Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 3, 2017, 239-255

Slant immersions in C_5-manifolds
by

Salvatore de Candia and Maria Falcitelli

Abstract

Odd-dimensional non anti-invariant slant submanifolds of an α-Kenmotsu manifold are studied. We relate slant immersions into a Kähler manifold with suitable slant submanifolds of an α-Kenmotsu manifold. More generally, in the framework of Chinea-Gonzalez, we specify the type of the almost contact metric structure induced on a slant submanifold, then stating a local classification theorem. The case of austere immersions is discussed. This helps in proving a reduction theorem of the codimension. Finally, slant submanifolds which are generalized Sasakian space-forms are described.

Key Words: slant submanifold, α-Kenmotsu manifold, warped product manifold, warped product immersion, generalized Sasakian space-form.
2010 Mathematics Subject Classification: Primary 53C25, Secondary 53C42, 53C15.

1 Introduction

The theory of slant submanifolds, started by B. Y. Chen in 1990 in the context of Hermitian Geometry, has been quickly developed in the last two decades.

In 1996, Lotta introduced the concept of a slant submanifold of an almost contact metric (a.c.m.) manifold, showing that there are two types of non anti-invariant slant submanifolds, depending on the position of the Reeb vector field ξ of the ambient space ([16]).
More precisely, if N is a non anti-invariant submanifold of an a.c.m. manifold M and N has dimension n, one has: n is odd (resp. n is even) if and only if ξ is tangent (resp. ξ is orthogonal) to N. Moreover, if n is odd, then N inherits from M an a.c.m. structure.

Slant immersions in a contact metric manifold, in particular in a Sasakian manifold, have been intensively studied ([3, 17]). Further results are known when the structure of the ambient space is a particular type, namely it is cosymplectic, or Kenmotsu or trans-Sasakian ([14, 12, 13] and References therein).

In this paper we relate immersions in almost Hermitian (a.H.) manifolds with submanifolds of a.c.m. manifolds.
Actually, starting by a slant submanifold of an a.H. manifold $\widehat{M}$, we construct a whole family of slant immersions in suitable a.c.m. manifolds, with the same slant angle. The ambient spaces are warped product manifolds $I \times_\lambda \widehat{M}$, I being an open interval of $\mathbb{R}$ and λ a positive real-valued smooth function, endowed with an a.c.m. structure naturally induced by the a.H. structure on $\widehat{M}$. In particular, if $\widehat{M}$ is a Kähler manifold, then $I \times_\lambda \widehat{M}$ turns out to be an α-Kenmotsu manifold, namely $I \times_\lambda \widehat{M}$ falls in the Chinea-Gonzalez class C_5.

So, we provide explicit examples of odd-dimensional slant submanifolds of a C_5-manifold, arising by the ones given in [4].

More generally, one considers a non anti-invariant slant submanifold (N, f) of a C_5-manifold M such that ξ is tangent to N. We specify the Chinea-Gonzalez class of N, which is endowed with the a.c.m. structure induced by M. Indeed, N turns out to be a $C_2 \oplus C_5$-manifold. This allows us to state that N is, locally, a warped product $]-\varepsilon, \varepsilon[\times_\lambda \mathcal{F}$, $\varepsilon > 0$, $\mathcal{F}$ being an almost Kähler manifold and $\lambda\colon]-\varepsilon, \varepsilon[\to \mathbb{R}$ a smooth positive function. Under suitable conditions, involving the second fundamental form of the immersion, one obtains that N is a C_5-manifold.

We also discuss the case of minimal, in particular austere, slant immersion. This allows us to prove a reduction theorem of codimension for submanifolds of the hyperbolic space.

Finally, we locally classify those submanifolds which are generalized Sasakian space-forms.

In this article all manifolds are assumed to be connected.

2 Preliminaries

Let $(\widehat{M}, \widehat{J}, \widehat{g})$ be an a.H. manifold and $\widehat{f}\colon (\widehat{N}, \widehat{g}') \to (\widehat{M}, \widehat{g})$ an isometric immersion. For any $x \in \widehat{N}$ and $X \in T_x\widehat{N}$, we make use of the identifications $x \equiv \widehat{f}(x)$ and $X \equiv (\widehat{f}_*)_x X$, where $(\widehat{f}_*)_x$ is the tangential map.

