{ij} + \sum_{k=1}^{n} a_{ik}\frac{\partial \chi_j}{\partial y_k}\big). \tag{1.6}$$

The second part of the paper is devoted to the corrector results, which are the main results of the present paper. The proof relies on the convergence of the energy functions associated to problem (1.1), see Proposition 5 for the details. The precise corrector results are stated as follows:

Theorem 2. *Let* $(u_{1\varepsilon}, u_{2\varepsilon})$ *be the solution of problem (1.1) and* u_1 *be the solution of the homogenized problem (1.2), then we have*

$$\begin{aligned} \big\|\nabla u_{1\varepsilon} - \nabla u_1 - \sum_{i=1}^{n} \mathcal{U}^\varepsilon_1\big(\frac{\partial u_1}{\partial x_i}\big)\mathcal{U}^\varepsilon_1(\nabla_y \chi_i|_{Y_1})\big\|_{L^2(\Omega_{1\varepsilon})} \longrightarrow 0, \\ \big\|\nabla u_{2\varepsilon} - \nabla u_1 - \sum_{i=1}^{n} \mathcal{U}^\varepsilon_2\big(\frac{\partial u_1}{\partial x_i}\big)\mathcal{U}^\varepsilon_2(\nabla_y \chi_i|_{Y_2})\big\|_{L^2(\Omega_{2\varepsilon})} \longrightarrow 0, \end{aligned} \tag{1.7}$$

where $\chi_j (j = 1, \cdots, n)$ *is the solution of the cell problem (1.4).*

This paper is organized as follows. In Section 2, we recall some elementary properties in the unfolding method and give some convergence results. Section 3 is devoted to the homogenization of problem (1.1). Section 4 focuses on the proof of the corrector results.

2 Preliminaries

2.1 Some notation and properties

Let Ω be a bounded connected open set with smooth boundary in $\mathbb{R}^n$ $(n \geq 3)$ and let $Y = [0, l_1) \times \cdots \times [0, l_n)$ be the reference cell with $l_i > 0$ $(i = 1, \cdots, n)$ in $\mathbb{R}^n$. To simplify the notations associated to the unfolding method, we denote by Y_2 an open subset of Y with smooth boundary such that $\overline{Y_2} \subset Y$. We shall refer to Y_2 as being the elementary hole.

The letter ε denotes the general term of a sequence of positive real numbers which converges to zero. For each ε, let $\tau_\varepsilon(\varepsilon Y_2)$ be the translated image of εY_2 by the vector εkl, namely:

$$\tau_\varepsilon(\varepsilon Y_2) = \varepsilon(kl + Y_2),$$

where $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ and $kl = (k_1 l_1, \cdots, k_n l_n)$. Let us denote by $\Omega_{2\varepsilon}$ the set of all the obstacles contained in Ω, i.e.

$$\Omega_{2\varepsilon} = \bigcup \{\tau_\varepsilon(\varepsilon Y_2) : \tau_\varepsilon(\varepsilon Y_2) \subset \Omega,\ k \in \mathbb{Z}^n\}.$$

Write the other component of Ω and the interface, respectively, by:

$$\Omega_{1\varepsilon} = \Omega \backslash \overline{\Omega}_{2\varepsilon} \quad \text{and} \quad \Gamma^\varepsilon = \partial \Omega_{2\varepsilon}.$$

Notice that $\Omega_{1\varepsilon}$ is a periodically perforated domain with the size of holes being the same as the period. Suppose that the obstacles do not intersect the boundary $\partial\Omega$. See Figure 1 for this domain in two dimensional case:

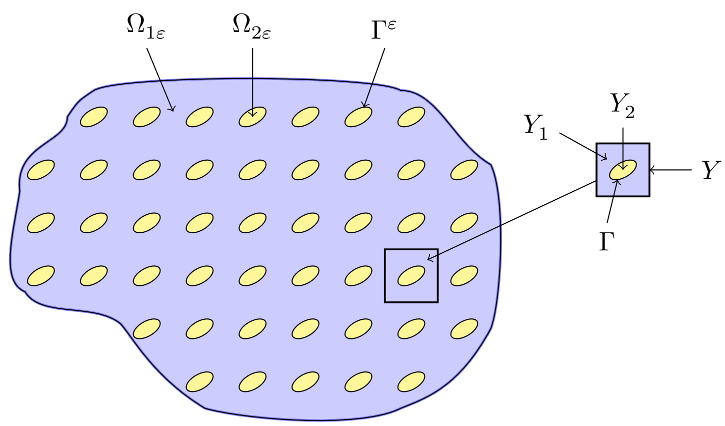

Figure 1: The perforated domain.

