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On the strong persistence property for monomial ideals

by
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Abstract

Let I be the edge ideal associated to a graph with loops, a weighted graph or a clutter. In this paper we study when I has the strong persistence property, this is $(I^{k+1}\colon I) = I^k$ for each $k \geq 1$.

Key Words: Strong persistence property, monomial ideals, clutters, weighted graphs, graphs with loops.
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1 Introduction

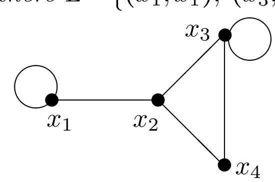
Let R be a commutative Noetherian ring. The *associated primes set* of an ideal I is $\mathrm{Ass}(I) = \{P \in \mathrm{Spec}(R) \mid P = (I\colon a) \text{ for some } a \in R\}$. If $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I, then $\mathrm{Ass}(I) = \{\mathrm{r}(Q_1), \ldots, \mathrm{r}(Q_s)\}$ where $\mathrm{r}(Q_i)$ is the radical of Q_i. I has the *persistence property* if $\mathrm{Ass}(I^k) \subseteq \mathrm{Ass}(I^{k+1})$ for each k. In [2] is showed that the edge ideal of a simple graph has the persistence property, and they use that these edge ideals satisfy $(I^{k+1}\colon I) = I^k$ for each k. Recently was proved that this concept implies the persistence property (see [1]) and it is called the *strong persistence property*. These concepts are not equivalent, in [2, Example 2.18] is given a squarefree monomial ideal with the persistence property, but it does not have the strong persistence property. Assuming this terminology, in [2, Lemma 2.12] was proved that the edge ideal of a simple graph has the strong persistence property. In this paper we study the strong persistence property for edge ideals of graphs with loops, weighted graphs, and clutters.

This paper is organized as follows: in Sect. 2 we prove that the edges ideals of graphs with loops have the strong persistence property. In Sect. 3 we prove that the edge ideal of a vertex–weighted graph (G, w) has the strong persistence property. Furthermore, we prove that $I(G)^k$ and $I(G, w)^k$ have the same associated primes. In Sect. 4 we study the edges ideals of clutters. In particular, we show that a clutter has the strong persistence property if and only if at less one of its connected component has the strong persistence property. Also, we prove that a König unmixed clutter without 4-cycles and squarefree monomials in four variables have the strong persistence property. Furthermore, we show that $(I^2\colon I) = I$, if I is a squarefree monomial ideal. Finally we prove that the strong persistence property is closed under c-minor and cones. In Sect. 5 we give some properties of the strong persistence property. Also, we introduce the symbolic persistence property and we show that an ideal has this property if it has the strong persistence property.

2 Graphs with loops

A *graph with loops* is a triplet $\mathcal{G} = (V, E, L)$ where $G = (V, E)$ is a simple graph with $V = \{x_1, \dots, x_n\}$ and $L \subseteq \{(x_i, x_i) \mid x_i \in V\}$, L is called the *set of loops* of $\mathcal{G}$. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring, as usual we use x^a as abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \dots, a_n)$ is an integer vector with $a_i \geq 0$. If $f = \{x_i, x_j\} \in E$ or $f = (x_i, x_i) \in L$, then we take the monomial $\widetilde{f} = x_i x_j$ or $\widetilde{f} = x_i^2$, respectively. The *edge ideal* of a graph with loops $\mathcal{G} = (V, E, L)$ is the ideal $I(\mathcal{G}) = \big(\{f_i \mid f_i \in E \cup L\}\big) = I(G) + \big(\{x_i^2 \mid (x_i, x_i) \in L\}\big)$ where $I(G) = \big(\{x_i x_j \mid \{x_i, x_j\} \in E\}\big)$ is the edge ideal of $G = (V, E)$.

Example 1. *Graph with loops, where* $L = \{(x_1, x_1),\ (x_3, x_3)\}$.



For an integer vector $a = (a_1, \dots, a_n)$ with $a_i \geq 0$, we define the simple graph $\mathcal{G}^a$ with vertex set is $V^a = \{x_1^1, \dots, x_1^{a_1}, \dots, x_i^1, \dots, x_i^{a_i}, \dots, x_n^1, \dots, x_n^{a_n}\}$, and whose edge set

$$E^a = \left\{ \{x_i^{k_i}, x_j^{k_j}\} \mid \begin{array}{c} \{x_i, x_j\} \in E,\ k_i \leq a_i, \text{ and } k_j \leq a_j; \text{ or} \\ (x_i, x_j) \in L \text{ and } 1 \leq k_i < k_j \leq a_i \end{array} \right\},$$

where j in x_i^j is only an index. Furthermore, if $x_i \in V(\mathcal{G})$, then we define the *duplication* of x_i in $\mathcal{G}^a$ as the simple graph $\mathcal{G}^{a x_i} = \mathcal{G}^{a+e_i}$ where e_i is the i-th unit vector in $\mathbb{R}^n$. This operation is commutative, that is $(\mathcal{G}^{a x_i})^{x_j} = (\mathcal{G}^{a x_j})^{x_i}$ for each $x_i, x_j \in V$. Furthermore, if $f = \{x_i, x_j\} \in E$, then we denote by $(\mathcal{G}^a)^f = \mathcal{G}^{a+e_i+e_j}$; and if $f = (x_i, x_i) \in L$, then $(\mathcal{G}^a)^f = \mathcal{G}^{a+e_i+e_i}$.

Definition 1. *Let G a simple graph. A matching of G is a set of pairwise disjoint edges. The matching number of G, denoted by $\nu(G)$, is the size of any maximum matching of G. A matching that covers all vertices of $V(G)$ is called a perfect matching of G.*

Notation 1. $\mathrm{Mon}(R)$ *is the set of monomials in* $R = K[x_1, \dots, x_n]$*. If* $I = (m_1, \dots, m_s)$ *with* $m_i \in \mathrm{Mon}(R)$*, then* $G(I)$ *is the minimal monomial generating set of* I*.*

Proposition 1. *Let $\mathcal{G}$ be a graph with loops whose vertex set is $V = \{x_1, \dots, x_n\}$. If $a = (a_1, \dots, a_n)$ is an integer vector where $a_i \geq 0$, then $\mathcal{G}^a$ has a matching of size l if and only if $x^a \in I(\mathcal{G})^l$.*

Proof. $\Rightarrow$) Let $P = \{g_1, \dots, g_l\}$ be a matching of $\mathcal{G}^a$ where $g_j = \{x_{i_j}^{k_{i_j}}, x_{r_j}^{s_{r_j}}\}$. Now, we consider the monomial $x^b = \prod_{j=1}^{l} x_{i_j} x_{r_j} \in I(\mathcal{G})^l$. If $b = (b_1, \dots, b_n)$, then $b_i = \big|\{r \mid x_i^r \in \bigcup_{j=1}^{l} g_j\}\big|$ for each $1 \leq i \leq n$. Since P is a matching, $b_i \leq a_i$. Therefore, $x^b \mid x^a$ and $x^a \in I(\mathcal{G})^l$.

