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On the generalized Ramanujan-Nagell equation
$x^2 + q^m = c^n$ with $q^r + 1 = 2c^2$
by

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Abstract

Let q be an odd prime, and let c, r be positive integers with $q^r + 1 = 2c^2$. For any nonnegative integer s, let $U_{2s+1} = \left(\alpha^{2s+1} + \beta^{2s+1}\right)/2$ and $V_{2s+1} = \left(\alpha^{2s+1} - \beta^{2s+1}\right)/2\sqrt{2}$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. In this paper we prove the following results: (i) If $r > 2$, then $(q, r, c) = (23, 3, 78)$ and the equation $x^2 + 23^m = 78^n$ has only the positive integer solution $(x, m, n) = (6083, 3, 4)$. (ii) If $r = 2$ and $(q, c) = (U_{2s+1}, V_{2s+1})$ with $s \not\equiv 0 (\text{mod} 4)$, then the equation $x^2 + q^m = c^n$ has only the positive integer solution $(x, m, n) = (c^2 - 1, 2, 4)$.

Key Words: exponential diophantine equation; generalized Ramanujan-Nagell equation; Pell number.
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1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers respectively. For coprime positive integers b and c, there were many papers investigated the generalized Ramanujan-Nagell equations of the form

$$x^2 + b^m = c^n, \ x, m, n \in \mathbb{N} \tag{1.1}$$

(see [1] and [3]-[19]).

Let q be an odd prime, and let c, r be positive integers with

$$q^r + 1 = 2c^2. \tag{1.2}$$

Recently, N. Terai [15] proved that the equation

$$x^2 + q^m = c^n, \ x, m, n \in \mathbb{N} \tag{1.3}$$

has only the solution $(x, m, n) = (c^2 - 1, r, 4)$ under some conditions. In this paper we shall deal with the solutions of (1.3) for $r > 1$.

By the Proposition 8.1 of [2], if $r > 2$, then from (1.2) we get only $(q, r, c) = (23, 3, 78)$. We completely solve (1.3) in this case as follows:

Theorem 1.1 If $(q, c) = (23, 78)$, then (1.3) has only the solution $(x, m, n) = (6083, 3, 4)$.

For any nonnegative integer k, let

$$U_k = \frac{1}{2}\left(\alpha^k + \beta^k\right), \ V_k = \frac{1}{2\sqrt{2}}\left(\alpha^k - \beta^k\right), \tag{1.4}$$

where
$$\alpha = 1 + \sqrt{2},\ \beta = 1 - \sqrt{2}. \tag{1.5}$$
It is well known that $(u, v) = (U_{2s}, V_{2s})(s = 1, 2, \cdots)$ and $(U, V) = (U_{2s+1}, V_{2s+1})$ $(s = 0, 1, \cdots)$ are all solutions of the Pell equations
$$u^2 - 2v^2 = 1,\ u, v \in \mathbb{N} \tag{1.6}$$
and
$$U^2 - 2V^2 = -1,\ U, V \in \mathbb{N}, \tag{1.7}$$
respectively. Therefore, if $r = 2$, then from (1.2) we get
$$q = U_{2s+1},\ c = V_{2s+1},\ s \in \mathbb{N}. \tag{1.8}$$
In this respect, we prove the following result:

Theorem 1.2 If q and c satisfy (1.8) with $s \not\equiv 0(\bmod 4)$, then (1.3) has only the solution $(x, m, n) = (c^2 - 1, 2, 4)$.

2 Proof of Theorem 1.1

Lemma 2.1 ([20]) Let X, n be positive integers with $\min\{X, n\} > 1$. Except when $(X, n) = (2, 3)$, $X^n + 1$ has a prime divisor p satisfies $p \nmid X^m + 1$ for any positive integer m with $m < n$.

Lemma 2.2 (1.3) has only the solution $(x, m, n) = (c^2 - 1, r, 4)$ with $2 \mid n$.