Let $Y_1 = Y \backslash \overline{Y}_2$. For $k \in \mathbb{Z}^n$, we denote

$$Y^k = kl + Y, \quad Y_i^k = kl + Y_i \ (i = 1, 2).$$

Now we recall some notations related to the unfolding method in [2], [4] and [11]:

$$\widehat{\Omega}_\varepsilon = \operatorname{int} \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon(k_l + \overline{Y}), \quad \Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon,$$

$$\widehat{\Omega}_{i\varepsilon} = \bigcup_{k\in\widehat{K}_\varepsilon} \varepsilon Y_i^k, \quad \Lambda_{i\varepsilon} = \Omega_{i\varepsilon}\backslash\widehat{\Omega}_{i\varepsilon}\ (i=1,2), \quad \widehat{\Gamma}^\varepsilon = \partial\widehat{\Omega}_{2\varepsilon},$$

where $\widehat{K}_\varepsilon = \{k \in \mathbb{Z}^n \mid \varepsilon Y^k \subset \Omega\}$. See Figure 2 for these notations:

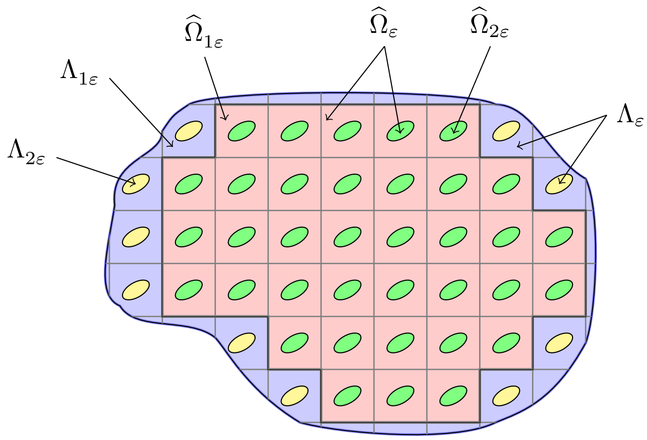

Figure 2: The sets $\widehat{\Omega}_\varepsilon,\ \widehat{\Omega}_{1\varepsilon},\ \widehat{\Omega}_{2\varepsilon},\ \Lambda_\varepsilon,\ \Lambda_{1\varepsilon},\ \Lambda_{2\varepsilon}$.

In the sequel, we will use the following notations:

- $\theta_i = |Y_i|/|Y|,\ \ i = 1, 2.$
- $\mathcal{M}_{\mathcal{O}}(v) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} v dx.$
- $\widetilde{f}$ is the zero extension to Ω of any function f defined on $\Omega_{i\varepsilon}$ for $i = 1, 2$.
- For $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$, $M(\alpha, \beta, \mathcal{O})$ denotes the set of $n \times n$ matrix-valued functions $B(x) \in (L^\infty(\mathcal{O}))^{n\times n}$ such that

$$(B(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \quad |B(x)\lambda| \leq \beta|\lambda|, \quad \text{for any } \lambda \in \mathbb{R}^n \text{ and a.e. on } \mathcal{O}.$$

- C denotes a generic constant which does not depend upon ε, but whose value may differ from line to line.
- The notation $L^p(\mathcal{O})$ will be used both for scalar and vector-valued functions defined on the set $\mathcal{O}$, when no ambiguity arises.

Next we list some properties in the unfolding method, which will play a key role in proving some important convergences needed in the proof of our main results. For other properties and related comments, we refer the reader to [2], [4] and [11].

For $i = 1, 2$, let $\mathcal{T}_i^\varepsilon$ and $\mathcal{U}_i^\varepsilon$ be the unfolding operator and the average operator, respectively.

Proposition 1. *Let $p \in [1, +\infty)$. For $i = 1, 2$,*

(i) *$\mathcal{T}_i^\varepsilon$ is linear and continuous from $L^p(\Omega_{i\varepsilon})$ to $L^p(\Omega \times Y_i)$.*

(ii) *For $\phi \in L^1(\Omega_{i\varepsilon})$,*

$$\frac{1}{|Y|}\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(\phi)(x,y)\,dx\,dy = \int_{\widehat{\Omega}_{i\varepsilon}} \phi(x)dx = \int_{\Omega_{i\varepsilon}} \phi(x)dx - \int_{\Lambda_{i\varepsilon}} \phi(x)dx.$$

