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# A new characterization for some extensions of PSL$(2, p)$ by order and some character degrees

by

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**Abstract**

Many authors were recently concerned with the following question: *What can be said about the structure of a finite group $G$, if some information is known about the arithmetical structure of the degrees of the irreducible characters of $G$?*

Let $G$ be a finite group and $X_1(G)$ be the set of all irreducible complex character degrees of $G$ counting multiplicities.

Let $p$ be an odd prime number and $M = \mathrm{PGL}(2, p)$, $M = \mathbb{Z}_2 \times \mathrm{PSL}(2, p)$ or $M = \mathrm{SL}(2, p)$. In this paper we prove that $M$ is uniquely determined by its order and some information on its character degrees. As a consequence of our results we prove that if $G$ is a finite group such that $X_1(G) = X_1(M)$, then $G \cong M$. This implies that $M$ is uniquely determined by the structure of its complex group algebra.

**Key Words**: Character degrees, order, projective special linear group, characterization

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# 1 Introduction and Preliminary Results

Let $G$ be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of $G$, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of $G$. The degree pattern of $G$, which is denoted by $X_1(G)$ is the set of all irreducible complex character degrees of $G$ counting multiplicities. We note that $X_1(G)$ is the first column of the ordinary character table of $G$. If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. If $G$ is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$.

Many authors were recently concerned with the following question: *What can be said about the structure of a finite group $G$, if some information is known about the arithmetical structure of the degrees of the irreducible characters of $G$?*

A finite group $G$ is called a $K_3$-group if $|G|$ has exactly three distinct prime divisors. Recently Chen et. al. in [28] proved that all simple $K_3$-groups are uniquely determined by their orders and one or both of its largest and second largest irreducible character degrees.

In [2, Problem 2*], R. Brauer asked whether two groups $G$ and $H$ are isomorphic given that two group algebras $\mathbb{F}G$ and $\mathbb{F}H$ are isomorphic for all fields $\mathbb{F}$. This is false in general. In fact, E. C. Dade [5] constructed two non-isomorphic metabelian groups $G$ and $H$ such that $\mathbb{F}G \cong \mathbb{F}H$ for all fields $\mathbb{F}$. In [6], M. Hertweck showed that this is not true even for the integral group rings. Note that if $\mathbb{Z}G \cong \mathbb{Z}H$, then $\mathbb{F}G \cong \mathbb{F}H$, for all fields $\mathbb{F}$, where $\mathbb{Z}$ is the ring of integer numbers. For nonabelian simple groups, W. Kimmerle obtained a positive answer in [20]. He outlined the proof asserting that if $G$ is a group and $H$ is a nonabelian simple group such that $\mathbb{F}G \cong \mathbb{F}H$ for all fields $\mathbb{F}$ then $G \cong H$.

Let $\mathbb{C}G$ be the complex group algebra of $G$. By Molien's Theorem (see [1, Theorem 2.13]), we know that $\mathbb{C}G = \oplus_{i=1}^{s} M_{n_i}(\mathbb{C})$ and thus knowing the structure of the complex group algebra is equivalent to knowing the first column of the ordinary character table of $G$. In [25], Tong-Viet proved that each classical simple group is uniquely determined by its complex group algebra.

Let $p$ be an odd prime number. In [14] the authors proved that the simple group $\mathrm{PSL}(2,p)$ is uniquely determined by its order and its largest and second largest irreducible character degrees. Also it is proved that the simple group $\mathrm{PSL}(2,p^2)$ is uniquely determined by its character degree graph and its order. In [18, 19] it is proved that some simple groups are uniquely determined by their character degree graphs and their orders. In [15], the authors proved that if $G$ is a finite group such that $|G| = 2|\mathrm{PSL}(2,p^2)|$, $p^2 \in \mathrm{cd(G)}$ and there does not exist any $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then $G$ has a unique nonabelian composition factor isomorphic to $\mathrm{PSL}(2,p^2)$. In [11] it is proved that the projective special linear group $\mathrm{PSL}(2,q)$ is uniquely determined by its group order and its largest irreducible character degree when $q$ is a prime or when $q = 2^a$ for an integer $a \geq 2$ such that $2^a - 1$ or $2^a + 1$ is a prime.

In [17] it is proved that if $p$ is an odd prime number and $G$ is a finite group such that $|G| = |\mathrm{PSL}(2,p^2)|$, $p^2 \in \mathrm{cd}(G)$ and there does not exist any $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then $G \cong \mathrm{PSL}(2,p^2)$.