Proof Let (x, m, n) be a solution of (1.3) with $2 \mid n$. Since q is an odd prime, by (1.3), we have
$$c^{n/2} + x = q^m,\ c^{n/2} - x = 1, \tag{2.1}$$
whence we get
$$q^m + 1 = 2c^{n/2}. \tag{2.2}$$
If $m < r$, then from (1.2) and (2.2) we get $n = 2$ and
$$q^m + 1 = 2c. \tag{2.3}$$
By (1.2) and (2.3), $q^r + 1$ has no prime divisor p satisfies $p \nmid q^m + 1$. But, since $q \neq 2$, by Lemma 2.1, it is impossible.

Similarly, if $m > r$, then $q^m + 1$ has no prime divisor p satisfies $p \nmid q^r + 1$. Therefore, it is impossible. Thus, (1.3) has only the solution $(x, m, n) = (c^2 - 1, r, 4)$ with $2 \mid n$. The lemma is proved.

Let D be a positive integer which is not a square.

Lemma 2.3 ([13, Theorem 8.1]) The Pell equation
$$u^2 - Dv^2 = 1,\ u, v \in \mathbb{N} \tag{2.4}$$
has positive integer solutions (u, v), and it has unique positive integer solution (u_1, v_1) such that $u_1 + v_1\sqrt{D} \leq u + v\sqrt{D}$, where (u, v) through all positive integer solutions of (2.4).

The solution (u_1, v_1) is called the least solution of (2.4). Every solution (u, v) of (2.4) can be expressed as

$$u + v\sqrt{D} = \lambda_1 \left(u_1 + \lambda_2 v_1 \sqrt{D}\right)^t, \ t \in \mathbb{Z}, \ t \geq 0.$$

Using the same method as in the proof of Lemma 3 of [8], we can obtain the following lemma immediately.

Lemma 2.4 Let p be an odd prime with $p \nmid D$. If the equation

$$X^2 - DY^2 = (-p)^Z, \ X, Y, Z \in \mathbb{Z}, \ \gcd(X, Y) = 1, \ Z > 0 \tag{2.5}$$

has solutions (X, Y, Z), then it has a unique solution (X_1, Y_1, Z_1) such that $X_1 > 0$, $Y_1 > 0$, $Z_1 \leq Z$ and

$$1 < \left|\frac{X_1 + Y_1\sqrt{D}}{X_1 - Y_1\sqrt{D}}\right| < u_1 + v_1\sqrt{D},$$

where Z through all solutions (X, Y, Z) of (2.5), (u_1, v_1) is the least solution of (2.4). The solution (X_1, Y_1, Z_1) is called the least solution of (2.5). Every solution (X, Y, Z) can be expressed as

$$Z = Z_1 t, \ t \in \mathbb{N},$$

$$X + Y\sqrt{D} = \left(X_1 + \lambda Y_1 \sqrt{D}\right)^t \left(u + v\sqrt{D}\right), \ \lambda \in \{1, -1\},$$

where (u, v) is a solution of (2.4).

Proof of Theorem 1.1 We now assume that (x, m, n) is a solution of the equation

$$x^2 + 23^m = 78^n, \ x, m, n \in \mathbb{N} \tag{2.6}$$

with $(x, m, n) \neq (6083, 3, 4)$. By Lemma 2.2, we have $2 \nmid n$. Obviously, (2.6) has no solutions (x, m, n) with $n \in \{1, 3\}$, we get

$$2 \nmid n, \ n \geq 5. \tag{2.7}$$

Further, by (2.6) and (2.7), we have $2 \nmid x$ and $0 \equiv 78^n \equiv x^2 + 23^m \equiv 1 + (-1)^m (\bmod 4)$. It implies that

$$2 \nmid m. \tag{2.8}$$

Since $2 \nmid n$, we see from (2.6) that the equation

$$X^2 - 78Y^2 = (-23)^Z, \ X, Y, Z \in \mathbb{Z}, \ \gcd(X, Y) = 1, \ Z > 0 \tag{2.9}$$