(iii) *For $w \in L^p(\Omega_{i\varepsilon})$, we have*

$$\|\mathcal{T}_i^\varepsilon(w)\|_{L^p(\Omega\times Y^*)} = |Y|^{1/p}\|w\|_{L^p(\widehat{\Omega}_{i\varepsilon})} \le |Y|^{1/p}\|w\|_{L^p(\Omega_{i\varepsilon})}.$$

(iv) *For $p \in (1, +\infty)$, let $\{\phi_\varepsilon\}$ and $\{\psi_\varepsilon\}$ be two sequences in $L^p(\Omega_{i\varepsilon})$ and $L^{p_0}(\Omega_{i\varepsilon})$ $(1/p + 1/p_0 < 1,\ 1/q + 1/q' = 1)$, respectively. If*

$$\|\phi_\varepsilon\|_{L^p(\Omega_{i\varepsilon})} \le C \quad \text{and} \quad \|\psi_\varepsilon\|_{L^{p_0}(\Omega_{i\varepsilon})} \le C,$$

then

$$\int_{\Omega_{i\varepsilon}} \phi_\varepsilon \psi_\varepsilon\,dx - \frac{1}{|Y|}\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(\phi_\varepsilon)\mathcal{T}_i^\varepsilon(\psi_\varepsilon)\,dx\,dy \longrightarrow 0.$$

Proposition 2. *Let $i = 1, 2$.*

(i) *For $p \in [1, \infty)$, let $\{\omega_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that*

$$\omega_\varepsilon \to \omega \text{ strongly in } L^p(\Omega),$$

then we have

$$\mathcal{T}_i^\varepsilon(\omega_\varepsilon) \to \omega \quad \text{strongly in } L^p(\Omega \times Y_i).$$

(ii) *For $p \in (1, \infty)$, let $\{\varphi_\varepsilon\}$ be a sequence in $L^p(\Omega_{i\varepsilon})$ such that $\|\varphi_\varepsilon\|_{L^p(\Omega_{i\varepsilon})} \le C$. If*

$$\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi \text{ weakly in } L^p(\Omega \times Y_i),$$

then we have

$$\widetilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\varphi) \text{ weakly in } L^p(\Omega).$$

(iii) *For $\omega \in L^p(\Omega)$,*

$$\|\mathcal{U}_i^\varepsilon(\omega) - \omega\|_{L^p(\Omega_{i\varepsilon})} \to 0.$$

(iv) *For $p \in [1, +\infty)$, let $\omega_\varepsilon \in L^p(\Omega_{i\varepsilon})$ and $\omega \in L^p(\Omega \times Y_i)$, then the following two assertions are equivalent:*

(a) *$\mathcal{T}_i^\varepsilon(\omega_\varepsilon) \to \omega$ strongly in $L^p(\Omega \times Y_i)$ and $\|\omega_\varepsilon\|_{L^p(\Lambda_{i\varepsilon})} \to 0$,*

(b) *$\|\omega_\varepsilon - \mathcal{U}_i^\varepsilon(\omega)\|_{L^p(\Omega_{i\varepsilon})} \to 0$.*

(v) *For $p \in [1, \infty)$, let $f \in L^p(\Omega)$ and $g \in L^p(Y_i)$. Then we have*

$$\|\mathcal{U}_i^\varepsilon(fg) - \mathcal{U}_i^\varepsilon(f)\mathcal{U}_i^\varepsilon(g)\|_{L^p(\Omega_{i\varepsilon})} \to 0.$$

2.2 Some convergence results

This subsection is devoted to some important convergences related to the solution of problem (1.1). Let us first introduce two spaces V^ε and H^ε.

Define V^ε by

$$V^\varepsilon := \{v \in H^1(\Omega_{1\varepsilon}) \mid v = 0 \text{ on } \partial\Omega\},$$

endowed with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_{1\varepsilon})}.$$

Let H^ε be the product space:

$$H^\varepsilon := \{u = (u_1, u_2) \mid u_1 \in V^\varepsilon,\ u_2 \in H^1(\Omega_{2\varepsilon}),\ u_1 = u_2 \text{ on } \Gamma^\varepsilon\}$$

with the norm:

$$\|u\|^2_{H^\varepsilon} = \|\nabla u_1\|^2_{L^2(\Omega_{1\varepsilon})} + \|\nabla u_2\|^2_{L^2(\Omega_{2\varepsilon})}.$$

The variational formulation of problem (1.1) is to find a $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \in H^\varepsilon$ such that

$$\begin{aligned} D_f \int_{\Omega_{1\varepsilon}} \nabla u_{1\varepsilon} \cdot \nabla v_1 \, dx + D_p \int_{\Omega_{2\varepsilon}} \nabla u_{2\varepsilon} \cdot \nabla v_2 \, dx \\ + a \int_{\Omega_{2\varepsilon}} g(u_{2\varepsilon}) v_2 \, dx = \int_{\Omega_{1\varepsilon}} f v_1 \, dx, \quad \forall (v_1, v_2) \in H^\varepsilon. \end{aligned} \tag{2.1}$$

For every fixed ε, by Proposition 3.4 in [6], we know that problem (2.1) has a unique u_ε such that:

$$\|u_{1\varepsilon}\|_{V^\varepsilon} + \|u_{2\varepsilon}\|_{H^1(\Omega_{2\varepsilon}))} < C, \tag{2.2}$$

where C is a positive constant, independent of ε.