Also in [13] groups with the same order and largest and second largest irreducible character degrees as $\mathrm{PGL}(2,9)$ are determined. In [3, 12] some characterizations for $\mathrm{PGL}(2,p)$, where $p$ is an odd prime number, are introduced.

The goal of this paper is to introduce some new characterizations for the finite groups $\mathrm{PGL}(2,p)$, $\mathbb{Z}_2 \times \mathrm{PSL}(2,p)$ and $\mathrm{SL}(2,p)$, where $p$ is an odd prime number. As a consequence of our results we show that these groups are uniquely determined by the structure of their complex group algebras.

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of $\theta$ in $G$ is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i \chi_i$, where for each $1 \leq i \leq k$, $\chi_i \in \mathrm{Irr}(G)$ and $e_i$ is a natural number, then each $\chi_i$ is called an irreducible constituent of $\chi$.

**Lemma 1.** (Gallagher's Theorem) *[9, Corollary 6.17] Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr(G)}$ be such that $\chi_N = \theta \in \mathrm{Irr(N)}$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr(G/N)}$ are irreducible distinct for distinct $\beta$ and all of the irreducible constituents of $\theta^G$.*

**Lemma 2.** (Itō's Theorem) *[9, Theorem 6.15] Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$, for all $\chi \in Irr(G)$.*

**Lemma 3.** *[9, Theorems 6.2, 6.8, 11.29] Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr(G)}$. Let $\theta$ be an irreducible constituent of $\chi_N$ and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of $\theta$ in $G$. Then $\chi_N = e\sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

**Lemma 4.** (Itō-Michler Theorem) *[7] Let $\rho(G)$ be the set of all prime divisors of the elements of $\mathrm{cd}(G)$. Then $p \notin \rho(G)$ if and only if $G$ has a normal abelian Sylow $p$-subgroup.*

**Lemma 5.** *[28, Lemma] Let $G$ be a nonsolvable group. Then $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H$ is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Let $q = p^f$ be a prime power. The outer automorphism group of $\mathrm{PSL}(2, q)$ is of order $(q-1, 2) \cdot f$, and is generated by a field automorphism $\varphi$ of order $f$ and, if $p$ is odd, a diagonal automorphism $\bar{\delta}$ of order 2 (see [4]). Also we note that $\mathrm{PGL}(2, q) = \mathrm{PSL}(2, q)\langle\bar{\delta}\rangle$.

**Lemma 6.** *[27, Theorem A] Let $S = PSL(2, q)$, where $q = p^f > 3$ for a prime $p$, $A = \mathrm{Aut}(S)$, and let $S \leq H \leq A$. Set $G = PGL(2, q)$ if $\bar{\delta} \in H$ and $G = S$ if $\bar{\delta} \notin H$, and let $|H : G| = d = 2^a m$, $m$ is odd. If $p$ is odd, let $\varepsilon = (-1)^{(q-1)/2}$. The set of irreducible character degrees of $H$ is*

$$\mathrm{cd}(H) = \{1, q, (q+\varepsilon)/2\} \cup \{(q-1)2^a i : i \mid m\} \cup \{(q+1)j : j \mid d\},$$

*with the following exceptions:*

1. *If $p$ is odd with $H \nleq S\langle\varphi\rangle$ or if $p = 2$, then $(q+\varepsilon)/2$ is not a degree of $H$.*
2. *If $f$ is odd, $p = 3$, and $H = S\langle\varphi\rangle$, then $i \neq 1$.*
3. *If $f$ is odd, $p = 3$, and $H = A$, then $j \neq 1$.*
4. *If $f$ is odd, $p = 2$, 3 or 5, and $H = S\langle\varphi\rangle$, then $j \neq 1$.*
5. *If $f \equiv 2 \pmod 4$, $p = 2$ or 3, and $H = S\langle\varphi\rangle$ or $H = S\langle\bar{\delta}\varphi\rangle$, then $j \neq 2$.*

If $n$ is an integer and $r$ is a prime number, then we write $r^\alpha \| n$, when $r^\alpha \mid n$ but $r^{\alpha+1} \nmid n$. Also if $r$ is a prime number we denote by $\mathrm{Syl}_r(G)$, the set of Sylow $r$-subgroups of $G$ and we denote by $n_r(G)$, the number of elements of $\mathrm{Syl}_r(G)$. All groups considered are finite and all characters are complex characters. We write $H$ ch $G$ if $H$ is a characteristic subgroup of $G$. All other notations are standard and we refer to [4].

