Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 3, 2017, 277–291

Some curvature properties of (α,β)-Metrics
by

Bezdaf Najafi[(1)] and Akbar Tayebi[(2)]

Abstract

We solve two open problems in Finsler geometry which have been proposed by Z. Shen about Finsler metrics with relatively isotropic Landsberg curvature and weakly Landsberg metrics. We define a new quantity which is closely related to the S-curvature. Then, we find some conditions for (α,β)-metrics under which the notions of relatively isotropic Landsberg curvature and relatively isotropic mean Landsberg curvature are equivalent. It extends Cheng-Shen's well-known theorem that proves the equality for the Randers metrics. As an application, we prove that every weakly Landsberg (α,β)-metric of non-Randers type with vanishing S-curvature is Berwaldian.

Key Words: Isotropic (mean) Landsberg curvature, (α,β)-metric

2010 Mathematics Subject Classification: Primary 53B40, Secondary 53C60.

1 Introduction

Let (M,F) be a Finsler manifold. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is the Cartan torsion $\mathbf{C}_y$ on T_xM. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. By definition, $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. F has relatively isotropic Landsberg curvature if $\mathbf{L}+cF\mathbf{C}=0$, where $c=c(x)$ is a scalar function on M. Taking a trace of $\mathbf{C}_y$ and $\mathbf{L}_y$ yield the mean Cartan torsion $\mathbf{I}_y$ and mean Landsberg curvature $\mathbf{J}_y$, respectively. Therefore, $\mathbf{J}/\mathbf{I}$ can be regarded as the relative rate of change of $\mathbf{I}$ along geodesics. F has relatively isotropic mean Landsberg curvature if $\mathbf{J}+cF\mathbf{I}=0$, where $c=c(x)$ is a scalar function on M.

Finsler metrics of relatively isotropic Landsberg curvature have important geometric meaning in Finsler geometry [2][3][4][8][14][18]. In [6], Cheng-Wang-Wang obtained a necessary and sufficient condition for an (α,β)-metric to be of relatively isotropic (mean) Landsberg curvature. Every Finsler metric of relatively isotropic Landsberg curvature has relatively isotropic mean Landsberg curvature. In [4], Cheng-Shen proved that every Randers metric of relatively isotropic mean Landsberg curvature is of relatively isotropic Landsberg curvature. But the converse might not be true in general (see page 325 in [14]). This motivates us to find some conditions under which the two notions of curvatures are equivalent for the class of (α,β)-metrics. Thus the following natural question arises:

Under which conditions, an (α,β)-metric of relatively isotropic mean Landsberg curvature has relatively isotropic Landsberg curvature?

An (α,β)-metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, in which $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on M and $b := \|\beta_x\|_\alpha$ (see [20] and [21]). For an (α,β)-metric $F := \alpha\phi(s)$, define $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Put

$$r_{ij} := \frac{1}{2}\big(b_{i|j} + b_{j|i}\big), \quad s_{ij} := \frac{1}{2}\big(b_{i|j} - b_{j|i}\big), \quad r_{00} := r_{ij}y^iy^j, \quad s_j := b^i s_{ij}.$$

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where $\phi = \phi(s)$ is a C^∞ function on the interval $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'', \tag{1.1}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Delta = 1 + sQ + (b^2 - s^2)Q'. \tag{1.2}$$

Then we have the following.

Theorem 1.1. *Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α,β)-metric on an n-dimensional manifold M. Then F has relatively isotropic Landsberg curvature $\mathbf{L} + cF\mathbf{C} = 0$, where $c = c(x)$ is a scalar function on M, if and only if it has relatively isotropic mean Landsberg curvature $\mathbf{J} + cF\mathbf{I} = 0$ and one of the following holds*

(i) β satisfies

$$r_{ij} = 0, \qquad s_{ij} = 0. \tag{1.3}$$

In this case, F is a Berwald metric.

(ii) β satisfies

$$r_{ij} = k(b^2 a_{ij} - b_ib_j) + \sigma b_ib_j, \qquad s_{ij} = 0, \tag{1.4}$$

where $k = k(x)$ and $\sigma = \sigma(x)$ are non-zero scalar functions on M and $\phi = \phi(s)$ satisfies the following system of ODEs

$$(n+1)\big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\big] = d\mathfrak{a}\mathcal{A}, \tag{1.5}$$

$$k\Psi_1 + s\sigma\Psi_3 + c\Phi(\phi - s\phi') = 0, \tag{1.6}$$

where d is a real constant and

$$\mathfrak{a} := \phi(\phi - s\phi'), \tag{1.7}$$

$$\mathcal{A} := \frac{3s\phi'' - (b^2 - s^2)\phi'''}{\phi - s\phi' + (b^2 - s^2)\phi''} + (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi}, \tag{1.8}$$

$$\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{1/2}\Big[\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{3/2}}\Big]', \tag{1.9}$$

$$\Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}, \tag{1.10}$$

$$\Psi_3 := \frac{s}{b^2 - s^2}\Psi_1 + \frac{b^2}{b^2 - s^2}\Psi_2. \tag{1.11}$$

Example 1. *Among the (α,β)-metrics, the Randers metric $F=\alpha+\beta$ is significant metric which constitute a majority of actual research. Every Randers metric with $k=2c/b^2$ and $\sigma=2c(1-b^2)/b^2$ satisfies (1.4) and (1.5) with $d=1$. For more details, see [4].*

A Finsler metric F is said to be weakly Landsbergian if $\mathbf{J}=0$ [24]. By Theorem 1.1, we have the following.