has the solution

$$(X, Y, Z) = (x, 78^{(n-1)/2}, m). \tag{2.10}$$

Notice that $(u_1, v_1) = (53, 6)$ is the least solution of the Pell equation

$$u^2 - 78v^2 = 1, \ u, v \in \mathbb{Z}, \tag{2.11}$$

and $(X, Y, Z) = (17, 2, 1)$ is a positive integer solution of (2.9) satisfies $1 < |(17 + 2\sqrt{78})/\left(17 - 2\sqrt{78}\right)| < 53 + 6\sqrt{78}$. It implies that $(X_1, Y_1, Z_1) = (17, 2, 1)$ is the least solution of (2.9). Therefore, applying Lemma 2.4 to (2.9) and (2.10), we get

$$x + 78^{(n-1)/2}\sqrt{78} = \left(17 + 2\lambda\sqrt{78}\right)^m \left(u + v\sqrt{78}\right), \ \lambda \in \{1, -1\}, \tag{2.12}$$

where (u, v) is a solution of (2.11).

We first consider the case of $\lambda = 1$. Then, by (2.12), we get

$$x + 78^{(n-1)/2}\sqrt{78} = \left(17 + 2\sqrt{78}\right)^m \left(u + v\sqrt{78}\right). \tag{2.13}$$

Since $x + 78^{(n-1)/2}\sqrt{78} > 0$ and $17 + 2\sqrt{78} > 0$, we have $u + v\sqrt{78} > 0$. Hence, by Lemma 2.3, we have

$$u + v\sqrt{78} = \left(53 + 6\lambda'\sqrt{78}\right)^t, \ \lambda' \in \{1, -1\}, \ t \in \mathbb{Z}, \ t \geq 0. \tag{2.14}$$

Let

$$X + Y\sqrt{78} = \left(17 + 2\sqrt{78}\right)^m. \tag{2.15}$$

Then X and Y are positive integers. By (2.13) and (2.15), we have

$$x + 78^{(n-1)/2}\sqrt{78} = \left(X + Y\sqrt{78}\right)^m \left(u + v\sqrt{78}\right). \tag{2.16}$$

whence we get

$$78^{(n-1)/2} = Xv + Yu. \tag{2.17}$$

By (2.14) and (2.15), we have

$$u \equiv 53^t (\bmod 78), \ v \equiv 53^{t-1} \cdot 6t (\bmod 78) \tag{2.18}$$

and

$$X \equiv 17^m (\bmod 78), \ Y \equiv 17^{m-1} \cdot 2m (\bmod 78), \tag{2.19}$$

respectively. Therefore, since $n \geq 5$ and $\gcd(17, 78) = \gcd(53, 78) = 1$, we get from (2.17), (2.18) and (2.19) that $0 \equiv 78^{(n-1)/2} \equiv 17^m \cdot 53^{t-1} \cdot 6t + 53^t \cdot 17^{m-1} \cdot 2m (\bmod 78)$ and

$$51t + 53m \equiv 0 (\bmod 39). \tag{2.20}$$

Further, since $3 \mid 39$, $3 \mid 51$ and $3 \nmid 53$, by (2.20), we have

$$3 \mid m. \tag{2.21}$$

Let $k = [t/3]$ be the integer part of $t/3$. Since $t \geq 0$, k is a nonnegative integer satisfies

$$t = 3k + l, \ l \in \{0, 1, 2\}. \tag{2.22}$$

Further let

$$a + b\sqrt{78} = \begin{cases} \left(17 + 2\sqrt{78}\right)^{m/3} \left(53 + 6\lambda'\sqrt{78}\right)^k, & \text{if } l \in \{0, 1\}, \\ \left(17 + 2\sqrt{78}\right)^{m/3} \left(53 + 6\lambda'\sqrt{78}\right)^{k+1}, & \text{if } l = 2. \end{cases} \tag{2.23}$$