By a similar argument of Theorem 4.5 in [12] and Theorem 2.20 in [11], we can get the following convergence properties:

Proposition 3. *Let u_ε be the solution of problem (1.1). Then, there exist $u_1 \in H^1_0(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H^1_{\text{per}}(Y_1))$ and $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$ such that*

$$\begin{aligned} &(i) \quad \mathcal{T}^\varepsilon_i(u_{i\varepsilon}) \to u_1 \quad \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ &(ii) \quad \mathcal{T}^\varepsilon_1(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1), \\ &(iii) \quad \mathcal{T}^\varepsilon_2(\nabla u_{2\varepsilon}) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2), \\ &(iv) \quad \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u_1 \quad \text{weakly in } L^2(\Omega), \end{aligned} \tag{2.3}$$

where $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ for $i = 1, 2$. Moreover, we have

$$\widehat{u}_1 = \widehat{u}_2 - y_\Gamma \nabla u_1 \quad \text{on } \Omega \times \Gamma,$$

where $y_\Gamma = y - \mathcal{M}_\Gamma(y)$.

Corollary 1. *Keep all the notations in Proposition 3. Let β^ε de defined by*

$$\beta^\varepsilon(x) = \begin{cases} u_{1\varepsilon}, & x \in \Omega_{1\varepsilon}, \\ u_{2\varepsilon}, & x \in \Omega_{2\varepsilon}. \end{cases}$$

Then we have

$$\beta^\varepsilon \rightharpoonup u_1 \quad \text{weakly in } H^1_0(\Omega).$$

Indeed, β^ε is bounded in $H_0^1(\Omega)$ (see [6] for more details). Hence, there exists $\beta \in H_0^1(\Omega)$ such that $\beta^\varepsilon \rightharpoonup \beta$ weakly in $H_0^1(\Omega)$. This implies

$$\widetilde{u}_{1\varepsilon} \rightharpoonup \theta_1 \beta \quad \text{weakly in } L^2(\Omega).$$

Together with (2.3)(iv), we obtain $\beta = u_1$.

Proposition 4. *Let $\{z_\varepsilon\}$ be a sequence such that*

$$z_\varepsilon \rightharpoonup z \quad \textit{weakly in } H_0^1(\Omega).$$

Suppose that g is a function satisfying (H_1) and (H_2), then we have

$$\mathcal{T}_2^\varepsilon(g(z_\varepsilon)) \rightharpoonup g(z) \quad \textit{weakly in } L^r(\Omega \times Y_2), \tag{2.4}$$

where $2 \leq r \leq \frac{2n}{\rho(n-2)}$.

Proof. By Theorem 2.4 in [6], we can get

$$g(z_\varepsilon) \to g(z) \quad \text{strongly in } L^{\bar{\rho}}(\Omega),$$

where $\bar{\rho} = \frac{2n}{(\rho-1)(n-2)+n}$. Since $\bar{\rho} \in (1, 2]$, Proposition 2(i) gives

$$\mathcal{T}_2^\varepsilon(g(z_\varepsilon)) \to g(z) \quad \text{strongly in } L^{\bar{\rho}}(\Omega \times Y_2). \tag{2.5}$$

On the other hand, from (H_2), we deduce

$$\int_\Omega |g(z_\varepsilon)|^r dx \leq C(|\Omega| + \|z_\varepsilon\|^{r\rho}_{L^{r\rho}(\Omega)}) \leq C(1 + \|z_\varepsilon\|^{r\rho}_{H_0^1(\Omega)}) \leq C.$$

As a result, $g(z_\varepsilon)$ is bounded in $L^r(\Omega)$. By Proposition 1(iii), this implies $\mathcal{T}_2^\varepsilon(g(z_\varepsilon))$ is bounded in $L^r(\Omega \times Y_2)$. Together with (2.5), we can complete the proof of Proposition 4. $\square$

3 Homogenization results

In this section, we study the asymptotic behavior, as $\varepsilon \to 0$, of the problem (1.1). The study was done by the oscillating test functions method. Here we use the unfolding method to study the homogenization, which will be crucial to the proof of the corrector results in Section 4. Moreover, we give the precise convergence of flux.