For any $X \in T\widehat{N}$ we put $\widehat{J}X = \widehat{P}X + \widehat{F}X$, where $\widehat{P}X$ and $\widehat{F}X$ denote the tangential and the normal components of $\widehat{J}X$, respectively. Also, for any $V \in T^\perp\widehat{N}$, we put $\widehat{J}V = \widehat{t}V + \widehat{n}V$, $\widehat{t}V$, $\widehat{n}V$ being the tangential and the normal components of $\widehat{J}V$. This allows us to define smooth maps $\widehat{P}\colon T\widehat{N} \to T\widehat{N}$, $\widehat{F}\colon T\widehat{N} \to T^\perp\widehat{N}$, $\widehat{t}\colon T^\perp\widehat{N} \to T\widehat{N}$ and $\widehat{n}\colon T^\perp\widehat{N} \to T^\perp\widehat{N}$ inducing linear maps on each fibre. Since $(\widehat{J}, \widehat{g})$ is an a.H. structure, for any $X, Y \in T\widehat{N}$, one has $\widehat{g}'(\widehat{P}X, Y) = -\widehat{g}'(X, \widehat{P}Y)$.

It follows that, for any $x \in \widehat{N}$, $\widehat{Q} = \widehat{P}^2\colon T_x\widehat{N} \to T_x\widehat{N}$ is a self-adjoint endomorphism, its non-zero eigenvalues have even multiplicity and belong to $[-1, 0[$.

The tensor fields of type $(1,1)$ on $\widehat{N}$ determined by $\widehat{P}$, $\widehat{Q}$ are denoted by the same symbol.

For any non-zero vector $X \in T_x\widehat{N}$, $x \in \widehat{N}$, the Wirtinger angle of X is the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between $\widehat{J}X$ and $T_x\widehat{N}$. The immersion $\widehat{f}\colon \widehat{N} \to \widehat{M}$ is said to be a slant immersion if the angle $\theta(X)$ is a constant θ, that is θ does not depend on the choice of X and $x \in \widehat{N}$. In this case, θ is called the slant angle of $\widehat{N}$ in $\widehat{M}$ ([5]). One says that $(\widehat{N}, \widehat{f})$ is a slant submanifold of $\widehat{M}$ and adopts the notation $sla(\widehat{N}) = \theta$. If $sla(\widehat{N}) = \theta \neq \frac{\pi}{2}$, then the dimension of $\widehat{N}$ is even and $(\frac{1}{\cos\theta}\widehat{P}, \widehat{g}')$ is an a.H. structure on $\widehat{N}$.

Holomorphic submanifolds and totally real submanifolds are nothing but slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold is called proper if it is neither holomorphic nor totally real.

Now, let $(M, \varphi, \xi, \eta, g)$ be an a.c.m. manifold and $f\colon (N, g') \to (M, g)$ an isometric immersion. For any $X \in TN$ we put $\varphi X = PX + FX$, where PX and FX denote the tangential and the normal components of φX, respectively. Also, for any $x \in N$, $V \in T_x^\perp N$ we put $\varphi V = tV + nV$, $tV \in T_xN$ and $nV \in T_x^\perp N$. So, one defines smooth maps $P\colon TN \to TN$, $F\colon TN \to T^\perp N$, $t\colon T^\perp N \to TN$ and $n\colon T^\perp N \to T^\perp N$ inducing linear maps on each fibre. In particular, P determines a tensor field on N, denoted again by P, that satisfies $g'(PX,Y)+g'(X,PY)=0$. Putting $Q=P^2$, at any $x\in N$, Q is a self-adjoint endomorphism whose non-zero eigenvalues belong to $[-1,0[$ and have even multiplicity.

As in [16], the immersion $f\colon N\to M$ is said to be a slant immersion (briefly, N is slant in M) if, for any $x\in N$, $X\in T_xN$ such that X, ξ are linearly independent, the Wirtinger angle $\theta(X)\in[0,\frac{\pi}{2}]$ between φX and T_xN is a constant θ. In this case, one adopts the notation $sla(N)=\theta$ and θ is called the slant angle of N in M. In particular, if $\theta=0$ (resp. $\theta=\frac{\pi}{2}$), then (N,f) turns out to be an invariant (resp. anti-invariant) submanifold of M. Moreover, if $\theta\neq\frac{\pi}{2}$ and ξ is tangent to N, putting $\varphi'=\frac{1}{\cos\theta}P$, $\eta'=f^*\eta$, $\xi'=(\eta')^\sharp$, one easily proves that (φ',ξ',η',g') is an a.c.m. structure, called the a.c.m. structure induced on N by f.