$\Leftarrow$) We take $E \cup L = \{f_1, \dots, f_q\}$. If $x^a \in I(\mathcal{G})^l$, then there exist an integer vector $\alpha = (\alpha_1, \dots, \alpha_q)$ such that $\alpha_1 + \cdots + \alpha_q = l$ and $x^a = m \widetilde{f}_1^{\alpha_1} \cdots \widetilde{f}_q^{\alpha_q}$ with $m \in \mathrm{Mon}(R)$. We can assume that $\alpha_1 > 0$. If $f_1 = \{x_r, x_s\} \in E$, then $\alpha_1 \leq a_r$ and $\alpha_1 \leq a_s$ since $\widetilde{f}_1^{\alpha_1} \mid x^a$. If

$P_1 = \{g_1, \ldots, g_{\alpha_1}\}$, where $g_j = \{x_r^{a_r-j+1}, x_s^{a_s-j+1}\}$ for $j \le \alpha_1$, then P_1 is a matching of $\mathcal{G}^a$ of size α_1. If $f_1 = (x_r, x_r) \in L$, then $2\alpha_1 \le a_r$ since $\widetilde{f}_1^{\alpha_1} \mid x^a$. Consequently, $P_1 = \{g_1, \ldots, g_{\alpha_1}\}$, where $g_j = \{x_r^{a_r-2j+2}, x_r^{a_r-2j+1}\}$ for $j \le \alpha_1$, is a matching of $\mathcal{G}^a$ of size α_1. Hence,

$$\mathcal{G}^b = \mathcal{G}^a \setminus \bigcup_{j=1}^{\alpha_1} g_j \text{ and } x^b = \frac{x^a}{\widetilde{f}_1^{\alpha_1}} = m\widetilde{f}_2^{\alpha_2} \cdots \widetilde{f}_q^{\alpha_q} \in I(\mathcal{G})^{l-\alpha_1},$$

where $b = a - \alpha_1(e_r + e_s)$ if $f_1 \in E$ or $b = a - 2\alpha_1 e_r$ if $f_1 \in L$. Following with the processes, we obtain matchings $P_1, \ldots, P_q$ such that

$$V(P_{i+1}) \cap \Big(\bigcup_{j=1}^{i} V(P_j)\Big) = \varnothing \text{ since } V(P_{i+1}) \subseteq V(\mathcal{G}^a) \setminus \bigcup_{j=1}^{i} V(P_j).$$

Therefore, $\bigcup_{j=1}^{q} P_j$ is a matching of $\mathcal{G}^a$ of size l. □

Corollary 1. *$x^a \in I(\mathcal{G})^k \setminus I(\mathcal{G})^{k+1}$ if and only if $k = \nu(\mathcal{G}^a)$.*

Definition 2. *The* deficiency *of a simple graph G is given by*

$$\operatorname{def}(G) = |V(G)| - 2\nu(G).$$

Theorem 1. *([2]) If G is a simple graph, then*

$$\operatorname{def}(G) = \max\{c_0(G \setminus S) - |S| \mid S \subseteq V(G)\},$$

where $c_0(G)$ denotes the number of odd components (components with an odd number of vertices) of G.

Proposition 2. *Let $\mathcal{G} = (V, E, L)$ be a graph with loops, so $\operatorname{def}(\mathcal{G}^{af}) = \delta$ for all $f \in F = E \cup L$ if and only if $\operatorname{def}(\mathcal{G}^a) = \delta$ and $\nu(\mathcal{G}^{af}) = \nu(\mathcal{G}^a) + 1$ for all $f \in F$.*

Proof. We take a maximum matching $g_1, \ldots, g_\ell$ of $\mathcal{G}^a$. If $f \in F$, then $g_1, \ldots, g_\ell, g$ is a matching of $\mathcal{G}^{af}$, where $g = \{x_i^{a_{i+1}}, x_j^{a_{j+1}}\}$ when $f = \{x_i, x_j\} \in E$ and $g = \{x_i^{a_i+1}, a_i^{a_i+2}\}$ when $f = (x_i, x_i) \in L$. Hence, $\nu(\mathcal{G}^{af}) \ge \nu(\mathcal{G}^a) + 1$. This implies $\operatorname{def}(\mathcal{G}^a) = |V(\mathcal{G}^a)| - 2\nu(\mathcal{G}^a) \ge |V(\mathcal{G}^{af})| - 2\nu(\mathcal{G}^{af})$ since $|V(\mathcal{G}^{af})| = |V(\mathcal{G}^a)| + 2$. Therefore, $\operatorname{def}(\mathcal{G}^a) \ge \operatorname{def}(\mathcal{G}^{af})$.

$\Rightarrow$) By contradiction, suppose $\operatorname{def}(\mathcal{G}^a) > \delta$. Thus, by Theorem 1, there is an $S \subseteq V(\mathcal{G}^a)$ such that $c_0(\mathcal{G}^a \setminus S) - |S| > \delta$. We set $r = c_0(\mathcal{G}^a \setminus S)$ and $H_1, \ldots, H_r$ the odd components of $\mathcal{G}^a \setminus S$. We take $x_i^{k_i} \in H_k$ for some $1 \le k \le r$ and $k_i \le a_i$. If $f = (x_i, x_i) \in L$, then we take the subgraph H'_k of $\mathcal{G}^{af} \setminus S$ induced by $V(H_k) \cup \{x_i^{a_i+1}, x_i^{a_i+2}\}$. We obtain that the odd connected components of $\mathcal{G}^{af} \setminus S$ are $H_1, H_2, \ldots, H_{k-1}, H'_k, H_{k+1} \ldots, H_r$. Consequently,

$$c_0(\mathcal{G}^{af} \setminus S) - |S| > \delta = \operatorname{def}(\mathcal{G}^{af}).$$

A contradiction. Now, assume $f = \{x_i, x_j\} \in E(\mathcal{G})$. If $\{x_i^{k_i}, x_j^{k_j}\} \in E(H_k)$, then we consider the subgraph $H_{k'}$ of $\mathcal{G}^{af} \setminus S$ induced by $V(H_k) \cup \{x_i^{a_i+1}, x_j^{a_j+1}\}$. We obtain that the odd connected components of $\mathcal{G}^{af} \setminus S$ are $H_1, H_2, \ldots, H_{k-1}, H'_k, H_{k+1} \ldots, H_r$. So,

$$c_0(\mathcal{G}^{af} \setminus S) - |S| > \delta = \operatorname{def}(\mathcal{G}^{af}).$$

This implies $V(H_k) = \{x_i^{k_i}\}$ and $a_j = 0$ or $x_j^{k_j} \in S$ for each $k_j \le a_j$. Hence, the odd components of $\mathcal{G}^{af} \setminus (S \cup \{x_j^{a_j+1}\})$ are $H_1, \dots, H_r, \{x_i^{a_i+1}\}$. Thus,

$$c_0(\mathcal{G}^{af} \setminus (S \cup \{x_j^{a_j+1}\})) - |S \cup \{x_j^{a_j+1}\}| = c_0(\mathcal{G}^{a} \setminus S) - |S| > \delta = \operatorname{def}(\mathcal{G}^{af}).$$

A contradiction, therefore $\operatorname{def}(\mathcal{G}^a) = \operatorname{def}(\mathcal{G}^{af})$ for all $f \in F$. Therefore, $\nu(G^{af}) = \nu(G^a) + 1$, since $|V(\mathcal{G}^{af})| = |V(\mathcal{G}^a)| + 2$ for all $f \in F$.

$\Leftarrow$) $\operatorname{def}(\mathcal{G}^{af}) = |V(\mathcal{G}^{af})| + 2\nu(\mathcal{G}^{af}) = |V(\mathcal{G}^a)| + 2 - 2(\nu(\mathcal{G}^a) + 1) = \operatorname{def}(\mathcal{G}^a) = \delta$. ◻

Theorem 2. *$I(G)$ has the strong persistence property if G is a simple graph.*

Proof. See [2, Lemma 2.12]. ◻

Theorem 3. *If $\mathcal{G}$ is a graph with loops, then $\left(I^{k+1}\colon I\right) = I^k$ with $I = I(\mathcal{G})$.*

Proof. We take a monomial $m = x^a \in (I^{k+1}\colon I)$. If $mf \in I^{k+2}$ for some $f = x_i x_j \in G(I)$, then $m(x_i x_j) = m' g_1 \cdots g_{k+2}$ with $g_i \in G(I)$ and $m' \in \operatorname{Mon}(R)$. Thus, $m \in I^k$. So, we can assume that $mf \in I^{k+1} \setminus I^{k+2}$ for each $f \in G(I)$. Consequently, by Corollary 1, $\nu(\mathcal{G}^{af}) = k + 1$ for each $f \in G(I)$. Hence, $\operatorname{def}(\mathcal{G}^{af}) = |V(\mathcal{G}^a)| + 2 - 2(k+1) = |V(\mathcal{G}^a)| - 2k$ for each $f \in G(I)$. Furthermore, by Proposition 2, $\operatorname{def}(\mathcal{G}^{af}) = \operatorname{def}(\mathcal{G}^a) = |V(\mathcal{G}^a)| - 2\nu(\mathcal{G}^a)$, then $\nu(\mathcal{G}^a) = k$. Therefore, by Proposition 1, $m = x^a \in I^k$. ◻

Corollary 2. *$I(\mathcal{G})$ has the persistence property if $\mathcal{G}$ is a graph with loops.*

Proof. By Theorem 3 and [1, Lemma 2.12]. ◻

3 Weighted monomial ideals

Let I be a monomial ideal, an irreducible monomial ideal J is I-minimal if J is minimal in the set of irreducible monomial ideals (have the form $\{x_{i_1}^{\alpha_{i_1}}, \dots, x_{i_s}^{\alpha_{i_s}}\}$), such that $I \subseteq J$. The set of I-minimal ideals is a minimal primary decomposition of I.