## 2 The Main Results

**Theorem 1.** *Let $G$ be a finite group such that $|G| = |\mathrm{PGL}(2, 5)| = |S_5| = 2^3 \cdot 3 \cdot 5$ and $\beta(1) = 5$ be the second largest element of $\mathrm{cd}(G)$. Then $G \cong \mathrm{PGL}(2, 5) \cong S_5$ or $G \cong 2 \cdot A_5 \cong \mathrm{SL}(2, 5)$.*

**Proof**: First we prove that $G$ is nonsolvable. Otherwise let $H$ be a Hall subgroup of $G$ of order $2^3 5$. Then $|G : H| = 3$ and so $G/H_G \hookrightarrow S_3$, where $H_G = Core_G(H) \trianglelefteq G$. Therefore $5 \mid |H_G|$ and if $P$ is a Sylow 5-subgroup of $H_G$, then $|P| = 5$ and $P$ ch $H_G \lhd G$, which implies that $P \lhd G$. Then by Itō's Theorem, it follows that $5 = \beta(1) \mid |G : P| = 24$, which is a contradiction. Hence $G$ is a nonsolvable group. Now by Lemma 5 we conclude that $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H$ is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Exercise 2.10(iii) in Chapter 3, page 309 in [24] states that $A_5$ is the only noncyclic simple group of order at most $120 = |\mathrm{PGL}(2, 5)|$. Therefore $K/H \cong A_5$ and $|G/K| \cdot |H| = 2$.

If $|G/K| = 1$, then $|H| = 2$ and so $H \subseteq Z(G)$. Since the Schur multiplier of $A_5$ is 2, we get that $G \cong \mathbb{Z}_2 \times A_5$ or $G \cong 2 \cdot A_5$. Since 5 is the greatest element of $\mathrm{cd}(\mathbb{Z}_2 \times A_5)$, we get that $G \cong 2 \cdot A_5 \cong \mathrm{SL}(2,5)$.

Also if $|G/K| = 2$, then $G \cong A_5 \cdot 2 \cong S_5 \cong \mathrm{PGL}(2,5)$. □

**Remark 1.** *We note that by [4],* $\mathrm{cd}(\mathrm{SL}(2,5)) = \{1,2,3,4,5,6\}$ *and* $\mathrm{cd}(\mathrm{PGL}(2,5)) = \{1,4,5,6\}$. *Therefore* $\mathbb{Z}_2 \times PSL(2,5)$, $\mathrm{SL}(2,5)$ *and* $\mathrm{PGL}(2,5)$ *are uniquely determined by their character degrees and their orders.*

**Lemma 7.** *Let $G$ be a group of order $p(p+1)$ with $p$ prime, and $n_p(G) = p+1$. Then $p+1$ is a power of a prime $q$, and a Sylow $q$-subgroup of $G$ is normal in $G$ and elementary abelian. (Hence either $p = 2$ and $G = S_3$, or $p$ is odd and $q = 2$.)*

**Proof**: Since $n_p(G) = p+1$, $G$ has exactly $(p-1)(p+1)$ elements of order $p$. Let $Q$ be the set of elements not of order $p$. So $|Q| = p+1$. If $P$ is a Sylow $p$-subgroup of $G$, then $N_G(P) = P$, so no element of $P$ can centralize any element of $Q - \{1\}$. Hence the $p$ elements of $Q - \{1\}$ are all conjugate under $P$, so they all have the same order, which must be a prime $q$. So all elements of $G$ have order 1, $q$ or $p$, so no other primes divide $|G|$ and hence $p+1$ is a power of $q$, and clearly $Q$ must be the unique (and hence normal) Sylow $q$-subgroup of $G$. Since all of its elements are conjugate in $G$, $Q$ is elementary abelian. □

**Theorem 2.** *Let $p > 5$ be an odd prime number. If $G$ is a finite group such that*

*(i)* $|G| = 2|\mathrm{PSL}(2,p)|$,

*(ii)* $p \in \mathrm{cd}(\mathrm{G})$,

*(iii) there does not exist any* $\theta \in \mathrm{Irr}(G)$ *such that* $2p \mid \theta(1)$,