In order to emphasize the link between the two concepts of a slant immersion, we focus on a particular class of a.c.m. manifolds strictly related to a.H. manifolds ([10]).

Given an a.H. manifold $(\widehat{M},\widehat{J},\widehat{g})$, an open interval $I\subset\mathbb{R}$ and a smooth function $\lambda\colon I\to\mathbb{R}$, $\lambda>0$, we consider the a.c.m. structure $(\varphi,\xi,\eta,g_\lambda)$ on $I\times\widehat{M}$ such that

$$\begin{aligned}&\varphi(a\frac{\partial}{\partial t},X)=(0,\widehat{J}X),\quad \eta(a\frac{\partial}{\partial t},X)=a,\quad a\in\mathfrak{F}(I\times\widehat{M}),X\in\Gamma(T\widehat{M})\\ &\xi=(\frac{\partial}{\partial t},0),\quad g_\lambda=\pi^*(dt\otimes dt)+\lambda^2\sigma^*(\widehat{g}),\end{aligned}\tag{2.1}$$

$\pi\colon I\times\widehat{M}\to I$, $\sigma\colon I\times\widehat{M}\to\widehat{M}$ denoting the canonical projections. Note that g_λ is the warped product metric of the Euclidean metric g_0 and $\widehat{g}$. The a.c.m. manifold $I\times_\lambda\widehat{M}=(I\times\widehat{M},\varphi,\xi,\eta,g_\lambda)$ is called the warped product manifold of (I,g_0) and $(\widehat{M},\widehat{J},\widehat{g})$ by λ. We identify any vector field X on $\widehat{M}$ with $(0,X)\in\Gamma(T(I\times\widehat{M}))$. The Levi-Civita connections ∇ of $I\times_\lambda\widehat{M}$ and $\widehat{\nabla}$ of $\widehat{M}$ are related by

$$\nabla_XY=\widehat{\nabla}_XY-g_\lambda(X,Y)\mathrm{grad}\log\lambda,\tag{2.2}$$

for any vector fields X,Y on $\widehat{M}$. Moreover, the following relations are well-known

$$\nabla_\xi\xi=0,\quad \nabla_\xi X=\nabla_X\xi=\xi(\log\lambda)X,\quad X\in\Gamma(T\widehat{M}).\tag{2.3}$$

Any warped product manifold $I\times_\lambda\widehat{M}$ belongs to the Chinea-Gonzalez class $\bigoplus_{1\le i\le 5}C_i$, briefly denoted by C_{1-5}. In particular, if $(\widehat{M},\widehat{J},\widehat{g})$ is a Kähler manifold, then $I\times_\lambda\widehat{M}$ falls in the Chinea-Gonzalez class C_5.
In any dimensions, $2m+1$, C_5-manifolds are characterized by

$$(\nabla_X\varphi)Y=-\frac{1}{2m}\delta\eta(g(\varphi X,Y)\xi-\eta(Y)\varphi X),\tag{2.4}$$

and are called α-Kenmotsu manifolds, $\alpha=-\frac{1}{2m}\delta\eta$. If $\alpha\equiv1$, one obtains Kenmotsu manifolds ([15]).

A local description of α-Kenmotsu manifolds is given in [10, 21].
Let (M,φ,ξ,η,g) be an α-Kenmotsu manifold, with $\dim M=2m+1\geq5$, and consider the integrable distribution D associated with the subbundle $\ker\eta$ of TM. Then D defines a spheric foliation, namely each leaf N of D is an extrinsic sphere of M, with mean curvature

vector field $H = -\alpha\xi|_N$, α being constant on N. Furthermore, M is, locally, almost contact isometric to a warped product manifold $]-\varepsilon,\varepsilon[\times_\lambda \widehat{M}$, $\widehat{M}$ being a Kähler manifold and $\lambda\colon]-\varepsilon,\varepsilon[\to \mathbb{R}$ a smooth function, $\lambda > 0$.