Definition 3. *For $m_1, m_2 \in \operatorname{Mon}(R)$, $m_1^s \parallel m_2$ if $m_1^s \mid m_2$ and $m_1^{s+1} \nmid m_2$.*

Proposition 3. *Let I be a monomial ideal. If $(x_{i_1}^{\alpha_{i_1}}, \dots, x_{i_s}^{\alpha_{i_s}})$ is a I-minimal ideal, then for each $1 \le t \le s$ there is $m \in G(I)$ such that $x_{i_t}^{\alpha_{i_t}} \parallel m$.*

Proof. Since $J = (x_{i_1}^{\alpha_{i_1}}, \dots, x_{i_s}^{\alpha_{i_s}})$ is an I-minimal ideal, then $I \subseteq J$. Thus, if $x_{i_t}^{\alpha_{i_t}} \nmid u$ for each $u \in G(I)$, then $I \subseteq \left(\{x_{i_1}^{\alpha_{i_1}}, \dots, x_{i_s}^{\alpha_{i_s}}\} \setminus \{x_{i_t}^{\alpha_{i_t}}\}\right)$. This contradicts the minimality of J. Hence, $x_{i_t}^{\alpha_{i_t}} \mid u$ for some $u \in G(I)$. Now, if $x_{i_t}^{\alpha_{i_t}+1} \mid m$ for each $m \in G(I)$ such that $x_{i_t}^{\alpha_{i_t}} \mid m$, then $I \subseteq (x_{i_1}^{\alpha_{i_1}}, \dots, x_{i_t}^{\alpha_{i_t}+1}, \dots, x_{i_s}^{\alpha_{i_s}}) \nsubseteq J$. A contradiction, therefore there is $m \in G(I)$ such that $x_{i_t}^{\alpha_{i_t}} \parallel m$. ◻

Definition 4. *A weight over a polynomial ring* $R = K[x_1,\dots,x_n]$ *is a function* $w : \{x_1,\dots,x_n\} \to \mathbb{N}$, $w_i = w(x_i)$ *is called the* weight *of the variable* x_i. *Given a monomial ideal* I *and a weight* w, *the* weighted ideal *of* I *and* w *is* $I_w = \big(h(m) \mid m \in G(I)\big)$ *where* h *is the isomorphism* $h : R \to K[x_1^{w_1},\dots,x_n^{w_n}]$ *given by* $x_i \mapsto x_i^{w_i}$.

Remark 1. *Since* h *is an isomorphism,* $G((I_w)^k) = G((I^k)_w)$, *so* $(I_w)^k = (I^k)_w$.

Theorem 4. *Let* I *be a monomial ideal and* w *a weight over* R, *then*

i) $\operatorname{Ass}(I_w^k) = \operatorname{Ass}(I^k)$ *for each* k;

ii) *I has the persistence property if and only if I_w has the persistence property;*

iii) *I has the strong persistence property if and only if I_w has the strong persistence property.*

Proof. i) If $(x_{i_1}^{\beta_{i_1}},\dots,x_{i_s}^{\beta_{i_s}})$ is an I_w^k-minimal ideal, then by Proposition 3 there is $m' \in G(I_w^k)$ such that $x_{i_j}^{\beta_{i_j}} \parallel m'$, so there is r_{i_j} such that $\beta_{i_j} = w_{i_j} r_{i_j}$ for $1 \le j \le s$. If $m = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \in G(I^k)$, then $h(m) = x_1^{\alpha_1 w_1}\cdots x_n^{\alpha_n w_n} \in G(I_w^k) \subseteq (x_{i_1}^{\beta_{i_1}},\dots,x_{i_s}^{\beta_{i_s}})$. Hence, there exist $t \le s$, such that $x_{i_t}^{\beta_{i_t}} \mid h(m)$. Thus, $w_{i_t} r_{i_t} = \beta_{i_k} \le w_{i_t}\alpha_{i_t}$ implies $r_{i_t} \le \alpha_{i_t}$ and $x_{i_t}^{r_{i_t}} \mid m$. Consequently $I^k \subseteq (x_{i_1}^{r_1},\dots,x_{i_s}^{r_s})$. Now, if $(x_{j_1}^{\alpha_{j_1}},\dots,x_{j_l}^{\alpha_{j_l}})$ is an I^k-minimal, then $I_w^k \subseteq (x_{j_1}^{w_{j_1}\alpha_{j_1}},\dots,x_{j_l}^{w_{j_l}\alpha_{j_l}})$. So, $(x_{i_1}^{r_1},\dots,x_{i_s}^{r_s}$ is I^k-minimal and $\big(x_{j_1}^{w_{j_1}\alpha_{j_1}},\dots,x_{j_l}^{w_{j_l}\alpha_{j_l}}\big)$ is I_w^k-minimal. Therefore, $(x_{i_1}^{\alpha_{i_1}},\dots,x_{i_s}^{\alpha_{i_s}})$ is I^k-minimal if and only if $(x_{i_1}^{w_{i_1}\alpha_{i_1}},\ \dots,x_{i_s}^{w_{i_s}\alpha_{i_s}})$ is I_w^k-minimal. Taking radicals of the I_w^k-minimal and I^k-minimal ideals we obtain $\operatorname{Ass}(I_w^k) = \operatorname{Ass}(I^k)$.

ii) By i).

iii) $\Rightarrow$) Since h is an isomorphism of k-algebras between $R' = K[x_1^{w_1},\dots,x_n^{w_n}]$ and R, $h(I)$ has the strong persistence property in $K[x_1^{w_1},\dots,x_n^{w_n}]$. Also, $m = x_1^{\lambda_1}\cdots x_n^{\lambda_n} \in R'$ if and only if $w_i \mid \lambda_i$ for each i. Thus, $I_w \cap K[x_1^{w_1},\dots,x_n^{w_n}] = h(I)$. Now, if $m \in (I_w^{k+1} \colon I_w)$, then $gm = \ell g_1\cdots g_{k+1}$ for each $g \in G(I_w)$ where $g_i \in G(I_w)$. We take $m = x_1^{a_1}\cdots x_1^{a_n}$ and $\ell = x_1^{b_1}\cdots x_n^{b_n}$. If r_i and t_i are the remainders obtained by dividing a_i and b_i by w_i respectively, then $w_i \mid r_i - t_i$, since $G(I_w) \subseteq K[x_1^{w_1},\dots,x_n^{w_n}]$. So, $r_i = t_i$ and we take $m' = x_1^{a_1-r_1}\cdots x_n^{a_n-r_n}$ and $\ell' = x_1^{b_1-r_1}\cdots x_n^{b_n-r_r}$. Hence, $m',\ell' \in K[x_1^{w_1},\dots,x_n^{w_n}]$ and $gm' = \ell' g_1\cdots g_{k+1}$. Since $G(I_w) = G(h(I))$, $m' \in \big(h(I)^{k+1} \colon h(I)\big) = h(I)^k$ implies $m' \in I_w^k$. Therefore $m \in I_w^k$, since $m' \mid m$.