Since $3 \mid m$, by Lemma 2.4, a and b are integers satisfy

$$a^2 - 78b^2 = -23^{m/3}, \ \gcd(a, b) = 1. \tag{2.24}$$

Since $1/\left(53+6\lambda'\sqrt{78}\right)=53-6\lambda'\sqrt{78}$, by (2.13), (2.14), (2.21), (2.22) and (2.23), we have

$$x+78^{(n-1)/2}\sqrt{78}=\begin{cases}\left(a+b\sqrt{78}\right)^3, & \text{if } l=0,\\ \left(a+b\sqrt{78}\right)^3\left(53+6\lambda''\sqrt{78}\right), & \text{if } l\neq 0.\end{cases} \tag{2.25}$$

where $\lambda''\in\{1,-1\}$.

If $l=0$, then from (2.25) we get

$$78^{(n-1)/2}=3b(a^2+26b^2). \tag{2.26}$$

Since $\gcd(a^2,78b^2)=1$ by (2.24), we have $\gcd(26,a^2+26b^2)=1$. Therefore, we see from (2.26) that

$$b\geq 26^{(n-1)/2},\ 0<a^2+26b^2\leq 3^{(n-3)/2}, \tag{2.27}$$

whence we get $3^{(n-3)/2}\geq a^2+26b^2>26b^2\geq 26^n$, a contradiction.

If $l\neq 0$, then we have

$$78^{(n-1)/2}=6\lambda''a^3+159a^2b+1404\lambda''ab^2+4134b^3. \tag{2.28}$$

Since $2\nmid a$ by (2.24), we get from (2.28) that $2\mid b$, $b=2d$ and

$$6^{(n-3)/2}\cdot 13^{(n-1)/2}=\lambda''a^3+53a^2b+936\lambda''ad^2+5512d^3. \tag{2.29}$$

Further, since $n\geq 5$, by (2.29), we have

$$\lambda''a^3+2a^2b+d^3\equiv 0(\text{mod}3). \tag{2.30}$$

Furthermore, since $\gcd(a,b)=1$, we get $\gcd(a,d)=1$ and $(a,d)\neq(0,0)(\text{mod}3)$. Therefore, since

$$\lambda''a^3+2a^2b+d^3\equiv\begin{cases}1(\text{mod}3), & \text{if } \lambda''=1 \text{ and } (a,d)\equiv(0,1),\ (1,0),\\ & (1,1),\ (1,2)(\text{mod}3) \text{ or } \lambda''=-1\\ & \text{and } (a,d)\equiv(0,1),\ (2,1)(\text{mod}3)\\ 2(\text{mod}3), & \text{if } \lambda''=1 \text{ and } (a,d)\equiv(2,1),\ (2,2)\\ & (\text{mod}3) \text{ or } \lambda''=-1 \text{ and } (a,d)\equiv(1,0)\\ & (1,1),\ (1,2),\ (2,2)(\text{mod}3).\end{cases}$$

(2.30) is impossible. Thus, (2.12) is false for $\lambda=1$.

Using the same method as in the above analysis, we can prove that (2.12) is false for $\lambda=-1$. Thus, (2.6) has the solution $(x,m,n)=(6083,3,4)$. The theorem is proved.

3 Proof of the Theorem 1.2

Lemma 3.1 ([11], [12]) The equation

$$X^4-2Y^2=\pm 1,\ X,Y\in\mathbb{N} \tag{3.1}$$

has only the solution $(X,Y)=(1,1)$.