Let $(\widehat{M}, \widehat{J}, \widehat{g})$ be an a.H. manifold, $\widehat{f}\colon (\widehat{N}, \widehat{g}') \to (\widehat{M}, \widehat{g})$ an isometric immersion, $I \subset \mathbb{R}$ an open interval and $\lambda\colon I \to \mathbb{R}$ a smooth function, $\lambda > 0$. The map

$$f_\lambda\colon I \times \widehat{N} \to I \times \widehat{M}, \quad f_\lambda(t,x) = (t, \widehat{f}(x)), \quad t \in I, x \in \widehat{N} \tag{2.5}$$

is an isometric immersion with respect to the warped product metrics $g'_\lambda = \pi^*(dt \otimes dt) + \lambda^2\sigma^*(\widehat{g}')$, $g_\lambda = \pi^*(dt\otimes dt) + \lambda^2\sigma^*(\widehat{g})$. Note that f_λ is a particular warped product immersion ([7, 18]), and $(I \times_\lambda \widehat{N}, f_\lambda)$ is a Riemannian submanifold of the a.c.m. manifold $I \times_\lambda \widehat{M}$. If $\lambda \equiv 1$, $(I \times \widehat{N}, f_1)$ is studied in [16]. For any $(t,x) \in I \times \widehat{N}$ we use the identification $T^\perp_{(t,x)}(I \times_\lambda \widehat{N}) \equiv T^\perp_x \widehat{N}$ and denote by A_V, $(A_\lambda)_V$ the Weingarten operators of $(\widehat{N}, \widehat{f})$, $(I \times_\lambda \widehat{N}, f_\lambda)$ with respect to any normal direction V. Analogous notation is used for the second fundamental forms h, h_λ and the mean curvature vector fields H, H_λ.
By direct calculus, applying the Gauss and Weingarten equations, (2.1), (2.2) and (2.3), for any $X, Y \in T(I \times \widehat{N})$ and $V \in T^\perp \widehat{N}$, one obtains:

$$\begin{aligned} &(A_\lambda)_V X = A_V(X - \eta(X)\xi), \\ &h_\lambda(X,Y) = h(X - \eta(X)\xi, Y - \eta(Y)\xi), \\ &H_\lambda = \frac{n}{n+1}\frac{1}{\lambda^2} H, \end{aligned} \tag{2.6}$$

where $n = \dim \widehat{N}$.

3 A class of odd-dimensional slant submanifolds of C_{1-5}-manifolds

In this section, we consider an isometric immersion $\widehat{f}\colon (\widehat{N}, \widehat{g}') \to (\widehat{M}, \widehat{J}, \widehat{g})$, an open interval $I \subset \mathbb{R}$, a smooth function $\lambda\colon I \to \mathbb{R}$, $\lambda > 0$ and the warped product immersion $f_\lambda\colon I \times_\lambda \widehat{N} \to I \times_\lambda \widehat{M}$ given in (2.5). By direct calculus, using (2.1), we obtain that the vector fields $\widehat{P}$, P_λ associated with $\widehat{f}$, f_λ, respectively, are related by

$$P_\lambda X = \widehat{P}(X - \eta(X)\xi), \quad X \in \Gamma(T(I \times \widehat{N})). \tag{3.1}$$

Analogously, the normal bundle-valued 1-forms $\widehat{F}$, F_λ associated with $\widehat{f}$, f_λ satisfy

$$F_\lambda X = \widehat{F}(X - \eta(X)\xi), \quad X \in \Gamma(T(I \times \widehat{N})). \tag{3.2}$$

Given $(t,x) \in I \times \widehat{N}$, we consider a vector $X \in T_{(t,x)}(I \times \widehat{N})$ such that X, ξ are linearly independent. Thus, $X - \eta(X)\xi$ is a non-zero vector in $T_x\widehat{N}$. By (2.1), (3.1), the Wirtinger angles $\theta_\lambda(X)$ and $\theta(X - \eta(X)\xi)$ satisfy

$$\cos\theta_\lambda(X) = \cos\theta(X - \eta(X)\xi). \tag{3.3}$$

It follows that $\widehat{f}$ is a slant immersion with $sla(\widehat{N}) = \theta$ if and only if f_λ is a slant immersion and $sla(I \times_\lambda \widehat{N}) = \theta$.