For later use, we introduce the matrix

$$A = (a_{ij})_{1 \leq i,j \leq n} := \begin{cases} D_f Id, & \text{in } Y_1, \\ D_p Id, & \text{in } Y_2. \end{cases}$$

Our unfolded formulation of the homogenization results is stated as follows:

Theorem 3. *Let u_ε be the solution of problem (1.1). Then, there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y_1))$ and $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$ such that*

$$
\begin{aligned}
&(i) \ \ \mathcal{T}_i^\varepsilon(u_{i\varepsilon}) \to u_1 \quad \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i=1,2,\\
&(ii) \ \ \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega\times Y_1),\\
&(iii) \ \ \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega\times Y_2),\\
&(iv) \ \ \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u_1 \quad \text{weakly in } L^2(\Omega),
\end{aligned}
\tag{3.1}
$$

where $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ for $i = 1, 2$. The pair $(u_1, \widehat{u})$ is the unique solution in $H_0^1(\Omega) \times L^2(\Omega, H^1_{\mathrm{per}}(Y))$ with $\mathcal{M}_\Gamma(\widehat{u}) = 0$, of the problem

$$
\begin{cases}
\dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y} A(\nabla u_1 + \nabla_y \widehat{u})(\nabla\Psi + \nabla_y\Phi)\,dx\,dy\\
+a\theta_2 \displaystyle\int_\Omega g(u_1)\Psi dx = \theta_1 \int_\Omega f\Psi dx\\
\text{for all } \Psi\in H_0^1(\Omega) \text{ and } \Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y)),
\end{cases}
\tag{3.2}
$$

where $\widehat{u} \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$ is the extension by periodicity of the following function (still denoted by $\widehat{u}$):

$$
\widehat{u}(\cdot, y) = \begin{cases} \widehat{u}_1(\cdot, y) & \text{when } \ y \in Y_1,\\ \widehat{u}_2(\cdot, y) - y_\Gamma \nabla u_1 & \text{when } \ y\in Y_2,\end{cases}
\tag{3.3}
$$

with $y_\Gamma = y - \mathcal{M}_\Gamma(y)$. Also, we have

$$
\widehat{u} = \sum_{j=1}^n \frac{\partial u_1}{\partial x_j}\chi_j,
\tag{3.4}
$$

where $\chi_j (j = 1, \cdots, n)$ is the solution of the cell problem (1.4).

Proofs of Theorem 3 and Theorem 1. In view of (2.2), Proposition 3 implies that convergences (3.1)(i)-(iii) hold at least for a subsequence (still denoted by ε). By the properties of the unfolding operators, we further obtain

$$
\begin{cases}
(i) \ \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i \mathcal{M}_{Y_i}(u_1) \ \ \text{weakly in } L^2(\Omega) \ \ \text{for} \ \ i=1,2,\\
(ii) \ D_f \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup \theta_1 \mathcal{M}_{Y_1}[A(\nabla u_1 + \nabla_y \widehat{u}_1)] \quad \text{weakly in } L^2(\Omega),\\
(iii) \ D_p \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup \theta_2 \mathcal{M}_{Y_2}[A(\nabla_y \widehat{u}_2)] \quad \text{weakly in } L^2(\Omega).
\end{cases}
\tag{3.5}
$$

Notice that u_1 is independent of y, we get convergence (3.1)(iv).

Let $\Psi, \phi \in \mathcal{D}(\Omega)$, $\psi \in H^1_{\mathrm{per}}(Y)$ and $\psi^\varepsilon(x) = \psi(x/\varepsilon)$. Set

$$
v_\varepsilon(x) = \Psi(x) + \varepsilon\phi(x)\psi^\varepsilon(x) \text{ and } v_{i\varepsilon} = v_\varepsilon|_{\Omega_{i\varepsilon}} \ (i = 1, 2).
$$

Then we have

$$
\begin{aligned}
&\mathcal{T}_i^\varepsilon(v_{i\varepsilon}) \to \Psi \quad \text{strongly in } L^2(\Omega\times Y_i),\\
&\mathcal{T}_i^\varepsilon(\nabla v_{i\varepsilon}) \to \nabla\Psi + \nabla_y\Phi \quad \text{strongly in } L^2(\Omega\times Y_i) \text{ with } \Phi(x,y) = \phi(x)\psi(y).
\end{aligned}
\tag{3.6}
$$