Corollary 1.1. *Let $F:=\alpha\phi(s)$, $s=\beta/\alpha$, be a regular weakly Landsberg (α,β)-metric of non-Randers type on a manifold M, i.e., $\phi\neq c_1\sqrt{1+c_2s^2}+c_3s$ for any constants $c_1>0$, c_2 and c_3. Suppose that F has vanishing S-curvature. Then F is a Berwald metric. Moreover, if the flag curvature satisfies $\mathbf{K}=0$ then F is locally Minkowskian.*

In [15], Shen studies Finsler metric of negatively flag curvature with constant S-curvature and emphasis on studying complete Finsler manifolds of dimension $n\geq 3$ with $\mathbf{J}=0$, $\mathbf{S}=0$ and $\mathbf{K}\leq 0$ (see page 631 in [15] or Problem 6 in [17]). Then Corollary 1.1 is an answer to the mentioned problem for the class of (α,β)-metrics non-Randers type.

In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2 Preliminary

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x\in M$, and by $TM=\cup_{x\in M}T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F:TM\to[0,\infty)$ which has the following properties:
(i) F is C^∞ on $TM_0:=TM\setminus\{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) for each $y\in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2(y+su+tv)\right]|_{s,t=0},\quad u,v\in T_xM.$$

See [12]. Now, let $x\in M$ and $F_x:=F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y:T_xM\otimes T_xM\otimes T_xM\to\mathbb{R}$ by

$$\mathbf{C}_y(u,v,w):=\frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0},\quad u,v,w\in T_xM.$$

The family $\mathbf{C}:=\{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. For $y\in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u):=I_i(y)u^i$, where $I_i:=g^{jk}C_{ijk}$.

For $y\in T_xM_0$, the Matsumoto torsion $\mathbf{M}_y:T_xM\otimes T_xM\otimes T_xM\to\mathbb{R}$ defined by $\mathbf{M}_y(u,v,w):=M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk}:=C_{ijk}-\frac{1}{n+1}\Big\{I_ih_{jk}+I_jh_{ik}+I_kh_{ij}\Big\}. \tag{2.1}$$

and $h_{ij}=g_{ij}-F_{y^i}F_{y^j}$ is the angular metric. F is said to be C-reducible if $\mathbf{M}_y=0$.

Lemma 2.1. ([11]) A Finsler metric F on a manifold of dimension $n\geq 3$ is a Randers metric if and only if $\mathbf{M}_y=0$, $\forall y\in TM_0$.

For $y \in T_xM_0$, define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ and $\mathbf{J}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k, \qquad \mathbf{J}_y(u) := J_i(y)u^i$$

where $L_{ijk} := C_{ijk|s}y^s$, $J_i := I_{i|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ and $\mathbf{J} := \{\mathbf{J}_y\}_{y\in TM_0}$ are called the Landsberg curvature and the mean Landsberg curvature, respectively. F is called a Landsberg metric and weakly Landsberg metric if $\mathbf{L} = 0$ and $\mathbf{J} = 0$, respectively.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big], \qquad y \in T_xM. \tag{2.2}$$

$\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$ [23].

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature. Then, F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1\cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in R^n \ \Big|\ F\big(y^i\frac{\partial}{\partial x^i}|_x\big) < 1\Big\}}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. Then for $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$, the S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}\big[\ln\sigma_F(x)\big],$$

where $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. The S-curvature is introduced by Shen for a comparison theorem on Finsler manifolds [13]. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [19][22].

In [1], Cheng consider regular (α,β)-metrics with isotropic S-curvature and prove the following.

Theorem 2.2. ([1]) A regular (α,β)-metric $F := \alpha\phi(\beta/\alpha)$, of non-Randers type on an n-dimensional manifold M is of isotropic S-curvature, $\mathbf{S} = (n+1)\sigma F$, if and only if β satisfies $r_{ij} = 0$ and $s_j = 0$. In this case, $\mathbf{S} = 0$, regardless of the choice of a particular $\phi = \phi(s)$.

3 Proof of Theorem 1.1

Here, we study a new quantity which is closely related to the Matsumoto torsion and S-curvature. For a non-Riemannian (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, let us define

$$\mathcal{P} := \frac{n+1}{\mathfrak{a}\mathcal{A}}\Big[s\phi\phi'' - \phi'(\phi - s\phi')\Big], \tag{3.1}$$

where $\mathfrak{a}=\mathfrak{a}(s)$ and $\mathcal{A}=\mathcal{A}(s)$ are given by (1.8). In the class of (α,β)-metrics, the quantity $\mathcal{P}=\mathcal{P}(s)$, $s=\beta/\alpha$, characterize Randers metrics. More precisely, we have the following.