Formulas (2.6) allow us to produce examples of suitable slant immersions f_λ, depending on the behavior of the second fundamental form of $\widehat{f}$. In particular, applying (2.3), we observe that the second fundamental form h_λ satisfies $h_\lambda(X, \xi) = 0$, for any $X \in \Gamma(T(I \times \widehat{N}))$. It follows that f_λ cannot be totally umbilical, unless it is totally geodesic. According to [23], f_λ is contact totally umbilical if there exists a normal vector field W such that

$$h_\lambda(X, Y) = \{g_\lambda(X, Y) - \eta(X)\eta(Y)\}W. \tag{3.4}$$

It is easy to verify that $W = \frac{n+1}{n} H_\lambda$, $n = \dim \widehat{N}$.

By (2.6) one gets the following equivalences:

i) $(\widehat{N}, \widehat{f})$ is totally umbilical if and only if $(I \times_\lambda \widehat{N}, f_\lambda)$ is contact totally umbilical.

ii) $(\widehat{N}, \widehat{f})$ is minimal if and only if $(I \times_\lambda \widehat{N}, f_\lambda)$ is minimal.

iii) $(\widehat{N}, \widehat{f})$ is totally geodesic if and only if $(I \times_\lambda \widehat{N}, f_\lambda)$ is totally geodesic.

Recalling that an immersed submanifold N of a Riemannian manifold (M, g) is said to be austere if, for any $V \in T^\perp N$, the set of eigenvalues of A_V is invariant under multiplication by -1, applying (2.6), we easily prove the equivalence:

iv) $(\widehat{N}, \widehat{f})$ is an austere submanifold of $\widehat{M}$ if and only if $(I \times_\lambda \widehat{N}, f_\lambda)$ is austere in $I \times_\lambda \widehat{M}$.

Now, we assume that $(\widehat{N}, \widehat{f})$ is a slant submanifold with $sla(\widehat{N}) = \theta \neq \frac{\pi}{2}$ and consider the a.H. structure $(J' = \frac{1}{\cos\theta}\widehat{P}, \widehat{g'})$ on $\widehat{N}$. By (3.1), one has that the a.c.m. structure $(\varphi'_\lambda = \frac{1}{\cos\theta} P_\lambda, \xi' = \xi|_{I\times\widehat{N}}, \eta' = f_\lambda^*\eta, g'_\lambda)$ on $I \times \widehat{N}$ is just of the a.c.m. structure defined in (2.1), which is associated with $(J', \widehat{g'})$. So, $I \times_\lambda \widehat{N}$ is the warped product manifold of (I, g_0) and $(\widehat{N}, J', \widehat{g'})$. It follows that the Gray-Hervella class of $(\widehat{N}, J', \widehat{g'})$ determines the Chinea-Gonzalez class of $I \times_\lambda \widehat{N}$ ([10]). In particular, if $(\widehat{M}, \widehat{J}, \widehat{g})$ is a Kähler manifold, we know that $(J', \widehat{g'})$ is an almost Kähler structure, namely $(\widehat{N}, J', \widehat{g'})$ falls in the Gray-Hervella class $\mathcal{W}_2$, so that $I \times_\lambda \widehat{N}$ is a $C_2 \oplus C_5$-manifold. Moreover, if $(\widehat{N}, \widehat{f})$ is Kählerian ([5]), then $I \times_\lambda \widehat{N}$ is a C_5-manifold. We remark that if $\dim \widehat{N} = 2$, $(\widehat{N}, J', \widehat{g'})$ is a Kähler manifold, equivalently $(\widehat{N}, \widehat{f})$ is Kählerian, thus $I \times_\lambda \widehat{N}$ is a C_5-manifold. This fits with next Theorems 1, 2 holding in a more general context.

We end this section giving some explicit examples, where the first two of them are obtained considering the main examples of slant immersions into the Kähler manifold $(\mathbb{R}^{2m}, J_0, g_0)$, (J_0, g_0) being the canonical Hermitian structure on $\mathbb{R}^{2m}$ ([3, 4]). The last two examples are obtained starting by an austere submanifold of $(\mathbb{R}^{2m}, g_0)$ ($m = 2$, $m = 4$) endowed with a suitable almost complex structure.