$\Leftarrow$) We take $m \in (I^{k+1} \colon I) \cap \operatorname{Mon}(R)$, then $mf = \ell f_1\cdots f_{k+1}$ with $f, f_1,\dots,f_{k+1} \in G(I)$. So $h(m)h(f) = h(\ell)h(f_1)\cdots h(f_{k+1}) \in I_w^{k+1}$. Thus, $h(m) \in (I_w^{k+1} \colon I_w) = I_w^k$ since $G(I_w) = G(h(I))$. This implies, $h(m) = \ell g_1\cdots g_k$ with $g_i = h(g_i') \in G(I_w)$ where $g_i' \in G(I)$. Since $h(m) \in R' = K[x_1^{w_1},\dots,x_n^{w_n}]$, $\ell \in R'$. Therefore, $m = h^{-1}(\ell) g_1'\cdots g_k' \in I^k$, since h is an isomorphism. ∎

Definition 5. *A* weighted graph (G,w) *consists of a simple graph* G *and a function* $w : V(G) \to \mathbb{N}$. *The* weight *of* $x \in V(G)$ *is* $w(x)$.

Definition 6. *The edge ideal of the weighted graph* (G,w) *denoted by* $I(G,w)$ *is the ideal generated by* $\{x_i^{w_i} x_j^{w_j} \mid x_i x_j \in E(G)\}$, *where* $w_k = w(x_k)$.

Corollary 3. *If $I = I(G)$ and $J = I(G, w)$, then $\operatorname{Ass}(J^k) = \operatorname{Ass}(I^k)$ for all k.*

Proof. By Theorem 4, since $J = I_w$. ∎

Theorem 5. *The edge ideal $I(G, w)$ has the strong persistence property.*

Proof. By Theorem 4, since $I(G)$ has the strong persistence property. ∎

4 Squarefree monomial ideal

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring. A monomial ideal I is *squarefree* if $G(I)$ consists of squarefree monomials. A *clutter* is a pair $\mathcal{C} = (V, E)$ where V is a finite set and E is a set of subsets of V such that if $a \subseteq b$ with $a, b \in E$, then $a = b$. The sets $V = V(\mathcal{C})$ and $E = E(\mathcal{C})$ are called vertex set and edge set, respectively. If $f = \{x_{i_1}, \ldots, x_{i_r}\} \in E(\mathcal{C})$, then we denote by $\widetilde{f}$ the squarefree monomial $x_{i_1} \cdots x_{i_r}$. Hence, if $f_1 \subseteq f_2 \subseteq X = \{x_1, \ldots, x_n\}$, then $\widetilde{f_1} \mid \widetilde{f_2}$. The edge ideal of the clutter $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal generated by $\{\widetilde{f} \mid f \in E(\mathcal{C})\}$. This assignment defines a natural bijection between squarefree monomial ideals of $K[x_1, \ldots, x_n]$ and clutters whose vertex set is X. Finally we say that a clutter $\mathcal{C}$ has the strong persistence property if $I(\mathcal{C})$ has the strong persistence property.

Lemma 1. *Let f, g be squarefree monomials, if there exists an integer $k \geq 2$ such that $f^k \mid mg$, then $f^{k-1} \mid m$.*

Proof. Since $f^k \mid mg$, $mg = f^k \ell$ with $\ell \in \operatorname{Mon}(R)$. We take $m' = \gcd(f, g)$, then $f = m'f'$ and $g = m'g'$ with $\gcd(f', g') = 1$. Hence, $\gcd(f, g') = \gcd(m', g') = u$. Consequently $u^2 \mid g$. But g is a squarefree monomial, so $\gcd(f, g') = 1$. Thus $g' \mid \ell$, since $mg' = f' f^{k-1} \ell$. Therefore $m = f^{k-1}(f'u')$ where $\ell = u'g'$ implies $f^{k-1} \mid m$. ∎

Corollary 4. *Let I be a squarefree monomial ideal. If $G(I)$ has at most two elements, then I has the strong persistence property.*

Proof. Let m be a monomial in $(I^{k+1} : I)$. So, for each $f \in G(I)$ there are monomials $\ell, g_1, \ldots, g_{k+1}$ with $g_i \in G(I)$, such that $mf = \ell g_1 \cdots g_{k+1}$. If $f = g_i$ for some i, then $m \in I^k$. Now, if $f \neq g_i$ for each i, then $g_i = g_1$ since $|G(I)| \leq 2$. Thus $g_1^{k+1} \mid mf$. Hence, by Lemma 1, $g_1^k \mid m$ and $m \in I^k$. ∎

Theorem 6. *If I is a squarefree monomial ideal, then $(I^2 : I) = I$.*

Proof. Let m be a monomial in $(I^2 : I)$, then for each $f_1 \in G(I)$ there are $h_1, g_1 \in G(I)$ and a monomial ℓ_1 such that $mf_1 = \ell_1 g_1 h_1$. Consequently,

$$m^2 f_1 = \ell_1 (m g_1) h_1 = \ell_1 \ell_2 g_2 h_2 h_1$$

where $mg_1 = \ell_2 g_2 h_2$ and $g_2, h_2 \in G(I)$. Follows, multiplying by m we obtain

$$m^r f_1 = \ell_1 \cdots \ell_r g_r h_r \cdots h_2 h_1,$$

where $mg_{i-1} = \ell_i g_i h_i$ and $g_i, h_i \in G(I)$ for $2 \le i \le r$. If $r \ge |G(I)|$, then $g_r = h_j$ or $h_j = h_i$ for some $1 \le i < j \le r$. Hence, $h_j^2 \mid m^r f_1$ and by Lemma 1, $h_j \mid m^r$. Thus, $h_j \mid m$, since h_j is squarefree. Therefore $m \in I$. □

Corollary 5. *If I is a squarefree monomial ideal and $k \ge 2$, then $(I^k{:}I) \subseteq I$.*

Proof. By Theorem 6, $(I^k{:}I) \subseteq I$. Hence, $I^k \subseteq I^2$ and $(I^k{:}I) \subseteq (I^2{:}I)$. □

Theorem 7. *A clutter has the strong persistence property if and only if some of its connected components has the strong persistence property.*

Proof. Let $\mathcal{C}_1, \dots, \mathcal{C}_r$ the connected components of $\mathcal{C}$ with $V_i = V(\mathcal{C}_i)$.
$\Leftarrow$) We can suppose that $\mathcal{C}_1$ has the strong persistence property. We take a monomial $m \in (I^{k+1}{:}I)$. We can write $m = m_1 \cdots m_r$ where $m_i \in \mathrm{Mon}(K[V_i])$ and we take a_i such that $m_i \in I_i^{a_i} \setminus I_i^{a_i+1}$. For each $f \in \mathcal{C}_1$ we consider s_f such that $m_1 f \in I_1^{s_f} \setminus I_1^{s_f+1}$ and $s_1 = \min\{s_f \mid f \in \mathcal{C}_1\}$. Thus $m_1 f \in I_1^{s_1}$ for each $f \in \mathcal{C}_1$, so $m_1 \in \left(I_1^{s_1}{:}I_1\right) = I_1^{s-1}$. Hence,

$$m \in I^{s_1 - 1 + \sum_{i=2}^{r} a_i} \text{ and } mf \in I^{s_f + \sum_{i=2}^{r} a_i} \setminus I^{s_f + 1 + \sum_{i=2}^{r} a_i} \text{ for each } f \in \mathcal{C}_1.$$