Lemma 3.1. *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian regular (α,β)-metric on a manifold M of dimension $n\geq 3$. Then $\mathbf{M}=0$ if and only if $\mathcal{P}=1$.*

Proof. The fundamental tensor of an (α,β)-metric $F=\alpha\phi(s)$ is given by

$$g_{ij} = \mathfrak{a}\ a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j, \tag{3.2}$$

where

$$\begin{aligned} \alpha_i &:= \ \alpha^{-1}a_{ij}y^j, \qquad \rho_0 := \phi\phi'' + \phi'\phi' \\ \rho_1 &:= \ -\alpha^{-1}\big[s(\phi\phi''+\phi'\phi') - \phi\phi'\big], \quad \rho_2 := s\alpha^{-2}\Big[s(\phi\phi''+\phi'\phi') - \phi\phi'\Big]. \end{aligned}$$

Taking a vertical derivation of (3.2) implies that

$$\begin{aligned} 2C_{ijk} := \quad & \alpha^{-1}\rho_0' b_i b_j b_k - \alpha^{-2}s\ \rho_2' y_i y_j y_k + \alpha^{-1}\rho_1'\Big[b_i b_j y_k + b_j b_k y_i + b_k b_i y_j\Big] \\ & + \rho_1\Big[a_{ij}b_k + a_{jk}b_i + a_{ki}b_j\Big] + \rho_2\Big[a_{ij}y_k + a_{jk}y_i + a_{ki}y_j\Big] \\ & + \alpha^{-1}\rho_2'\Big[b_i y_j y_k + b_j y_k y_i + b_k y_i y_j\Big]. \end{aligned} \tag{3.3}$$

By plugging $\mathfrak{a}'=\alpha\rho_1$, $-s\mathfrak{a}'=\alpha^2\rho_2$, $-s\rho_0'=\alpha\rho_1'$ and $-s\rho_1'=\alpha\rho_2'$ in (3.3), it follows that

$$\begin{aligned} 2C_{ijk} = \quad & (\rho_1 - \alpha\epsilon\rho_2)\Big[a_{ij}b_k + a_{jk}b_i + a_{ki}b_j\Big] + \alpha\Big[a_{ij}Y_k + a_{jk}Y_i + a_{ki}Y_j\Big]\rho_2 \\ & + \alpha^{-1}b_i b_j b_k\rho_0' - \alpha^{-2}y_i y_j y_k\rho_2' s + \alpha^{-1}\Big[b_i b_j y_k + b_j b_k y_i + b_k b_i y_j\Big]\rho_1' \\ & + \alpha^{-1}\Big[b_i y_j y_k + b_j y_k y_i + b_k y_i y_j\Big]\rho_2', \end{aligned} \tag{3.4}$$

where $Y_i := \alpha_i + b_i\rho_1/\rho_2$. The angular metric $h_{ij} := g_{ij} - F_{y^i}F_{y^j}$ of an (α,β)-metric $F=\alpha\phi(s)$ is in the following form

$$h_{ij} = \mathfrak{a}\ a_{ij} + \phi\phi'' b_i b_j - s\phi\phi''\ \big[b_i\alpha_j + b_j\alpha_i\big] - \phi\big[(\phi - s\phi') - s^2\phi''\big]\ \alpha_i\alpha_j. \tag{3.5}$$

The mean Cartan torsion an (α,β)-metric $F=\alpha\phi(s)$ is given by

$$I_i = \frac{s}{2\alpha}\mathcal{A}Y_i. \tag{3.6}$$

Plugging (3.5) and (3.6) into (3.4) and considering $dim(M)\geq 3$ imply that

$$C_{ijk} = \frac{\mathcal{P}}{1+n}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{1-\mathcal{P}}{\|\mathbf{I}\|^2}I_i I_j I_k. \tag{3.7}$$

where $\mathcal{P}=\mathcal{P}(x,y)$ is a scalar function on TM and given by (3.1) (see [21]). By (3.7), it is easy to see that if $\mathcal{P}=1$ then Matsumoto torsion satisfies $\mathbf{M}=0$.

Now, suppose that $\mathbf{M}=0$. Then by (2.1) and (3.7), we get

$$(1-\mathcal{P})\Big[C_{ijk}-\frac{1}{\|\mathbf{I}\|^2}I_iI_jI_k\Big]=0. \tag{3.8}$$

If $\mathcal{P}\neq 1$, then by (3.8) it follows that

$$C_{ijk}=\frac{1}{\|\mathbf{I}\|^2}I_iI_jI_k. \tag{3.9}$$

By (3.7) and (3.9), we get $\mathcal{P}=0$. According to (3.1), $\mathcal{P}=0$ if and only if ϕ satisfies following

$$s(\phi\phi''+\phi'\phi')-\phi\phi'=0. \tag{3.10}$$

By solving (3.10), we get $\phi=\sqrt{c_1s^2+c_2}$, where c_1 and c_2 are two real constant. In this case, $F=\alpha\phi(s)$, $s=\beta/\alpha$, reduces to a Riemannian metric which contradicts with our assumptions. Thus $\mathcal{P}=1$. ∎

By Lemmas 2.1 and 3.1, we get the following.