*(iv) If* $p = 2^\alpha - 1$, *for some* $\alpha > 0$, *then there exists an element* $b \in \mathrm{cd}(\mathrm{G})$ *such that* $4 \mid b$,

*then $G$ has a unique nonabelian composition factor isomorphic to PSL$(2,p)$.*

**Proof**: Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = p$. We prove the result in the following steps:

**Step 1.** *First we prove that if $|G|$ has a divisor of the form $n_p = kp+1$, then $n_p = 1$ or $n_p = p+1$.*

By assumption $n_p = (1+kp) \mid (p-1)(p+1)$. If $k \geq 1$, then there exists a natural number $t$ such that $t(1+kp) = p^2 - 1$. Hence $p \mid (t+1)$ and so there exists $s \in \mathbb{N}$ such that $t+1 = ps$. Therefore $p^2 - 1 = (ps-1)(kp+1)$. Now it is obvious that if $s > 1$ or $k > 1$, then $p^2 - 1 < (sp-1)(kp+1)$ and so $k = s = 1$. Therefore $n_p = p+1$.

**Step 2.** *Now we prove that $G$ is a nonsolvable group.*

On the contrary let $G$ be a solvable group. Now we consider two cases:

**Case 1.** *Let $p+1$ have an odd prime divisor $r$ and $r^\alpha \mid |G|$ and $r^{\alpha+1} \nmid |G|$.*

Then let $H$ be a Hall subgroup of order $p(p-1)(p+1)/r^\alpha$ of $G$. Then $|G : H| = r^\alpha$ and so $G/H_G \hookrightarrow S_{r^\alpha}$. By assumptions, since $r^\alpha \leq (p+1)/2$, it follows that $p \mid |H_G|$. Let

$Q \in \mathrm{Syl}_p(H_G)$. Then $n_p(H_G) = 1$, since $(p+1) \nmid |H_G|$. Therefore $Q$ ch $H_G \lhd G$ and so $Q \lhd G$ and $Q$ is abelian. Now using Itō's Theorem we get a contradiction since $\chi(1) = p$.

**Case 2.** *Let $p + 1 = 2^\alpha$, for some $\alpha > 0$.*

In this case we see that if $R \in \mathrm{Syl}_2(G)$, then $|R| = 2^{\alpha+1}$. Let $H$ be a Hall subgroup of $G$ of order $2p(p+1) = 2^{\alpha+1}p$. Then $|G : H| = (p-1)/2$ and so $G/H_G \hookrightarrow S_{(p-1)/2}$. Hence $p \mid |H_G|$ and $|H_G| \mid 2^{\alpha+1}p$. Therefore $|H_G| = 2^\gamma p$, where $0 \leq \gamma \leq \alpha + 1$. If $\gamma \leq \alpha - 1$, then $|H_G| < p(p+1)$ and so $n_p(H_G) = 1$. Therefore if $Q \in \mathrm{Syl}_p(H_G)$, then $Q \lhd G$ and $Q$ is abelian. Now using Itō's Theorem we get a contradiction. So in the sequel we consider two possibilities: $|H_G| = p(p+1)$ and $|H_G| = 2p(p+1)$.

**(2.1)** Let $|H_G| = p(p+1) = 2^\alpha p$.

Similarly to the previous discussion we get that if $O_p(H_G) \neq 1$, then $O_p(G) \neq 1$ and we get a contradiction by Itō's Theorem. So $n_p(H_G) = p+1$. Let $R \in \mathrm{Syl}_2(H_G)$. By the above discussion and Lemma 7, it follows that $R$ is elementary abelian and $R$ ch $H_G \lhd G$ and so $R \lhd G$. Now since there exists an irreducible character $\beta$ such that $4 \mid \beta(1)$, we get a contradiction by Itō's Theorem.