Lemma 3.2 ([13, Section 15.2]) Let n be an odd positive integer with $n>1$. Every solution (X,Y,Z) of the equation

$$X^2+Y^2=Z^n, X,Y,Z\in\mathbb{N},\ \gcd(X,Y)=1 \tag{3.2}$$

can be expressed as

$$Z=a^2+b^2,\ a,b\in\mathbb{N},\ \gcd(a,b)=1,$$
$$X+Y\sqrt{-1}=\lambda_1\left(a+\lambda_2 b\sqrt{-1}\right)^n,\ \lambda_1,\lambda_2\in\{1,-1\}.$$

Lemma 3.3 For any nonnegative integer s, we have

$$U_{2s+1}=\begin{cases}1(\mathrm{mod}8), & \text{if } s\equiv 0(\mathrm{mod}2),\\ 7(\mathrm{mod}8), & \text{if } s\equiv 1(\mathrm{mod}2),\end{cases} \tag{3.3}$$

$$V_{2s+1}=\begin{cases}1(\mathrm{mod}8), & \text{if } s\equiv 0 \text{ or } 3(\mathrm{mod}4),\\ 5(\mathrm{mod}8), & \text{if } s\equiv 1 \text{ or } 2(\mathrm{mod}4).\end{cases} \tag{3.4}$$

Proof By (1.4) and (1.5), we get

$$U_{2s+5}-U_{2s+1}=4(3U_{2s+2}+U_{2s+1}). \tag{3.5}$$

Since $2\nmid U_{2s+1}U_{2s+2}$, we see from (3.5) that $U_{2s+5}\equiv U_{2s+1}(\mathrm{mod}8)$. Therefore, by the initial values $U_1=1$ and $U_3=7$, we obtain (3.3).

Similarly, since $2\nmid V_{2s+1}V_{2s+3}$ and $V_{2s+9}-V_{2s+1}=4(51V_{2s+3}-9V_{2s+1})$, we have $V_{2s+9}\equiv V_{2s+1}(\mathrm{mod}8)$. Therefore, by the initial values $V_1=1$, $V_3=5$, $V_5=29$ and $V_7=169$, we get (3.4). The lemma is proved.

Lemma 3.4 For any nonnegative integer s, we have $(V_{8s+5}/U_{8s+5})=-1$, where $(*/*)$ is the Jacobi symbol.

Proof By (1.4) and (1.5), we have

$$U_{8s+5}+V_{8s+5}=V_{8s+6}=2U_{4s+3}V_{4s+3}. \tag{3.6}$$

Hence, by (3.6), we get

$$\begin{aligned}\left(\frac{V_{8s+5}}{U_{8s+5}}\right)&=\left(\frac{U_{8s+5}+V_{8s+5}}{U_{8s+5}}\right)=\left(\frac{V_{8s+6}}{U_{8s+5}}\right)\\&=\left(\frac{2}{U_{8s+5}}\right)\left(\frac{U_{4s+3}}{U_{8s+5}}\right)\left(\frac{V_{4s+3}}{U_{8s+5}}\right).\end{aligned} \tag{3.7}$$

By Lemma 3.3, we have $U_{8s+5}\equiv 1(\mathrm{mod}8)$ and

$$\left(\frac{2}{U_{8s+5}}\right)=1. \tag{3.8}$$

Using (1.4) and (1.5) again, we have

$$U_{8s+5}=4U_{4s+2}U_{4s+3}-1=4V_{4s+2}V_{4s+3}+1. \tag{3.9}$$

Since $U_{8s+5} \equiv 1(\mathrm{mod}4)$ and $U_{4s+3} \equiv 3(\mathrm{mod}4)$, by (3.9), we get

$$\left(\frac{U_{4s+3}}{U_{8s+5}}\right) = \left(\frac{U_{8s+5}}{U_{4s+3}}\right) = \left(\frac{-1}{U_{4s+3}}\right) = -1,$$

$$\left(\frac{V_{4s+3}}{U_{8s+5}}\right) = \left(\frac{U_{8s+5}}{U_{4s+3}}\right) = \left(\frac{1}{V_{4s+3}}\right) = 1. \tag{3.10}$$

Therefore, by (3.7), (3.8) and (3.10), the lemma is proved.