Choosing $(v_{1\varepsilon}, v_{2\varepsilon})$ as test function in the variational formulation (2.1), we get

$$\begin{aligned} D_f \int_{\Omega_{1\varepsilon}} \nabla u_{1\varepsilon} \cdot \nabla v_{1\varepsilon}\, dx + D_p \int_{\Omega_{2\varepsilon}} \nabla u_{2\varepsilon} \cdot \nabla v_{2\varepsilon}\, dx \\ + a \int_{\Omega_{2\varepsilon}} g(u_{2\varepsilon}) v_{2\varepsilon}\, dx = \int_{\Omega_{1\varepsilon}} f v_{1\varepsilon}\, dx. \end{aligned} \tag{3.7}$$

By (2.3) and (3.6), we have

$$\begin{aligned} &\int_{\Omega_{1\varepsilon}} \nabla u_{1\varepsilon} \cdot \nabla v_{1\varepsilon}\, dx \to \frac{1}{|Y|} \int_{\Omega \times Y_1} (\nabla u_1 + \nabla_y \widehat{u}_1) \cdot (\Psi + \nabla_y \Phi)\, dx\, dy, \\ &\int_{\Omega_{2\varepsilon}} \nabla u_{2\varepsilon} \cdot \nabla v_{2\varepsilon}\, dx \to \frac{1}{|Y|} \int_{\Omega \times Y_2} \nabla_y \widehat{u}_2 \cdot (\Psi + \nabla_y \Phi)\, dx\, dy, \\ &\int_{\Omega_{1\varepsilon}} f v_{1\varepsilon}\, dx \to \theta_1 \int_{\Omega} f \Psi\, dx. \end{aligned} \tag{3.8}$$

By Corollary 1 and Proposition 4,

$$\mathcal{T}_2^{\varepsilon}(g(\beta_\varepsilon)) \rightharpoonup g(u_1) \quad \text{weakly in } L^r(\Omega \times Y_2), \tag{3.9}$$

where $2 \le r \le \frac{2n}{\rho(n-2)}$. Together with (3.6), we use Proposition 1(iv) to get

$$\int_{\Omega_{2\varepsilon}} g(u_{2\varepsilon}) v_{2\varepsilon}\, dx = \int_{\Omega_{2\varepsilon}} g(\beta_\varepsilon) v_{2\varepsilon}\, dx \to \theta_2 \int_{\Omega} g(u_1) \Psi\, dx. \tag{3.10}$$

Passing to the limit in (3.7) and using (3.8) and (3.10), we obtain

$$\begin{aligned} \frac{D_f}{|Y|} \int_{\Omega \times Y_1} (\nabla u_1 + \nabla_y \widehat{u}_1) \cdot (\Psi + \nabla_y \Phi)\, dx\, dy + \frac{D_p}{|Y|} \int_{\Omega \times Y_2} \nabla_y \widehat{u}_2 \cdot (\Psi + \nabla_y \Phi)\, dx\, dy \\ + a\theta_2 \int_{\Omega} g(u_1) \Psi\, dx = \theta_1 \int_{\Omega} f \Psi\, dx. \end{aligned} \tag{3.11}$$

On the other hand, by Proposition 3, we have $\widehat{u}_1 = \widehat{u}_2 - y_{{}_\Gamma} \nabla u_1$ on $\Omega \times \Gamma$. Consequently, we can define $\widehat{u}$ by (3.3) and extend it by periodicity (still denoted by $\widehat{u}$). Together with the density, we get (3.2).

Setting $\Psi = 0$ in (3.2), we have

$$\mathrm{div}_y A(\nabla u_1 + \nabla_y \widehat{u}) = 0.$$

Notice that u_1 is independent of y and $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$. Hence we get (3.4). Then by a standard computation, we get the convergence (1.5) from (3.5) and the following identity:

$$\frac{1}{|Y|} \int_Y A(\nabla u_1 + \nabla_y \widehat{u}) \nabla \Psi dy = A^0 \nabla u_1 \nabla \Psi \tag{3.12}$$

with A^0 defined by (1.3). Moreover, we obtain (1.2).

By the standard arguments, we derive the ellipticity of A^0 and the uniqueness of the solution of the homogenized problem. Hence we get that the pair $(u_1, \widehat{u})$ is the unique solution of problem (3.2). This implies that each convergence in Theorem 3 holds for the whole sequence. □

4 Corrector results

This section is devoted to the proof of the corrector results. Our starting point is the convergence of the energy functions associated to problem (1.1). We first recall the following crucial lemma due to Cioranescu, Damlamian, Donato, Griso and Zaki [2].