**(2.2)** Let $|H_G| = 2p(p+1) = 2^{\alpha+1}p$, i.e. $H_G = H$.

Similarly to above $O_p(H) = 1$, and so $O_2(H) \neq 1$. Let $Z = Z(O_2(H))$. Then $Z \lhd G$ and $|Z| = 2^i$, for some $i > 0$. Since there exists an irreducible character $\beta$ such that $4 \mid \beta(1)$, we get that $1 < |Z| \leq 2^{\alpha-1} = (p+1)/2$, by Itō's Theorem. Also $H/Z \lhd G/Z$. Now if $|Z| > 2$, then $|H/Z| < p(p+1)$, $n_p(H/Z) = 1$ and if $Q/Z \in \mathrm{Syl}_p(H/Z)$ and $P \in \mathrm{Syl}_p(Q)$, then $P \lhd G$, which is a contradiction. Otherwise $|Z| = 2$ and $|H/Z| = p(p+1)$. If $Q/Z \in \mathrm{Syl}_p(H/Z)$ is a normal subgroup of $H/Z$, then similarly to the previous cases we get a contradiction by Itō's Theorem. Therefore $n_p(H/Z) = p+1$. Let $R/Z \in \mathrm{Syl}_2(H/Z)$. Analogy to the above discussion and using Lemma 7 it follows that $R/Z$ is elementary abelian and $R/Z$ ch $H/Z \lhd G/Z$ and so $R \lhd G$. Then $R$ is nilpotent of class 2 and $|Z(R)| = 2$. Also $R' = Z(R)$. So $R$ is an extraspecial group. Now the commutator map is non-degenerate, bilinear and alternating and we get an induced map $R/Z(R) \times R/Z(R) \to Z(R)$, which forces $R/Z(R)$ to be elementary abelian of even dimension over the field of order 2. Hence $R$ is of order $2^k$, where $k$ is odd, which is a contradiction since $p = 2^\alpha - 1 > 5$ is a prime and so $\alpha$ is an odd prime number.

Therefore $G$ is not a solvable group.

**Step 3.** *Now we prove that $G$ has a unique nonabelian composition factor isomorphic to $PSL(2, p)$.*

By the above discussion and using Lemma 5 we get that $G$ has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that $K/H$ is a direct product of $m$ copies of a nonabelian simple group $S$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

First we claim that $p \nmid |G/K|$. Otherwise $p \mid |G/K|$ and since $\mathrm{Out}(K/H) \cong \mathrm{Out}(S) \wr S_m$, it follows that $p \mid |S_m|$ or $p \mid |\mathrm{Out}(S)|$. If $p \mid |S_m|$, then $m \geq p$. Now since the smallest order of a nonabelian simple group is 60, it follows that $(p^2 - 1) \geq |K/H| \geq 60^p$, which is impossible. Hence $p \mid |\mathrm{Out}(S)|$ and $p \geq 7$. Then by [4] we get that $S$ is not isomorphic to a sporadic simple group or an alternating group. Therefore $S$ is a simple group of Lie type over $\mathrm{GF}(q)$, where $q = p_0^f$. By assumption, $p \nmid |S|$ and $p \mid |\mathrm{Out}(S)| = dfg$, where $d$, $f$ and $g$ are the orders of diagonal, field and graph automorphisms of $S$ (see [4]). Since $\pi(dg) \subset \pi(S)$, it follows that $p \mid f$. Then $2^p \leq q \leq |S| \leq p^2 - 1$, which is a contradiction. Therefore $p \nmid |G/K|$.

Now let $p \mid |H|$. Let $\eta \in \mathrm{Irr}(H)$ such that $[\chi_H, \eta] \neq 0$. Then $\chi(1)/\eta(1) \mid |G : H|$, which implies that $\eta(1) = p$. Therefore $\chi_H = \eta \in \mathrm{Irr}(H)$ and by Gallagher's Theorem $\beta\chi \in \mathrm{Irr}(G)$, for each $\beta \in \mathrm{Irr}(G/H)$. Since $K/H$ is a normal subgroup of $G/H$, each character degree of $G/H$ will be $\theta(1) \cdot j$, for some irreducible character $\theta$ of $K/H$ and some divisor $j$ of $|G : K|$. Also $K/H$ is a direct product of $m$ copies of a nonabelian simple group $S$ and using Itō-Michler Theorem we know that $S$ has an even irreducible character degree and so $K/H$ has such a character degree. Therefore there exists an irreducible character $\eta \in \mathrm{Irr}(G)$ such that $2p \mid \eta(1)$, which is a contradiction.

Therefore $p \mid |K/H|$. Since $p^2 \nmid |G|$, it follows that $K/H$ is a nonabelian simple group, say $S$, such that $p$ is the largest prime divisor of $|S|$ and $|S| \mid p(p^2-1)$. Now we use the classification of finite simple groups in [4]. The orders of sporadic simple groups show that $S$ is not a sporadic simple group. If $S \cong A_n$, where $n \geq 5$, then $7 \leq p \leq n$ and so $n!/2 = |A_n| \leq p(p^2-1) \leq n(n^2-1)$, which is a contradiction. Therefore $K/H$ is isomorphic to a simple group of Lie type.