Example 1. *For any $k \in \mathbb{R}$, $k > 0$, the map $\widehat{f}\colon \mathbb{R}^4 \to \mathbb{R}^8$ acting as*

$$\widehat{f}(x^1, x^2, x^3, x^4) = (x^1, x^2, k\sin x^3, k\sin x^4, kx^3, kx^4, k\cos x^3, k\cos x^4)$$

defines a slant submanifold with $sla(\mathbb{R}^4) = \frac{\pi}{4}$*, and* $(\mathbb{R}^4, \widehat{f})$ *is a Kählerian submanifold. Hence, for any smooth function* $\lambda\colon I \to \mathbb{R}$, $\lambda > 0$*, the map* $f_\lambda\colon I \times \mathbb{R}^4 \to I \times \mathbb{R}^8$, $f_\lambda(t,x) = (t, f(x))$*, defines a slant submanifold of the* C_5*-manifold* $I \times_\lambda \mathbb{R}^8$ *with slant angle* $\frac{\pi}{4}$ *and the a.c.m. manifold* $I \times_\lambda \mathbb{R}^4$ *falls in the class* C_5*.*

Example 2. *For any* $\theta \in [0, \frac{\pi}{2}[$*, the map* $\widehat{f}\colon \mathbb{R}^4 \to \mathbb{R}^8$ *acting as*

$$\widehat{f}(x^1, x^2, x^3, x^4) = (x^1, 0, x^3, 0, x^2\cos\theta, x^2\sin\theta, x^4\cos\theta, x^4\sin\theta)$$

defines a totally geodesic slant submanifold with $sla(\mathbb{R}^4) = \theta$*. The metric on* $\mathbb{R}^4$ *induced by* $\widehat{f}$ *is the Euclidean metric* g_0 *and* $(\mathbb{R}^4, J' = \frac{1}{\cos\theta}\widehat{P}, g_0)$ *is a Kähler manifold. So, for any smooth function* λ, $\lambda > 0$, f_λ *is a totally geodesic immersion and* $I \times_\lambda \mathbb{R}^4$ *is a* C_5*-manifold.*

Example 3. *The map* $\widehat{f}\colon \mathbb{R}^2 \to \mathbb{R}^4$ *acting as*

$$\widehat{f}(x^1, x^2) = (x^1, x^2, e^{x^1}\cos x^2, e^{x^1}\sin x^2)$$

defines an holomorphic submanifold of the Kähler manifold $(\mathbb{R}^4, \widehat{J}, g_0)$*,* $\widehat{J}$ *being the almost complex structure defined by* $\widehat{J}(y^1, y^2, y^3, y^4) = (-y^2, y^1, -y^4, y^3)$*. We know that* $(\widehat{\nabla'}_X\widehat{F})Y = \widehat{n}h(X,Y) - h(X, \widehat{P}Y)$ *for any* X, Y *tangent to* $\mathbb{R}^2$*,* $\widehat{\nabla'}$ *being the Levi-Civita connection of* $(\mathbb{R}^2, \widehat{f})$ *(see* [4], Chapter II, *formula* (3.3)*). Since* $\widehat{F} \equiv 0$*, it follows that* $h(X, \widehat{P}Y) = \widehat{n}h(X,Y) = h(\widehat{P}X, Y)$ *. Thus, for any normal vector* V*, it is possible to verify that* $A_V \circ \widehat{P} = -\widehat{P} \circ A_V$*. This implies that* $(\mathbb{R}^2, \widehat{f})$ *is an austere submanifold. By direct computation we obtain that the second fundamental form* h *does not vanish, namely* $(\mathbb{R}^2, \widehat{f})$ *is not totally geodesic. Hence, for any smooth function* $\lambda\colon I \to \mathbb{R}$, $\lambda > 0$*, the map* f_λ *is an austere, but not totally geodesic, invariant immersion into the* C_5*-manifold* $I \times_\lambda \mathbb{R}^4$*.*

Example 4. *Let* $\widehat{f}\colon \mathbb{R}^4 \to \mathbb{R}^8$ *be the map acting as*

$$\widehat{f}(x^1, x^2, x^3, x^4) = (x^1, x^2, e^{x^1}\cos x^2, e^{x^1}\sin x^2, 0, x^4, 0, x^3).$$

This map defines an holomorphic submanifold of the Kähler manifold $(\mathbb{R}^8, \widehat{J}, g_0)$*,* $\widehat{J}$ *acting as*

$$\widehat{J}(y^1, y^2, y^3, y^4, y^5, y^6, y^7, y^8) = (-y^2, y^1, -y^4, y^3, -y^7, -y^8, y^5, y^6).$$

As in Example 3*, the submanifold* $(\mathbb{R}^4, \widehat{f})$ *is austere and* $\widehat{f}$ *induces a Kähler structure on* $\mathbb{R}^4$*. Hence, for any smooth function* $\lambda\colon I \to \mathbb{R}$, $\lambda > 0$*, the map* $f_\lambda\colon I \times \mathbb{R}^4 \to I \times \mathbb{R}^8$ *is an austere invariant immersion into the* C_5*-manifold* $I \times_\lambda \mathbb{R}^8$*.*

4 Slant submanifolds of a C_5-manifold

The aim of this section is to determine the type of the a.c.m. structure induced on an odd-dimensional slant submanifold of a C_5-manifold. This allows us to give a local description of such submanifolds.