Since $mf \in I^{k+1}$, $s_f + \sum_{i=2}^{r} a_i \ge k+1$. Then, $s_1 + \sum_{i=2}^{r} a_i \ge k+1$. Therefore $m \in I^k$.
$\Rightarrow$) If $I_i = I(\mathcal{C}_i)$ has no the strong persistence property, then there is k_i and a monomial $m_i \in (I_i^{k_i+1}{:}I_i) \setminus I_i^{k_i}$. We take a_i such that $m_i \in I_i^{a_i} \setminus I_i^{a_i+1}$, then $a_i \le k_i - 1$. Now, we consider $m = m_1 \cdots m_r$, then $m \in I^b \setminus I^{b+1}$, for $b = \sum_{i=1}^{r} a_i$. If we take $f_i \in E(\mathcal{C}_i)$, then $mf_i \in I^{s_i}$, where $s_i = a_1 + \cdots + k_i + 1 + \cdots + a_r$. But $s_i \ge \sum_{j=1}^{r} a_j + 2$, thus $s = \min\{s_1, \dots, s_r\} \ge \sum_{j=1}^{r} a_j + 2$. Therefore $m \in (I^s{:}I) \setminus I^{s-1}$. □

Example 2. *Let $\mathcal{C}$ be a clutter. If $f_1, f_2 \in \{A \subseteq V(G) \mid A \cap f = \varnothing \text{ if } f \in E(\mathcal{C})\}$, then by Theorem 7 and Corollary 4, $\mathcal{C} \cup \{f_1, f_2\}$ has the strong persistence property.*

Lemma 2. *Let $\mathcal{C}$ be a clutter. If there exists an edge $f \in E(\mathcal{C})$ such that $A = \{g \cap f \mid g \in E(\mathcal{C})\}$ is a chain, then $I(\mathcal{C})$ has the strong persistence property.*

Proof. If m is a monomial in $(I^{k+1}{:}I)$, then $m\widetilde{f} = \widetilde{g}\widetilde{f}_1 \cdots \widetilde{f}_{k+1}$ where $f_i \in E(\mathcal{C})$ and $g \subseteq V(\mathcal{C})$. So, $f \subseteq g \cup f_1 \cup \cdots \cup f_{k+1}$. Since A is a chain, we can assume $f_{k+1} \cap f \subseteq f_k \cap f \subseteq \cdots \subseteq f_1 \cap f$. Thus, $f \subseteq g \cup f_1$ and $\widetilde{f} \mid \widetilde{g}\widetilde{f}_1$. Therefore $m \in I^k$. □

Corollary 6. *If $\mathcal{C}$ is a clutter without the strong persistence property, then for $f \in E(\mathcal{C})$ there are $f_1, f_2 \in E(\mathcal{C})$ such that $f \cap f_1 \nsubseteq f \cap f_2$ and $f \cap f_2 \nsubseteq f \cap f_1$.*

Definition 7. *Let $\mathcal{C}$ be a clutter, $A \subseteq V(\mathcal{C})$ is a vertex cover if $A \cap e \neq \varnothing$ for each $e \in E(\mathcal{C})$. The cover number of $\mathcal{C}$ is $\tau(\mathcal{C}) = \min\{|A| \mid A \text{ is a vertex cover}\}$. $\mathcal{C}$ is unmixed if $|B| = \tau(\mathcal{C})$ for each minimal vertex cover B. A* matching *is a set of disjoint edges $\{e_1, \dots, e_s\}$ of $\mathcal{C}$. It is* perfect *if $\cup_{i=1}^{s} e_i = V(\mathcal{C})$. Furthermore, $\mathcal{C}$ is König if there is a matching with $\tau(\mathcal{C})$ edges.*

Proposition 4. *Let $\mathcal{C}$ be a König clutter, then $\mathcal{C}$ is unmixed if and only if there is a perfect matching $e_1, \dots, e_g$ with $g = \tau(\mathcal{C})$, such that for any two edges $e \neq e'$ and for any two distinct vertices $x \in e$, $y \in e'$ contained in some e_i, one has that $(e \setminus \{x\}) \cup (e' \setminus \{y\})$ contains an edge.*

Proof. See Corollary 2.11 in [3]. ∎

Definition 8. *The incidence matrix of a clutter $\mathcal{C}$, denoted by $A_{\mathcal{C}}$, is the matrix whose columns are the characteristic vectors of the edges of $\mathcal{C}$. A r-cycle of $\mathcal{C}$ is a $r \times r$-submatrix of $A_{\mathcal{C}}$ with exactly two 1's in each row and each column.*

Theorem 8. *Let $\mathcal{C}$ be a König unmixed clutter. If $\mathcal{C}$ does not contain 4-cycles, then $\mathcal{C}$ has the strong persistence property.*

Proof. By Proposition 4, $\mathcal{C}$ has a perfect matching $e_1, \dots, e_s$ where $s = \tau(\mathcal{C})$. If $\mathcal{C}$ does not have the strong persistence property, then by Corollary 6 there exist $f_1, f_2 \in E(\mathcal{C})$ and vertices $x_1 \in (f_1 \cap e_1) \setminus f_2$ and $x_2 \in (f_2 \cap e_1) \setminus f_1$. Now by Proposition 4, there exist $f \in E(\mathcal{C})$ such that $f \subseteq (f_1 \setminus x_1) \cup (f_2 \setminus x_2)$. We can assume $e_1 \cap (f_2 \cup f_1)$ is minimal in

$$B = \{e_1 \cap (g_2 \cup g_1) \mid g_1, g_2 \in E(\mathcal{C}),\ g_2 \cap e_1 \notin g_1 \cap e_1, \text{ and } g_1 \cap e_1 \notin g_2 \cap e_1\}.$$

Thus, $(e_1 \cap f) \subseteq e_1 \cap ((f_1 \setminus x_1) \cup (f_2 \setminus x_2)) = e_1 \cap (f_1 \cup f_2 \setminus x_1 x_2)$. Hence, $e_1 \cap (f_i \cup f) \subseteq (e_1 \cap (f_1 \cup f_2)) \setminus x_j$ where $\{i, j\} = \{1, 2\}$. Since $e_1 \cap (f_1 \cap f_2)$ is monomial in B, $e_1 \cap f \subseteq e_1 \cap f_2$ or $e_1 \cap f_2 \subseteq e_1 \cap f$. But $x_2 \in (e_1 \cap f_2) \setminus (e_1 \cap f)$, then $e_1 \cap f \subseteq e_1 \cap f_2$. Now, if $(f_1 \cap f) \subseteq (e_1 \cup f_2)$, then $f \subseteq (f_1 \cup f_2) \cap f \subseteq (f_1 \cap f) \cup (f_2 \cap f) \subseteq (e_1 \cup f_2) \cup (f_2 \cap f) \subseteq (e_1 \cup f_2)$. So, $f \subseteq (e_1 \cap f) \cup f_2 \subseteq (e_1 \cap f_2) \cup f_2 \subseteq f_2$. But $x_2 \in f_2 \setminus f$, a contradiction. Hence, there is $y_1 \in (f_1 \cap f) \setminus (e_1 \cup f_2)$. Similarly there is $y_2 \in (f_2 \cap f) \setminus (e_1 \cup f_1)$. Consequently, the matrix

$$\begin{array}{c} \\ f_1 \\ f_2 \\ e_1 \\ f \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

is a 4-cycle. A contradiction, therefore $\mathcal{C}$ has the strong persistence property. ∎

Example 3 ([1])**.** *Let $\mathcal{C}_0$ be the clutter with vertex set $\{x_1, \dots, x_6\}$ whose edges are $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_5$, $x_1x_4x_6$, $x_1x_5x_6$, $x_2x_3x_6$, $x_2x_4x_5$, $x_2x_5x_6$, $x_3x_4x_5$ and $x_3x_4x_6$. $\mathcal{C}_0$ is an unmixed shellable clutter. But $\big(I(\mathcal{C}_0)^3 : I(\mathcal{C}_0)\big) \neq I(\mathcal{C}_0)^2$, then $\mathcal{C}_0$ has no the strong persistence property.*

Definition 9. *The* cone *over a clutter $\mathcal{C}$, denoted by $\mathcal{C}x$, is the clutter whose vertex set is $V(\mathcal{C}) \cup \{x\}$ and edge set $\{f \cup \{x\} \mid f \in E(\mathcal{C})\}$, where x is a new vertex.*

Proposition 5. *$\mathcal{C}$ has the strong persistence property if and only if $\mathcal{C}x$ has the strong persistence property.*