Lemma 1. *Let $\{D_\varepsilon\}$ be a sequence of $n \times n$ matrices in $M(\alpha, \beta, \mathcal{O})$ for some open set $\mathcal{O}$, such that $D_\varepsilon \to D$ a.e. on $\mathcal{O}$ (or more generally, in measure in $\mathcal{O}$). If $\zeta_\varepsilon \rightharpoonup \zeta$ weakly in $L^2(\mathcal{O})$, then*

$$\int_{\mathcal{O}} D\zeta\zeta\, dx \leq \liminf_{\varepsilon\to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon\, dx.$$

Furthermore, if $\limsup\limits_{\varepsilon\to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon\, dx \leq \int_{\mathcal{O}} D\zeta\zeta\, dx$, then

$$\int_{\mathcal{O}} D\zeta\zeta\, dx = \lim_{\varepsilon\to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon\, dx \text{ and } \zeta_\varepsilon \to \zeta \text{ strongly in } L^2(\mathcal{O}).$$

For each ε, the energy E_ε is defined by

$$E_\varepsilon = \int_{\Omega_{1\varepsilon}} D_f \nabla u_{1\varepsilon} \cdot \nabla u_{1\varepsilon}\, dx + \int_{\Omega_{2\varepsilon}} D_p \nabla u_{2\varepsilon} \cdot \nabla u_{2\varepsilon}\, dx.$$

Then we have the following convergence of energy, which is crucial to the proof of our corrector results.

Proposition 5. *Let u_ε be the solution of problem (1.1). Then*

$$\lim_{\varepsilon\to 0} E_\varepsilon = \frac{1}{|Y|} \int_{\Omega\times Y} A(\nabla u_1 + \nabla_y \widehat{u}) \cdot (\nabla u_1 + \nabla_y \widehat{u})\, dx\, dy. \tag{4.1}$$

Proof. By (3.3), we have

$$\begin{aligned} &\frac{1}{|Y|} \int_{\Omega\times Y} A(\nabla u_1 + \nabla_y \widehat{u}) \cdot (\nabla u_1 + \nabla_y \widehat{u})\, dx\, dy \\ =&\frac{1}{|Y|} \int_{\Omega\times Y_1} A(\nabla u_1 + \nabla_y \widehat{u}_1) \cdot (\nabla u_1 + \nabla_y \widehat{u}_1)\, dx\, dy \\ &+ \frac{1}{|Y|} \int_{\Omega\times Y_2} A(\nabla_y \widehat{u}_2) \cdot (\nabla_y \widehat{u}_2)\, dx\, dy. \end{aligned} \tag{4.2}$$

Together with (3.1), we use Lemma 1 to get

$$\begin{aligned} &\frac{1}{|Y|} \int_{\Omega\times Y} A(\nabla u_1 + \nabla_y \widehat{u}) \cdot (\nabla u_1 + \nabla_y \widehat{u})\, dx\, dy \\ \leq&\liminf_{\varepsilon\to 0} \frac{1}{|Y|} \int_{\Omega\times Y_1} A\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \cdot \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon})\, dx\, dy \\ &+ \liminf_{\varepsilon\to 0} \frac{1}{|Y|} \int_{\Omega\times Y_2} A\mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \cdot \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon})\, dx\, dy. \end{aligned}$$

Then Proposition 1(ii) gives

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y} & A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy\\ &\le \liminf_{\varepsilon\to 0}\Big[\int_{\widehat{\Omega}_{1\varepsilon}} D_f\nabla u_{1\varepsilon}\cdot\nabla u_{1\varepsilon}\,dx+\int_{\widehat{\Omega}_{2\varepsilon}} D_p\nabla u_{2\varepsilon}\cdot\nabla u_{2\varepsilon}\,dx\Big].\end{aligned}\tag{4.3}$$

Let

$$\widehat{E}_\varepsilon=\int_{\widehat{\Omega}_{1\varepsilon}} D_f\nabla u_{1\varepsilon}\cdot\nabla u_{1\varepsilon}\,dx+\int_{\widehat{\Omega}_{2\varepsilon}} D_p\nabla u_{2\varepsilon}\cdot\nabla u_{2\varepsilon}\,dx.$$

Furthermore,

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy\le\liminf_{\varepsilon\to 0}\widehat{E}_\varepsilon\le\limsup_{\varepsilon\to 0}\widehat{E}_\varepsilon\le\limsup_{\varepsilon\to 0}E_\varepsilon \tag{4.4}$$