Lemma 3.5 The equation

$$X^2 + 1 = V_{2s+1},\ X, s \in \mathbb{N} \tag{3.11}$$

has only the solution $(X, s) = (2, 1)$.

Proof Let (X, s) be a solution of (3.11). By (1.4) and (1.5), we have

$$V_{2s+1} - 1 = \begin{cases} 2U_{s+1}V_s, & \text{if } 2 \mid s, \\ 2U_sV_{s+1}, & \text{if } 2 \nmid s. \end{cases} \tag{3.12}$$

Since $2 \nmid U_sU_{s+1}$ and $\gcd(U_{s+1}, V_s) = \gcd(U_s, V_{s+1}) = 1$, we see from (3.11) and (3.12) that either

$$U_{s+1} = a^2,\ V_s = 2b^2,\ X = 2ab,\ a, b \in \mathbb{N},\ 2 \mid s \tag{3.13}$$

or

$$U_s = a^2,\ V_{s+1} = 2b^2,\ X = 2ab,\ a, b \in \mathbb{N},\ 2 \nmid s \tag{3.14}$$

Using Lemma 3.1, by (1.6) and (1.7), (3.13) is false and (3.14) holds if and only if $s = 1$. Therefore, (3.11) has only the solution $(X, s) = (2, 1)$. The lemma is proved.

Proof of Theorem 1.2 We now assume that (x, m, n) is a solution of (1.3) with $(x, m, n) \neq (c^2 - 1, 2, 4)$. By Lemma 2.2, we have $2 \nmid n$.

If $2 \mid m$, then (3.2) has the solution

$$(X, Y, Z) = (x, q^{m/2}, c). \tag{3.15}$$

Applying Lemma 3.2 to (3.15), we get

$$c = a^2 + b^2,\ a, b \in \mathbb{N},\ \gcd(a, b) = 1, \tag{3.16}$$

$$x + q^{m/2}\sqrt{-1} = \lambda_1 \left(a + \lambda_2 b\sqrt{-1}\right)^n,\ \lambda_1, \lambda_2 \in \{1, -1\}. \tag{3.17}$$

By (3.17), we have $b \mid q^{m/2}$. When $b > 1$, since q is an prime, we have $q \mid b$ and $b \geq q$. But, by (1.2) and (3.16), we get $c = a^2 + b^2 \geq 1 + b^2 \geq 1 + q^2 = 2c^2$, a contradiction. When $b = 1$, by Lemma 3.5, we get from (1.8) and (3.16) that $q = U_3 = 7$ and $c = V_3 = 5$. However, since $2 \nmid n$, by (1.3), we get $1 = (5/7) = (7/5) = (2/5) = -1$, a contraction. So we have $2 \nmid m$.

By Lemma 3.3, we have $U_{2s+1} \equiv (-1)^s(\mathrm{mod}4)$ and $V_{2s+1} \equiv 1(\mathrm{mod}4)$. Since $2 \mid x$ and $2 \nmid mn$, by (1.3), we get $1 \equiv c^n \equiv x^2 + q^m \equiv 0 + q \equiv q \equiv U_{2s+1} \equiv (-1)^s(\mathrm{mod}4)$. It implies that $2 \mid s$. Further, since $s \not\equiv 0(\mathrm{mod}4)$, we have $2s + 1 = 8l + 5$, where l is a nonnegative integer. Therefore, by (1.3) and (1.8), we get

$$\left(\frac{c}{q}\right) = \left(\frac{V_{8l+5}}{U_{8l+5}}\right) = 1. \tag{3.18}$$

But, by Lemma 3.4, (3.18) is false. Thus, if $r = 2$ and $s \not\equiv 0(\mathrm{mod}4)$, then (1.3) has only the solution $(x, m, n) = (c^2 - 1, 2, 4)$. The theorem is proved.

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