Proof. ⇒) If $m = x^{\alpha}m' \in \big(I(\mathcal{C}x)^{k+1}{:}I(\mathcal{C}x)\big)$ with $\gcd(m',x) = 1$, then $\widetilde{f}m \in I(\mathcal{C}x)^{k+1}$ for $f \in E(\mathcal{C}x)$. Furthermore $\widetilde{f} = \widetilde{g}x$ with $g \in E(G)$ then $x^{k+1} \mid \widetilde{g}xm$ implying $x^k \mid m$. Thus, $\alpha \geq k$ and $\widetilde{g}m' \in I(\mathcal{C})^{k+1}$. Hence $m' \in \big(I(\mathcal{C})^{k+1}{:}I(\mathcal{C})\big) = I(\mathcal{C})^k$, *i. e.*, $m' = \ell\widetilde{f}_1\cdots\widetilde{f}_k$ where $f_i \in E(\mathcal{C})$. Therefore, $m = x^{\alpha}\ell\widetilde{f}_1\cdots\widetilde{f}_k = x^{\alpha-k}\ell(\widetilde{f}_1x)\cdots(\widetilde{f}_kx)$, so $m \in I(\mathcal{C}x)^k$.
⇐) If $m \in \big(I(\mathcal{C})^{k+1}{:}I(\mathcal{C})\big)$, then $fm = \ell g_1\cdots g_{k+1}$ for each $f \in I(\mathcal{C})$ and $g_i \in I(\mathcal{C})$. Thus, $(fx)(mx^k) = \ell(xg_1)\cdots(xg_{k+1}) \in I(\mathcal{C}x)^{k+1}$. So, $mx^k \in \big(I(\mathcal{C}x)^{k+1}{:}I(\mathcal{C}x)\big) = I(\mathcal{C}x)^k$. Hence, $mx^k = \ell(f_1x)\cdots(f_kx)$ for $f_i \in I(\mathcal{C})$. Therefore $m \in I(\mathcal{C})^k$. □

Proposition 6. *$\mathcal{C}$ has the persistence property if and only if $\mathcal{C}x$ has the persistence property.*

Proof. If $Q_1,\ldots,Q_r$ is the monomial minimal primary decomposition of $I(\mathcal{C})^k$ and $Q_i' = R[x]\cdot Q_i$, then $Q_1',\ldots,Q_r',(x^k)$ is the monomial minimal primary decomposition of $I(\mathcal{C}x)^k$. Hence, $\operatorname{Ass}(I(\mathcal{C}x)^k) = \operatorname{Ass}(I(\mathcal{C})^k) \cup \{(x)\}$. □

Proposition 7. *$\mathcal{C} = (V,E)$ has the strong persistence property if and only if $\mathcal{C}' = (V,E')$ has the strong persistence property, where $E' = \big\{f \setminus \cap_{g\in E} g \mid f \in E\big\}$.*

Proof. Set $A = \cap_{g\in E} g$. By induction on $k = |A|$. If $k = 0$, then $\mathcal{C} = \mathcal{C}'$. Now if $k \geq 1$ and $x \in A$, then $\mathcal{C} = \mathcal{C}_1x$ where $\mathcal{C}_1 = \mathcal{C} \setminus x$. So, by induction hypothesis $\mathcal{C}_1$ has the strong persistence property if and only if $\mathcal{C}'$ has the strong persistence property. Therefore, we obtain the result by Proposition 5. □

Proposition 8. *A clutter $\mathcal{C}$ with 3 edges has the strong persistence property.*

Proof. We assume $E(\mathcal{C}) = \{f_1,f_2,f_3\}$ and $V(X) = \{x_1,\ldots,x_n\}$. By Proposition 7, we can suppose that $f_1 \cap f_2 \cap f_3 = \varnothing$. If $\mathcal{C}$ is not connected, then it has a component with one edge. Hence, by Corollary 4 and Theorem 7, $\mathcal{C}$ has the strong persistence property. Now, we assume that $\mathcal{C}$ is connected. If $f_i \cap f_j = \varnothing$ for some $i \neq j$, then $\mathcal{C}$ has the strong persistence property by Lemma 2. Consequently, we suppose $a_{ij} = f_i \cap f_j \neq \varnothing$ for $i \neq j$. We set b_i such that $f_i = a_{ij} \uplus b_i \uplus a_{ir}$ for $\{i,j,r\} = \{1,2,3\}$. So, each pair of $b_1,b_2,b_3,a_{12},a_{13},a_{23}$ are disjoint. We take $m \in (I^{k+1}{:}I)$ where $I = I(\mathcal{C})$, then $m\widetilde{f}_1 = \ell\widetilde{f}_1^{\alpha_1}\widetilde{f}_2^{\alpha_2}\widetilde{f}_3^{\alpha_3}$ with $\alpha_1+\alpha_2+\alpha_3 = k+1$. If $\alpha_1 > 0$, then $m \in I^k$. Now, if $\alpha_1 = 0$, then $b_1 \mid \ell$ since b_1, f_1, f_3 are disjoint pairs. This implies, $\ell = b_1\ell'$ and $ma_{12}a_{13} = \ell'\widetilde{f}_2^{\alpha_2}\widetilde{f}_3^{\alpha_3}$. If $\alpha_2 = 0$, then $\ell' = u_1a_{12}$ and $m = u_1u_2\widetilde{f}_3^{\alpha_k-1}$ where $\widetilde{f}_3 = u_2a_{13}$. Thus, $m \in I^k$. Similarly if $\alpha_3 = 0$, then we suppose $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Consequently $m = \ell'(b_2b_3a_{23}^2)\widetilde{f}_2^{\alpha_2-1}\widetilde{f}_3^{\alpha_3-1}$, implying $a_{23}^{k+1} \mid m$ and $b_2^{\alpha_2}b_3^{\alpha_3} \mid m$. Similarly, we can assume $a_{12}^{k+1} \mid m$ and $a_{13}^{k+1} \mid m$. Hence, $(a_{12}a_{13}a_{23})^{k+1}b_2^{\alpha_2}b_3^{\alpha_3} \mid m$ so $\widetilde{f}_2^{\alpha_2}\widetilde{f}_3^{\alpha_3} \mid m$, since $\alpha_2+\alpha_3 = k+1$. Therefore, $m \in I^{k+1} \subseteq I^k$. □

Proposition 9. *If X is a set $A \subseteq X$ and $x \notin X$, then the clutter $\mathcal{C}$ whose edge set is $\{X\} \cup \{xx_i \mid x_i \in A\}$ has the strong persistence property.*

Proof. We set $A = \{x_1, \ldots, x_r\}$, $f_0 = X$ and $f_i = \{x, x_i\}$. Since $\mathcal{C}$ is clutter, $r > 1$. We take $m \in (I^{k+1}\colon I)$ where $I = I(\mathcal{C})$, then $m\widetilde{f_i} = \ell_i \widetilde{f_0}^{\alpha_{0i}} \widetilde{f_1}^{\alpha_{1i}} \cdots \widetilde{f_r}^{\alpha_{ri}}$ where $\sum_{j=0}^{r} \alpha_{ji} = k+1$. If $\alpha_{0i} = 0$ for each $i \geq 1$, then $m \in \left(J^{k+1}\colon J\right)$, where $J = (\widetilde{f_1}, \ldots, \widetilde{f_r})$. But J is an edge ideal of a graph so, by Theorem 2, $m \in J^k \subseteq I^k$. Thus, we can assume $\alpha_{01} > 0$ and we take $\alpha_i = \alpha_{i1}$. If $\alpha_1 = 0$ and $x \nmid \ell_1$, then $x^{k-\alpha_0} \parallel m$ and $\widetilde{f_0}^{\alpha_0 - 1} \parallel m$, since

$$m = \ell_1 \frac{\widetilde{f_0}^{\alpha_2}}{x_1} \cdot \frac{\widetilde{f_2}^{\alpha_2} \cdots \widetilde{f_n}^{\alpha_n}}{x} \quad \text{and} \quad \widetilde{f_0} \nmid \widetilde{f_2}^{\alpha_2} \cdots \widetilde{f_n}^{\alpha_n}.$$