From (2.1), we have

$$\lim_{\varepsilon\to 0}E_\varepsilon=\lim_{\varepsilon\to 0}\Big[\int_{\Omega_{1\varepsilon}} fu_{1\varepsilon}\,dx-a\int_{\Omega_{2\varepsilon}} g(u_{2\varepsilon})u_{2\varepsilon}\,dx\Big]\tag{4.5}$$

By (3.1)(i), we get

$$\int_{\Omega_{1\varepsilon}} fu_{1\varepsilon}\,dx\to\theta_1\int_\Omega fu_1dx.$$

For the last term in (4.5), by (3.1)(i) and (3.9), we use Proposition 1(iv) to obtain

$$\int_{\Omega_{2\varepsilon}} g(u_{2\varepsilon})u_{2\varepsilon}\,dx=\int_{\Omega_{2\varepsilon}} g(\beta_\varepsilon)u_{2\varepsilon}\,dx\to\theta_2\int_\Omega g(u_1)u_1\,dx.$$

Together with (4.4) and (4.5), we get

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy\le\lim_{\varepsilon\to 0}E_\varepsilon=\theta_1\int_\Omega fu_1dx-a\theta_2\int_\Omega g(u_1)u_1dx.$$

On the other hand, Choosing $\Psi=u_1$, $\Phi=\widehat{u}$ in (3.2), we have

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy=\theta_1\int_\Omega fu_1dx-a\theta_2\int_\Omega g(u_1)u_1dx.$$

This implies (4.1). Consequently, the proof of Proposition 5 is finished. □

Based on this proposition, we can obtain that the convergences on the gradient in (3.1) are strong.

Corollary 2. *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}&(i)\ \int_{\Lambda_{i\varepsilon}}|\nabla u_{i\varepsilon}|^2dx\to 0,\ \ i=1,2,\\ &(ii)\ \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon})\to\nabla u_1+\nabla_y\widehat{u}_1\quad\textit{strongly in } L^2(\Omega\times Y_1),\\ &(iii)\ \mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon})\to\nabla_y\widehat{u}_2\quad\textit{strongly in } L^2(\Omega\times Y_2).\end{aligned}\tag{4.6}$$

In fact, by (4.4), we obtain

$$\lim_{\varepsilon\to 0}\widehat{E}_\varepsilon=\lim_{\varepsilon\to 0}E_\varepsilon,$$

which implies (4.6)(i). Also, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big[\int_{\Omega\times Y_1}A\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon})\cdot\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon})\,dx\,dy+\int_{\Omega\times Y_2}A\mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon})\cdot\mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon})\,dx\,dy\Big]\\&\quad=\int_{\Omega\times Y}A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy.\end{aligned}$$

Combining this with (3.1)(ii)(iii), a direct computation shows that

$$\begin{aligned}&\int_{\Omega\times Y_1}A\big[\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon})-(\nabla u_1+\nabla_y\widehat{u}_1)\big]\cdot\big[\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon})-(\nabla u_1+\nabla_y\widehat{u}_1)\big]\,dx\,dy\\&\quad+\int_{\Omega\times Y_2}A\big[\mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon})-\nabla_y\widehat{u}_2\big]\cdot\big[\mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon})-\nabla_y\widehat{u}_2\big]\,dx\,dy\\&\longrightarrow\int_{\Omega\times Y}A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy\\&\qquad-\int_{\Omega\times Y}A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy\\&\qquad-\int_{\Omega\times Y}A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy\\&\qquad+\int_{\Omega\times Y}A(\nabla u_1+\nabla_y\widehat{u})\cdot(\nabla u_1+\nabla_y\widehat{u})\,dx\,dy=0.\end{aligned}$$

This allows us to obtain (4.6)(ii)(iii).

With Corollary 2 at our disposal, we are in a position to complete the proof of the corrector results.

Proof of Theorem 2. By (4.6)(i)(ii), we use Proposition 2(iv) to get

$$\|\nabla u_{1\varepsilon}-\mathcal{U}_1^\varepsilon(\nabla u_1+\nabla_y\widehat{u})\|_{L^2(\Omega_{1\varepsilon})}\to 0. \tag{4.7}$$

Since ∇u_1 is independent of y, from Proposition 2(iii), we have

$$\|\nabla u_1-\mathcal{U}_1^\varepsilon(\nabla u_1)\|_{L^2(\Omega_{1\varepsilon})}\to 0.$$

Combining this with (4.7), we obtain

$$\|\nabla u_{1\varepsilon}-\nabla u_1-\mathcal{U}_1^\varepsilon(\nabla_y\widehat{u})\|_{L^2(\Omega_{1\varepsilon})}\to 0.$$

By (3.3) and (3.4), Proposition 2(v) gives the first convergence in (1.7). The second one can be also similarly proved. □

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