Corollary 3.1. *Every non-Riemannian regular (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, on a manifold of dimension $n\geq 3$ is a Randers metric if and only if $\mathcal{P}=1$.*

For an (α,β)-metric $F:=\alpha\phi(s)$, $s=\beta/\alpha$, let us put

$$\Theta:=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi\big[(\phi-s\phi')+(b^2-s^2)\phi''\big]},\qquad \Psi:=\frac{1}{2}\frac{\phi''}{(\phi-s\phi')+(b^2-s^2)\phi''}.$$

Then, we have the following.

Lemma 3.2. *Let $F:=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian regular (α,β)-metric on a manifold M of dimension $n\geq 3$. Then $\mathcal{P}=\mathcal{P}(s)$ is constant along any Finslerian geodesic if and only if one of the following holds*

(i) β satisfies

$$r_{00}=2\alpha Qs_0. \tag{3.11}$$

(ii) $\phi=\phi(s)$ satisfies

$$(n+1)\Big[s(\phi\phi''+\phi'\phi')-\phi\phi'\Big]=d\mathfrak{a}\mathcal{A}, \tag{3.12}$$

where $d\in\mathbb{R}$ is a real constant.

Proof. Let $G^i=G^i(x,y)$ and $\bar{G}^i=\bar{G}^i(x,y)$ denote the spray coefficients of F and α respectively in the same coordinate system. By (2.2), we have

$$G^i=\bar{G}^i+Py^i+Q^i, \tag{3.13}$$

where

$$P := \alpha^{-1}\Theta(r_{00} - 2Q\alpha s_0), \quad Q^i := \alpha Q s^i{}_j y^j + \Psi(r_{00} - 2Q\alpha s_0)b^i.$$

Taking a horizontal derivation of (3.1) implies that

$$\nabla_0\mathcal{P} = \mathcal{P}_{|i}y^i = \mathcal{P}'(s)s_{|i}y^i = \mathcal{P}'(s)\Big[\frac{\tilde{r}_{00}}{\alpha} - \frac{\beta}{\alpha^2}\ \alpha_{|i}y^i\Big], \tag{3.14}$$

where

$$\tilde{r}_{00} = r_{00} - 2P\beta - 2Q^k b_k. \tag{3.15}$$

Hence, $\nabla_0\mathcal{P} = 0$ if and only if $\mathcal{P}$ is constant or the following holds

$$\tilde{r}_{00}\alpha - \beta\alpha_{|i}y^i = 0. \tag{3.16}$$

Suppose that $\mathcal{P}'(s) \neq 0$. It is sufficent to show that (3.16) is equievalent to (3.11). Note that, we have

$$\alpha_{|i}y^i = y^i\frac{\partial\alpha}{\partial x^i} - 2G^i\frac{\partial\alpha}{\partial y^i}. \tag{3.17}$$

By (3.13) and (3.17), we get

$$\begin{aligned}\alpha_{|i}y^i &= y^i\frac{\partial\alpha}{\partial x^i} - 2\bar{G}^i\frac{\partial\alpha}{\partial y^i} - 2Py^i\frac{\partial\alpha}{\partial y^i} - 2Q^i\frac{\partial\alpha}{\partial y^i}\\ &= -2P\alpha - 2\Psi(-2Q\alpha s_0 + r_{00})s\\ &= -2\Theta(-2Q\alpha s_0 + r_{00}) - 2\Psi(-2Q\alpha s_0 + r_{00})s\\ &= -2(\Theta + s\Psi)r_{00} + 4(\Theta + s\Psi)Q\alpha s_0.\end{aligned} \tag{3.18}$$

Plugging (3.18) into (3.16) yields

$$\Big[1 - 2(b^2 - s^2)\Psi\Big](r_{00} - 2\alpha Q s_0) = 0. \tag{3.19}$$

Since $1 - 2(b^2 - s^2)\Psi \neq 0$, then we get (3.11). ∎

Now, we are going to consider (α,β)-metrics with relatively isotropic mean Landsberg curvature such that the quantity $\mathcal{P} = \mathcal{P}(s)$ is constant along any Finslerian geodesic.

Lemma 3.3. *Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that $\mathcal{P} = \mathcal{P}(s)$ is constant along any Finslerian geodesic. If F has relatively isotropic mean Landsberg curvature, then it has relatively isotropic Landsberg curvature.*

Proof. By Lemma 3.1, the Cartan torsion of an (α,β)-metric is given by following

$$C_{ijk} = \frac{\mathcal{P}}{1+n}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{1-\mathcal{P}}{\|\mathbf{I}\|^2}I_iI_jI_k. \tag{3.20}$$