So, $x^{k-\alpha_0+1} \parallel m\widetilde{f_j}$ and $\widetilde{f_0}^{\alpha_0-1} \parallel m\widetilde{f_j}$ for $j \neq 1$. Hence, $m\widetilde{f_j} \notin I^{k+1}$ a contradiction. Now if $\alpha_1 \neq 0$ or $x \mid \ell_1$, then $m = \ell_1 \widetilde{f_0}^{\alpha_0} \widetilde{f_1}^{\alpha_1 - 1} \widetilde{f_2}^{\alpha_2} \cdots \widetilde{f_n}^{\alpha_n}$ or $m = ab\widetilde{f_0}^{\alpha_0-1} \widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_n}^{\alpha_n}$, where $\ell_1 = xa$ and $\widetilde{f_0} = x_1 b$. Therefore $m \in I^k$. □

Theorem 9. *If I is a squarefree monomial ideal in $K[x_1, x_2, x_3, x_4]$, then I has the strong persistence property.*

Proof. Let $\mathcal{C}$ be the clutter associated to I. By Proposition 8 and Theorem 7 we can assume that $|E(\mathcal{C})| > 3$ and $\mathcal{C}$ has no edges of cardinality 1. If $\mathcal{C}$ has only edges of cardinality 3, then $4 \leq |E(\mathcal{C})| \leq \binom{4}{3} = 4$. Hence, $\mathcal{C}$ is a complete clutter, implies $\mathcal{C}$ is a base set of a polymatroid. Consequently, by [1, Proposition 2.4] $\mathcal{C}$ has the strong persistence property. If $\mathcal{C}$ has only one edge of cardinality 2, then $|E(\mathcal{C})| \leq 3$. A contradiction, so there are $f_1, f_2 \in E(\mathcal{C})$ such that $|f_1| = |f_2| = 2$. By Theorem 2 we can suppose $f = \{x_1, x_2, x_3\} \in E(\mathcal{C})$. So, if $f' \in E(\mathcal{C}) \setminus \{f\}$, then $x_4 \in f'$. Hence, we can assume $f_1 = \{x_1, x_4\}$ and $f_2 = \{x_2, x_4\}$. Thus, if $f' \in E(\mathcal{C}) \setminus \{f_1 f_1, f_2\}$, then $f' = \{x_3, x_4\}$. Therefore, by Proposition 9, $\mathcal{C}$ has the strong persistence property. □

Corollary 7. *If $I \subseteq K[x_1, \ldots, x_n]$ is a squarefree monomial ideal without the strong persistence property, then $n \geq 5$ and there is $k \geq 3$ such that $(I^k\colon I) \neq I^{k-1}$.*

Proof. By Theorem 9 and Theorem 6. □

Definition 10. *Let $\mathcal{C} = (V, E)$ be a clutter with $x \in V$, the deleting of x is the clutter $\mathcal{C} \setminus x$ with vertex set $V \setminus \{x\}$ and edge set $\{f \in E \mid x \notin f\}$. Furthermore, the contraction of x is the clutter $\mathcal{C} / x$ with vertex set $V \setminus \{x\}$ and whose edges are $f \setminus \{x\}$ with $f \in E$ and there is not $f' \in E$ such that $f' \setminus \{x\} \subset f \setminus \{x\}$.*

Example 4. *We consider the clutter $\mathcal{C}$ with vertex set $V(\mathcal{C}_0) \cup \{x\}$ and edge set $E(\mathcal{C}_0) \cup \{xx_1\}$, where $\mathcal{C}_0$ is the clutter in Example 3. By Theorem 7, $I(\mathcal{C})$ has the strong property but $\mathcal{C} \setminus x = \mathcal{C}_0$ has no the strong persistence property.*

Proposition 10. *Let $\mathcal{C}$ be a clutter and $x \in V(\mathcal{C})$. If $\mathcal{C}$ has the (strong) persistence property, then $\mathcal{C} / x$ has the (strong) persistence property.*

Proof. We set $E(\mathcal{C}) = \{f_1, \dots, f_r\}$. We can suppose $\{f_i \mid x \in f_i\} = \{f_1, \dots, f_{r_1}\}$ and $\{f_i \mid f_j \setminus \{x\} \nsubseteq f_i$ for each $j \le r_1\} = \{f_{r_1+1}, \dots, f_{r_2}\}$. We define $f'_i = f_i \setminus \{x\}$ for $i \le r_2$ and $A = \bigcup_{i \le r_2} f'_i$. Also, we set $I = I(\mathcal{C} \diagup x)$ and $J = I(\mathcal{C})$. Thus, $f'_1, \dots, f'_{r_2}$ are the edges of $\mathcal{C} \diagup x$ and $f'_i = f_i$ for $r_1 + 1 \le i$. Furthermore, if $i > r_2$, then $f'_j \subseteq f_i$ for some j. So, for each $1 \le i \le r$ there is $j \le r_2$ such that $\widetilde{f}'_j \mid f_i$. Consequently, if $m \in G(J^k)$, then there is $m' \in G(I^k)$ such that $m' \mid m$. We take $\mathcal{L} = (z_1, \dots, z_s)$ where $z_j = x_j^{\beta_{i_j}}$. Hence if $\mathcal{L}$ is an I^k-minimal ideal, then $J^k \subseteq \mathcal{L}$. Furthermore, $G(\mathrm{r}(\mathcal{L})) = \{x_{i_1}, \dots, x_{i_s}\} \subseteq A$ since $\mathcal{L}$ is I^k-minimal. Now, we suppose $\mathcal{L}$ is J^k-minimal and $G(\mathrm{r}(\mathcal{L})) \subseteq A$. If $m \in G(I^k)$, then $m = \widetilde{f}_1'^{a_1} \cdots \widetilde{f}_{r_2}'^{a_{r_2}}$ with $a_1 + \cdots + a_{r_2} = k$. So, $x^\alpha m = \widetilde{f}_1^{a_1} \cdots \widetilde{f}_{r_2}^{a_{r_2}} \in J^k$, where $\alpha = a_1 + \cdots + a_{r_1}$. Thus, $z_j \mid x^\alpha m$ for some $j \le s$. Since $x \notin A$, $\gcd(x, z_j) = 1$, and $z_j \mid m$. Therefore $I^k \subseteq \mathcal{L}$. Now, we will prove that $\mathcal{L}$ is an I^k-minimal ideal if and only if $\mathcal{L}$ is an J^k-minimal ideal and $G(\mathrm{r}(\mathcal{L})) \subseteq A$. Assume $\mathcal{L}$ is I^k-minimal so, $\mathrm{r}(\mathcal{L}) \subseteq A$. If $\mathcal{L}$ is not J^k-minimal, then there is $\mathcal{L}'$ such that $J^k \subseteq \mathcal{L}' \subset \mathcal{L}$ and $\mathrm{r}(\mathcal{L}') \subseteq \mathrm{r}(\mathcal{L}) \subseteq A$. Consequently, $I^k \subseteq \mathcal{L}'$. A contradiction, therefore $\mathcal{L}$ is J^k-minimal. Now suppose $\mathcal{L}$ is J^k-minimal and $\mathcal{L}$ is not I^k-minimal, then there is $\mathcal{L}'$ such that $I^k \subseteq \mathcal{L}' \subset \mathcal{L}$. This implies $J^k \subseteq \mathcal{L}'$, a contradiction, since $\mathcal{L}$ is J^k-minimal.

Hence, $\mathrm{Ass}(I^k) = \{P \in \mathrm{Ass}(J^k) \mid G(P) \subseteq A\}$ for each k. Since J has the persistence property, if $P \in \mathrm{Ass}(I^k)$, then $P \in \mathrm{Ass}(J^{k+1})$ and $G(P) \subseteq A$. Thus, $P \in \mathrm{Ass}(I^{k+1})$. Therefore, I has the persistence property.