If $S$ is a nonabelian simple group of Lie type over $\mathrm{GF}(q)$, where $p \nmid q$, then there exists $i > 0$ such that $p \mid (q^i - 1)$ or $p \mid (q^i + 1)$. Now the order of $S$ shows that the only possibility for $S$ is $\mathrm{PSL}(3, 2) \cong \mathrm{PSL}(2, 7)$.

If $S$ is a nonabelian simple group of Lie type over a field of characteristic $p$, say $\mathbb{F}$, then $\mathbb{F} = \mathrm{GF}(p)$. Now by easy computation we see that $p^2 \nmid |S|$ and $|S| \mid p(p^2-1)$ imply that $S \cong \mathrm{PSL}(2, p)$.

Therefore $K/H \cong \mathrm{PSL}(2, p)$ and so $|H| = 2$ or $|G/K| = 2$. Therefore the main theorem is proved. ∎

**Theorem 3.** *Let $G$ be a finite group satisfying the hypothesis of Theorem 2 and $\varepsilon = (-1)^{(p-1)/2}$. Then the following hold:*

1. *If $(p+\varepsilon)/2 \notin \mathrm{cd(G)}$, then $G \cong PGL(2, p)$;*
2. *if $(p+\varepsilon)/2 \in \mathrm{cd(G)}$ and $(p-\varepsilon)/2 \notin \mathrm{cd(G)}$, then $G \cong \mathbb{Z}_2 \times PSL(2, p)$;*
3. *if $(p+\varepsilon)/2, (p-\varepsilon)/2 \in \mathrm{cd(G)}$, then $G \cong \mathrm{SL}(2, p)$.*

**Proof**: Using Theorem 2 we know that $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2, p)$ and $|H| \cdot |G/K| = 2$. If $|G/K| = 2$, then $G \cong \mathrm{PGL}(2, p)$. Let $|H| = 2$. Since the Schur multiplier of $\mathrm{PSL}(2, p)$ is 2, it follows that $G \cong \mathbb{Z}_2 \times \mathrm{PSL}(2, p)$ or $G \cong 2 \cdot \mathrm{PSL}(2, p) \cong \mathrm{SL}(2, p)$, and in each case $(p+\varepsilon)/2 \in \mathrm{cd}(G)$. Also $(p-\varepsilon)/2 \notin \mathrm{cd(PSL(2, p))}$ and $(p-\varepsilon)/2 \in \mathrm{cd(SL(2, p))}$. So we get the result in each case. ∎

In [25] Tong-Viet posed the following question:
**Question.** *Which groups can be uniquely determined by the structure of their complex group algebras?*

It was shown in [22] that the symmetric groups are uniquely determined by the structure of their complex group algebras. It was conjectured that all nonabelian simple groups are uniquely determined by the structure of their complex group algebras. This conjecture was verified in [20, 23, 26] for the alternating groups, the sporadic simple groups, the Tits group and the simple exceptional groups of Lie type. We note that abelian groups are not determined by the structure of their complex group algebras. In fact the complex group algebras of any two abelian groups of the same orders are isomorphic. There are also examples of nonabelian p-groups with isomorphic complex group algebras, for example the dihedral group of order 8 and the quaternion group of order 8.

As a consequence of our results we get the following result:

**Corollary 1.** *Let $p$ be a prime number and $M = PGL(2, p)$, $M = \mathbb{Z}_2 \times PSL(2, p)$ or $M = \mathrm{SL}(2, p)$. If $G$ is a finite group such that $X_1(G) = X_1(M)$, then $G \cong M$.*

The following result is an answer to the above question in [25].

**Corollary 2.** *Let $p$ be a prime number and $M = PGL(2, p)$, $M = \mathbb{Z}_2 \times PSL(2, p)$ or $M = \mathrm{SL}(2, p)$. If $G$ is a group such that $\mathbb{C}G \cong \mathbb{C}M$, then $G \cong M$. Thus $PGL(2, p)$, $\mathbb{Z}_2 \times PSL(2, p)$ and $\mathrm{SL}(2, p)$ are uniquely determined by the structure of their complex group algebras.*

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