Taking a horizontal covariant derivation of (3.20) yields

$$\begin{aligned} L_{ijk} = \ & \frac{\mathcal{Q}'}{\|\mathbf{I}\|^2} I_i I_j I_k - \frac{2\mathcal{Q}}{\|\mathbf{I}\|^4} J_m I^m I_i I_j I_k + \frac{\mathcal{Q}}{\|\mathbf{I}\|^2}\Big\{J_i I_j I_k + I_i J_j I_k + I_i I_j J_k\Big\} \\ & + \frac{\mathcal{P}'}{n+1}\Big\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\Big\} + \frac{\mathcal{P}}{n+1}\Big\{h_{ij} J_k + h_{jk} J_i + h_{ki} J_j\Big\}, \end{aligned} \tag{3.21}$$

where $\mathcal{Q} := 1-\mathcal{P}$, $\mathcal{P}' := \mathcal{P}_{|l} y^l$ and $\mathcal{Q}' := \mathcal{Q}_{|l} y^l$. By assumptions, we have $\mathcal{P}' = \mathcal{Q}' = 0$. Then (3.21) reduces to following

$$\begin{aligned} L_{ijk} = \frac{\mathcal{P}}{n+1}\Big\{h_{ij} J_k + h_{jk} J_i + h_{ki} J_j\Big\} + \ & \frac{\mathcal{Q}}{\|\mathbf{I}\|^2}\Big\{J_i I_j I_k + I_i J_j I_k + I_i I_j J_k\Big\} \\ - \ & \frac{2\mathcal{Q}}{\|\mathbf{I}\|^4} J_m I^m I_i I_j I_k. \end{aligned} \tag{3.22}$$

Putting $\mathbf{J} + cF\mathbf{I} = 0$ in (3.22) and considering (3.20), we get $\mathbf{L} + cF\mathbf{C} = 0$. □

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{\mathcal{P}}{1+n}\Big\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\Big\} + \frac{\mathcal{Q}}{\|\mathbf{I}\|^2} I_i I_j I_k. \tag{3.23}$$

where $\mathcal{P} = \mathcal{P}(x,y)$ and $\mathcal{Q} = \mathcal{Q}(x,y)$ are scalar function on TM and $C^2 = I^i I_i$. Contracting the last relation by g^{jk} shows that $\mathcal{P}$ and $\mathcal{Q}$ satisfy $\mathcal{P} + \mathcal{Q} = 1$. In [9], Matsumoto proved that every (α,β)-metric is semi-C-reducible.

Proposition 3.1. ([9]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

Since $\mathcal{P} + \mathcal{Q} = 1$, then taking a horizontal derivation of it implies that $\mathcal{P}' = -\mathcal{Q}'$. Thus $\mathcal{P}' = 0$ if and only if $\mathcal{Q}' = 0$. In [18], the following is proved (see Proposition 3.1 in [18]).

Proposition 3.2. ([18]) Let (M, F) be a semi-C-reducible manifold. Suppose that F has relatively isotropic Landsberg curvature, $\mathbf{L} + cF\mathbf{C} = 0$, for some scalar function $c = c(x)$ on M. Then the quantity $\mathcal{P}$ is constant along any Finslerian geodesic.

By Lemma 3.3 and Proposition 3.2, we get the following theorem.

Theorem 3.4. *Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F has relatively isotropic Landsberg curvature if and only if $\mathcal{P}$ is constant along any Finslerian geodesic and F has relatively isotropic mean Landsberg curvature.*

Now, we prove that (3.11) implies that β is a constant length Killing one-form. More perecisely, we have the following lemma.

Lemma 3.5. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that (3.11) holds. Then β is constant length Killing one-form.*

Proof. To simplify the equation (3.11), we change the y-coordinates (y^i) at a point to "polar" coordinates (s, u^A), where $i = 1, \cdots, n$ and $A = 2, \cdots, n$ (for more details see [5], [7] and [16]).

Fix an arbitrary point $x \in M$. Take an orthonormal basis e_i at x such that

$$\alpha = \sqrt{\sum_{i=1}^{n} (y^i)^2}, \qquad \beta = by^1, \tag{3.24}$$

where $b := ||\beta||_\alpha$.

Fix an arbitrary number s with $|s| < b$. It follows from $\beta = s\alpha$ that

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \qquad y^A = u^A, \tag{3.25}$$

where

$$\bar{\alpha} = \sqrt{\sum_{A=2}^{n} (y^A)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}. \tag{3.26}$$

Let us put

$$\bar{r}_{10} := \sum_{A=2}^{n} r_{1A} y^A, \qquad \bar{s}_{10} := \sum_{A=2}^{n} s_{1A} y^A \qquad \bar{r}_{00} := \sum_{A,B=2}^{n} r_{AB} y^A y^B,$$

$$\bar{r}_0 := \sum_{A=2}^{n} r_A y^A \qquad \bar{s}_0 := \sum_{A=2}^{n} s_A y^A.$$

Then we get the following

$$r_1 = br_{11}, \qquad r_A = br_{1A}, \tag{3.27}$$

$$s_1 = 0, \qquad s_A = bs_{1A}, \tag{3.28}$$

$$r_{00} = \frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + \frac{2s\bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00}, \tag{3.29}$$

$$r_{10} = \frac{s\bar{\alpha}}{\sqrt{b^2 - s^2}} r_{11} + \bar{r}_{10} \qquad s_0 = \bar{s}_0 = b\bar{s}_{10}. \tag{3.30}$$