(Strong). Now, we set $m \in (I^{k+1} \colon I)$. If $1 \le i \le r_2$, then $mf'_i = \ell_i f_1'^{\alpha_{i1}} \cdots f_r'^{\alpha_{ir_2}}$ where $\ell \in \mathrm{Mon}(R)$ and $\alpha_{i1} + \cdots + \alpha_{ir_2} = k + 1$. We take $u_i = \alpha_{i1} + \cdots + \alpha_{ir_1}$. If $i \le r_1$, then

$$x^{k+1} m f_i = x^{k+2} m f'_i = x^{k+2} \ell_i (f'_1)^{\alpha_{i1}} \cdots (f'_{r_2})^{\alpha_{ir_2}} = x^{k+2-u_i} \ell_i f_1^{\alpha_{i1}} \cdots f_{r_2}^{\alpha_{ir_2}}.$$

Now if $r_1 + 1 \le i \le r_2$, then $x^{k+1} m f_i = x^{k+1} m f'_i = x^{k+1-u} \ell f_1^{\alpha_{i1}} \cdots f_{r_2}^{\alpha_{ir_2}}$. Finally if $r_2 + 1 \le i \le r$, then there exist $j \le r_1$ such that $f'_j \mid f_i$. So,

$$x^{k+1} m f_i = \frac{f_i}{f'_j} x^{k+1} m f'_j = \frac{f_i}{f'_j} x^{k+1-u_j} \ell_j f_1^{\alpha_{j1}} \cdots f_{r_2}^{\alpha_{jr_2}}.$$

Consequently, $x^{k+1} m \in (J^{k+1} \colon J) = J^k$. This implies $x^{k+1} m = \ell f_1^{\beta_1} \cdots f_r^{\beta_r}$ with $\ell \in \mathrm{Mon}(R)$ and $\beta_1 + \cdots + \beta_r = k$. Since $x \nmid f_j$ for $j \ge r_1 + 1$, $x^w \mid \ell$, where $w = k + 1 - (\beta_1 + \cdots + \beta_{r_1})$. Therefore, $\ell = x^w \ell'$ where $\ell' \in \mathrm{Mon}(R)$ and $m = \ell' (f'_1)^{\beta_1} \cdots (f'_{r_1})^{\beta_{r_1}} (f_{r_1+1})^{\beta_{r_1+1}} \cdots f_r^{\beta_r} \in I^k$. ∎

Remark 2. *The converse affirmation of Proposition 10 is not true. We take $\mathcal{C}_0$ as in Example 3. So, $\mathcal{C}_0 \diagup \{x_i\}$ is a simple graph for each i. Hence, by Theorem 3, $\mathcal{C}_0 \diagup \{x_i\}$ has the strong persistence property.*

Definition 11. *Let $\mathcal{C} = (V, E)$ be a clutter and $\sigma \in S_V$ a permutation. We consider the clutter $\sigma(\mathcal{C}) = (V, E')$ where $E' = \{x_{\sigma(i_1)} \cdots x_{\sigma(i_s)} \mid x_{i_1} \cdots x_{i_s} \in E\}$.*

Proposition 11. *If $\mathcal{C}$ has the strong persistence property and $\sigma \in S_{V(\mathcal{C})}$, then $\sigma(\mathcal{C})$ also has the strong persistence property.*

Proof. We take a morphism of k-algebras $\phi : R = K[x_1, \dots, x_n] \to R$ given by $\phi(x_i) = x_{\sigma(i)}$. Hence, ϕ is an automorphism of R, with $\phi(I(\mathcal{C})) = I(\sigma(\mathcal{C}))$. Therefore, $I(\mathcal{C})$ and $I(\sigma(\mathcal{C}))$ are isomorphic. ∎

5 The symbolic strong persistence property

In this section we study some properties of the strong persistence property in a general ring. Furthermore, we introduce the symbolic strong persistence property and we prove that the strong persistence property implies the symbolic strong persistence property.

Theorem 10. *An ideal I has the strong persistence property if and only if $(I^t{:}I^s) = I^{t-s}$ for all $s \le t$.*

Proof. We proceed by induction on s. For $s = 1$ we recover the strong persistence property. Now, we take $a \in (I^t{:}I^{s+1})$ with $t \ge s+1$ and $x \in I$, then $axb \in I^t$ for all $b \in I^s$. Hence $ax \in (I^t{:}I^s)$. By induction hypothesis $ax \in I^{t-s}$. Consequently $a \in (I^{t-s}{:}I)$ and, by induction, $a \in I^{t-s-1}$. Therefore $(I^t{:}I^{s+1}) = I^{t-s-1}$. ∎

Corollary 8. *If I has the strong persistence property, then I^t has the strong persistence property.*

Proof. By Theorem 10 $(I^{kt}{:}I^t) = I^{kt-t} = I^{t(k-1)}$ for all $k \ge 1$. Therefore, I^t has the strong persistence property. ∎

By [4] normal ideals in an integer domain satisfies $(I^r{:}I^s) = I^{r-s}$ for all $s \le r$. Hence, by Theorem 10 a normal ideal has the strong persistence property, but the converse affirmation is not true.

Example 5 ([5])**.** *Let G be a simple connected graph, the $I(G)$ has the strong persistence property but if $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $E(G) = \{x_1x_2,\ x_2x_3, x_1x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_5x_7\}$, then $I(G)$ is not normal.*

Definition 12. *Let $P_1, \dots, P_r$ be the minimal primes of I. The i-th symbolic power of I is $I^{(i)} = q_1 \cap \cdots \cap q_r$, where q_i is the P_i-primary component of I^n.*

Remark 3. *$I^{(i)} \subseteq (I^{(i+1)}{:}I^{(1)})$ for each i.*

Definition 13. *I has the symbolic strong persistence property if $\left(I^{(i+1)}{:}I^{(1)}\right) = I^{(i)}$ for each i.*

Theorem 11. *Strong persistence property implies the symbolic strong persistence property.*

Proof. Let $\mathrm{Min}(I) = \{P_1, \dots, P_r\}$ be the set of minimal primes containing I. We take $I^d = Q_{1d} \cap \cdots \cap Q_{s_d d}$ a minimal primary decomposition of I^d for each d. We can suppose that there exists $r_d \le s_d$ such that $\mathrm{r}(Q_{id}) \in \mathrm{Min}(I)$ if and only if $i \le r_d$. Now for $j > r_{k+1}$, then $\mathrm{r}(Q_{j\,k+1})$ is not minimal. Consequently, $\mathrm{r}(Q_{j\,k+1}) \nsubseteq \mathrm{r}(Q_{i\,k+1})$ with $i \le r_{k+1}$. This implies $\mathrm{r}(Q_{j\,k+1}) \nsubseteq B$, where $B = \bigcup_{i=1}^{r_{k+1}} \mathrm{r}(Q_{i\,k+1})$. Thus, there is $a_j \in \mathrm{r}(Q_{j\,k+1}) \setminus B$. So, $b_j = a_j^{s_1} \in Q_{j\,k+1}$ for some s_j. Hence, $b_j \in Q_{jk+1} \setminus B$. Now, we take $a \in (I^{(k+1)}{:}I^{(1)})$, then $ax \in I^{(k+1)}$ for all $x \in I^{(1)}$. Consequently, if $c = \prod_{j \ge r_{k+1}} b_j$, then $axc \in I^{k+1}$ for all $x \in I$ since $I \subseteq I^{(1)}$. So, $ac \in I^k$, since I has the strong persistence property. Furthermore, if $j > r_{k+1}$, then $b_j \notin \mathrm{r}(Q_{ik})$ for $i \le r_k$. Thus, $a \in Q_{ik}$ for $1 \le i \le r_k$, since $ac \in Q_{ik}$ and Q_{ik} is primary. Therefore, $a \in I^{(k)}$. ∎

Proposition 12. *An ideal I has the symbolic strong persistence property if and only if $\left(I^{(r)}\colon I^{(s)}\right) = I^{(r-s)}$ for all $s \leq r$.*

Proof. Similar to proof of Theorem 10. ∎

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