By (3.11), (3.26), (3.28), (3.29) and (3.30), we have

$$\frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + \frac{2s\bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00} = \frac{2b\bar{\alpha}}{\sqrt{b^2 - s^2}} Qb\bar{s}_{10}. \tag{3.31}$$

(3.31) is equivalent to the following two equations

$$\frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + \bar{r}_{00} = 0, \tag{3.32}$$

$$s\bar{r}_{10} - b^2 Q\bar{s}_{10} = 0. \tag{3.33}$$

(3.26) implies that

$$\frac{s^2\bar{\alpha}^2}{b^2-s^2}=\frac{\beta^2}{b^2}. \tag{3.34}$$

By (3.32) and (3.34), we get

$$\bar{r}_{00}+\frac{\beta^2}{b^2}r_{11}=0. \tag{3.35}$$

Since

$$\frac{\partial\bar{r}_{00}}{\partial y^1}=0,\qquad \frac{\partial\beta}{\partial y^1}=b,$$

then differentiating (3.35) with respect to y^1 yields

$$\frac{\beta}{b}r_{11}=0. \tag{3.36}$$

Thus

$$r_{11}=0. \tag{3.37}$$

By plugging (3.37) in (3.35), we have

$$\bar{r}_{00}=0. \tag{3.38}$$

Plugging (3.37) and (3.38) in (3.29) and (3.30) imply that

$$r_{00}=\frac{2s\bar{\alpha}}{\sqrt{b^2-s^2}}\bar{r}_{10}=\frac{2\beta}{b}\bar{r}_{10}, \tag{3.39}$$

$$r_{10}=\bar{r}_{10}. \tag{3.40}$$

We shall divide the problem into two cases: (a) $\bar{r}_{10}=0$ and (b) $\bar{r}_{10}\neq 0$.

Case (a): $\bar{r}_{10}=0$. In this case, by (3.39) we get

$$r_{00}=0. \tag{3.41}$$

Plugging (3.41) in (3.11) implies that $s_i=0$. In this case, β is constant length Killing one-form.

Case (b): $\bar{r}_{10}\neq 0$. In this case, by (3.33) we have

$$s\bar{r}_{10}-b^2Q\bar{s}_{10}=0. \tag{3.42}$$

Since

$$\frac{\partial\bar{r}_{10}}{\partial y^1}=0,\qquad \frac{\partial\bar{s}_{10}}{\partial y^1}=0,$$

then differentiating (3.42) with respect to y^1 yields

$$(s)_{y^1}\bar{r}_{10}-b^2(Q)_{y^1}\bar{s}_{10}=0. \tag{3.43}$$

Multiplying (3.42) with $(s)_{y^1}$ implies that

$$s(s)_{y^1}\bar{r}_{10} - b^2 Q(s)_{y^1}\bar{s}_{10} = 0. \tag{3.44}$$

By (3.43) and (3.44), it follows that

$$b^2\Big[Q(s)_{y^1} - s(Q)_{y^1}\Big]\bar{s}_{10} = 0. \tag{3.45}$$

From (3.45), we get two cases:

$$\bar{s}_{10} = 0 \tag{3.46}$$

or

$$Q(s)_{y^1} = s(Q)_{y^1}. \tag{3.47}$$

Subcase (b1). Let (3.46) holds. Then (3.42) reduces to $s = 0$, which is impossible.

Subcase (b2). Let (3.47) holds. Then

$$\frac{(Q)_{y^1}}{Q} = \frac{(s)_{y^1}}{s}. \tag{3.48}$$

On the other hand, we have

$$(Q)_{y^1} = (Q)_s \ s_{y^1}. \tag{3.49}$$

Since $s_{y^1} \neq 0$, then by (3.48) and (3.49), we get

$$\frac{(Q)_s}{Q} = \frac{1}{s}. \tag{3.50}$$

(3.50) implies that $\ln(Q) = \ln(s) + c$, where c is a real constant. Thus

$$\ln\Big(\frac{Q}{s}\Big) = c, \tag{3.51}$$

or equivalently $Q = ks$, where k is a non-zero real constant. In this case, it follows that F is Riemannian. This is a contradiction. Then the case (a) holds, only. ∎

By Theorem 2.2 and Lemmas 3.2 and 3.5, we conclude the following.

Corollary 3.2. *Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi = \phi(s)$ dose not satisfy (3.12). Then $\mathcal{P} = \mathcal{P}(s)$ is constant along any Finslerian geodesic if and only if $\mathbf{S} = 0$.*

For Finsler surfaces, we have the following.

Lemma 3.6. *Let (M,F) be a 2-dimensional Finsler manifold. Then F has relatively isotropic Landsberg curvature if and only if it has relatively isotropic mean Landsberg curvature.*

Proof. The Cartan torsion of every 2-dimensional Finsler manifold satisfies

$$C_{ijk}=\frac{1}{3}\big\{h_{ij}I_k+h_{jk}I_i+h_{ki}I_j\big\}. \tag{3.52}$$

See page 485 in [10]. Taking a horizontal derivation of (3.52) implies that

$$L_{ijk}=\frac{1}{3}\big\{h_{ij}J_k+h_{jk}J_i+h_{ki}J_j\big\}. \tag{3.53}$$

By putting $\mathbf{J}=cF\mathbf{I}$ in (3.53), we get $\mathbf{L}=cF\mathbf{C}$. ∎

Theorem 3.7. *Let $F:=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Riemannian regular (α,β)-metric on an n-dimensional manifold M. Then F has relatively isotropic Landsberg curvature $\mathbf{L}+cF\mathbf{C}=0$, where $c=c(x)$ is a scalar function on M, if and only if it has relatively isotropic mean Landsberg curvature $\mathbf{J}+cF\mathbf{I}=0$ and one of the following holds*

(i) β satisfies

$$r_{ij}=0 \quad \text{and} \quad s_i=0. \tag{3.54}$$

In this case, $\mathbf{S}=0$.

(ii) β satisfies

$$r_{ij}\neq 0 \quad \text{or} \quad s_i\neq 0 \tag{3.55}$$

and $\phi=\phi(s)$ satisfies

$$(n+1)\big[s(\phi\phi''+\phi'\phi')-\phi\phi'\big]=d\mathfrak{a}\mathcal{A}, \tag{3.56}$$

where d is a real constant and $\mathfrak{a}$ and $\mathcal{A}$ are given by (1.7) and (1.8).

Proof. By Lemmas 3.2, 3.3, 3.5, 3.6 and Proposition 3.2, we get the proof. ∎

Proof of Corollary 1.1: Since $\mathbf{J}=0$ and $\mathbf{S}=0$, then by Theorems 3.7 and 2.2 it follows that F is a Landsberg metric. In [16], Shen proved that every regular Landsbergian (α,β)-metric is a Berwald metric. On the other hand, every Berwald metric with vanishing flag curvature is locally Minkowskian. This completes the proof. ∎

In [6], the authors find a necessary and sufficient condition for an (α,β)-metric to be of relatively isotropic mean Landsberg curvature.

Theorem 3.8. ([6]) Let $F:=\alpha\phi(s)$, $s=\beta/\alpha$, be a regular (α,β)-metric on a manifold M of dimension $n\geq 3$. Then F has relatively isotropic mean Landsberg curvature $\mathbf{J}+cF\mathbf{I}=0$, where $c=c(x)$ is a scalar function on M, if and only if β satisfies

$$r_{ij}=k(b^2a_{ij}-b_ib_j)+\sigma b_ib_j, \qquad s_{ij}=0, \tag{3.57}$$

where $k=k(x)$ and $\sigma=\sigma(x)$ are scalar functions on M and $\phi=\phi(s)$ satisfies

$$k\Psi_1+s\sigma\Psi_3+c\Phi(\phi-s\phi')=0, \tag{3.58}$$

where Ψ_1, Ψ_2 and Ψ_3 are given by (1.9), (1.10) and (1.11).

Proof of Theorem 1.1: By Theorems 3.7 and 3.8, we get the proof. ∎

4 Some Solutions of the ODE (3.12)

In this section, we are going to find some solutions of the ODE (3.12). It is equal to following

$$\frac{s(\phi\phi''+\phi'\phi')-\phi\phi'}{\phi(\phi-s\phi')}=\frac{d}{n+1}\left[\frac{3s\phi''-(b^2-s^2)\phi'''}{\phi-s\phi'+(b^2-s^2)\phi''}+(n-2)\frac{s\phi''}{\phi-s\phi'}-(n+1)\frac{\phi'}{\phi}\right], \tag{4.1}$$

where d is a real constant. The solution of (4.1) is given by following

$$\ln\big(\phi(\phi-s\phi')\big)=\frac{d}{n+1}\Big[\ln\big(\phi-s\phi'+(b^2-s^2)\phi''\big)+\ln(\phi-s\phi')^{(n-2)}+\ln(\phi)^{(n+1)}\Big] \tag{4.2}$$

or equivalently

$$\ln\big(\phi(\phi-s\phi')\big)=\ln\Big[\phi^{n+1}(\phi-s\phi')^{n-2}\big(\phi-s\phi'+(b^2-s^2)\phi''\big)\Big]^{\frac{d}{n+1}}. \tag{4.3}$$

Thus

$$\phi^{n+1}(\phi-s\phi')^{n+1}=\Big[\phi^{n+1}(\phi-s\phi')^{n-2}\big(\phi-s\phi'+(b^2-s^2)\phi''\big)\Big]^{d}. \tag{4.4}$$

Simplifying (4.4) yields

$$\phi^{(n+1)(1-d)}(\phi-s\phi')^{n(1-d)+2d+1}=\big(\phi-s\phi'+(b^2-s^2)\phi''\big)^{d}. \tag{4.5}$$

If $d=1$, then we get

$$(\phi-s\phi')^3=\phi-s\phi'+(b^2-s^2)\phi''. \tag{4.6}$$

Let $\psi:=\phi-s\phi'$. Then (4.6) can be written as follows

$$s\psi^3=s\psi-(b^2-s^2)\psi'. \tag{4.7}$$

Dividing (4.7) by ψ^3 and putting $v=\psi^{-2}$, we get the following first order ODE in terms of v

$$2s=2sv+(b^2-s^2)v'. \tag{4.8}$$

The general solution of (4.8) is given by

$$v=c(b^2-s^2)+1. \tag{4.9}$$

Consequently, we get

$$\phi-s\phi'=\frac{1}{\sqrt{c(b^2-s^2)+1}}. \tag{4.10}$$

By (4.10), we have

$$\phi=-s\int\frac{1}{s^2\sqrt{c_1(b^2-s^2)+1}}\,ds+c_2s, \tag{4.11}$$

where c_1 and c_2 are real constants. For example, if we put $c_1=0$ and $c_2=1$ then we get the Randers metric $\phi=1+s$.

References

[1] X. Cheng, The (α, β)-metrics of scalar flag curvature, *Differ. Geom. Appl*, **35**, 361-369 (2014).

[2] X. Cheng, X. Mo, Z. Shen, On the flag curvature of Finsler metrics of scalar curvature, *J. London Math. Soc.*, **68**, 762-780 (2003).

[3] X. Cheng, Z. Shen, *Finsler Geometry, An Approach via Randers Spaces*, Springer-Verlag (2012).

[4] X. Cheng, Z. Shen, Randers metrics with special curvature properties, *Osaka J. Math.*, **40**, 87-101 (2003).

[5] X. Cheng, Z. Shen, A class of Finsler metrics with isotropic S-curvature, *Israel J. Math.*, **169**, 317-340 (2009).

[6] X. Cheng, H. Wang, M. Wang, (α, β)-metrics with relatively isotropic mean Landsberg curvature, *Publ. Math. Debrecen*, **72**, 475-485 (2008).

[7] B. Li, Z. Shen, On a class of weakly Landsberg metrics, *Science in China, Series A: Mathematics*, **50**, 573-589 (2007).

[8] X. Mo, *An Introduction to Finsler Geometry*, Would Scientific Press (2006).

[9] M. Matsumoto, Theory of Finsler spaces with (α, β)-metric,*Rep. Math. Phys.*, **31**, 43-84 (1992).

[10] M. Matsumoto, On Finsler spaces with Randers' metric and special forms of important tensors, *J. Math. Kyoto Univ.*, **14**(3), 477-498 (1974).

[11] M. Matsumoto, S. Hōjō , *A conclusive theorem for C-reducible Finsler spaces*, Tensor. N. S. **32**(1978), 225-230.

[12] L-I. Pişcoran , From Finsler geometry to noncommutative geometry, *General. Math.*, **12**(4), 29-38 (2004).

[13] Z. Shen, Volume comparison and its applications in Riemann-Finsler geometry, *Advances. Math.*, **128**, 306-328 (1997).

[14] Z. Shen, Landsberg curvature, S-curvature and Riemann curvature, In *A Sampler of Finsler Geometry*, MSRI series, Cambridge University Press (2004).

[15] Z. Shen, Finsler manifolds with nonpositive flag curvature and constant S-curvature, *Math. Z.*, **249**(3) , 625-639 (2005).

[16] Z. Shen, On a class of Landsberg metrics in Finsler geometry, *Canad. J. Math.*, **61**, 1357-1374 (2009).

[17] Z. Shen, Some open problems in Finsler geometry, http://www.math.iupui.edu/ zshen/ Research/preprintindex.htm

[18] A. Tayebi, E. Peyghan, B. Najafi, On semi-C-reducibility of (α, β)-metrics, *Int. J. Geom. Meth. Modern. Phys.*, **9**(4), 1250038 (2012).

[19] A. Tayebi, M. Rafie. Rad, S-curvature of isotropic Berwald metrics, *Sci. China. Series A: Math.*, **51** 2198-2204 (2008).

[20] A. Tayebi, H. Sadeghi, On generalized Douglas-Weyl (α, β)-metrics, *Acta Mathematica Sinica, English Series*, **31**(10), 1611-1620 (2015).

[21] A. Tayebi, H. Sadeghi, On Cartan torsion of Finsler metrics, *Publ. Math. Debrecen*, **82**(2), 461-471 (2013).

[22] A. Tayebi, H. Sadeghi, Generalized P-reducible (α, β)-metrics with vanishing S-curvature, *Ann. Polon. Math.*, **114**(1), 67-79 (2015).

[23] A. Tayebi, M. Shahbazi Nia, A new class of projectively flat Finsler metrics with constant flag curvature $\mathbf{K} = 1$, *Differ. Geom. Appl.*, **41**, 123-133 (2015).

[24] L. Zhou, The Finsler surface with $\mathbf{K} = 0$ and $\mathbf{J} = 0$, *Differ. Geom. Appl.*, **35**, 370-380 (2014).

Received: 10.02.2016
Accepted: 26.09.2016

(1) Department of Mathematics and Computer Sciences
Amirkabir University
Tehran. Iran
Email: behzad.najafi@aut.ac.ir

(2) Department of Mathematics, Faculty of Science
University of Qom
Qom. Iran
Email: akbar.tayebi@gmail.com