Let $f\colon N \to M$ be an isometric immersion in an α-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$. Assume that ξ is tangent to N and, whenever there is no danger of confusion, denote again by ξ the restriction $\xi_{|_N}$. Also, we denote by η the 1-form $f^*\eta$, by g the induced metric on

N and write α instead of $\alpha \circ f$.
The maps P, F, t and n associated with f satisfy the following relations

$$\begin{aligned} &P^2 + tF = -I_{TN} + \eta \otimes \xi, \quad Pt + tn = 0, \\ &FP + nF = 0, \quad Ft + n^2 = -I_{T^\perp N}. \end{aligned} \tag{4.1}$$

Applying (2.4), the Gauss and Weingarten equations, one gets

$$\begin{aligned} &\nabla'_X \xi = \alpha(X - \eta(X)\xi), \quad h(X,\xi) = 0, \\ &(\nabla'_X P)Y = A_{FY}X + th(X,Y) + \alpha(g(PX,Y)\xi - \eta(Y)PX), \end{aligned} \tag{4.2}$$

where ∇' is the Levi-Civita connection of N, h is the second fundamental form and A_{FY} the Weingarten operator with respect to FY.

For any $X, Y \in \Gamma(TN)$, $V \in \Gamma(T^\perp N)$ one puts

$$\begin{aligned} (\overline{\nabla}_X F)Y &= \nabla^\perp_X FY - F(\nabla'_X Y), \\ (\overline{\nabla}_X t)V &= \nabla'_X tV - t(\nabla^\perp_X V), \\ (\nabla^\perp_X n)V &= \nabla^\perp_X nV - n(\nabla^\perp_X V), \end{aligned}$$

$\nabla^\perp$ denoting the normal connection. Thus, applying (2.4), the Gauss and Weingarten equations, we obtain

$$\begin{aligned} (\overline{\nabla}_X F)Y &= nh(X,Y) - h(X,PY) - \alpha\eta(Y)FX, \\ (\overline{\nabla}_X t)V &= A_{nV}X - P(A_V X) + \alpha g(FX,V)\xi, \\ (\nabla^\perp_X n)V &= -h(X,tV) - F(A_V X). \end{aligned} \tag{4.3}$$

Now, we assume that (N,f) is a slant submanifold with $sla(N) = \theta$. Since $Q = P^2 = (-\cos^2\theta)(I_{TN} - \eta \otimes \xi)$, one gets

$$g(PX,PY) = (\cos^2\theta)g(\varphi X, \varphi Y), \quad g(FX,FY) = (\sin^2\theta)g(\varphi X,\varphi Y). \tag{4.4}$$

We also recall the relation ([13])

$$(\nabla'_X Q)Y = \alpha(\cos^2\theta)(g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X). \tag{4.5}$$

Proposition 1. *Let (N,f) be a slant submanifold of an α-Kenmotsu manifold (M,φ,ξ,η,g) such that ξ is tangent to N. For any $X, Y \in \Gamma(TN)$ we have*

$$A_{FPY}X + P(A_{FY}X) + th(X,PY) + Pth(X,Y) = 0. \tag{4.6}$$

Proof: By (4.2), (4.4), for any $X, Y \in \Gamma(TN)$ we have

$$\begin{aligned} (\nabla'_X Q)Y =& (\nabla'_X P)PY + P((\nabla'_X P)Y) = A_{FPY}X + th(X,PY) + P(A_{FY}X) \\ &+ Pth(X,Y) + \alpha(\cos^2\theta)(g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X). \end{aligned}$$

Hence, the statement follows by (4.5). □

Remark 1. *It is easy to check that* (4.6) *is equivalent to*

$$A_{FPY}X + A_{FPX}Y - A_{FY}PX - A_{FX}PY = 0, \tag{4.7}$$

for any $X, Y \in \Gamma(TN)$.

Theorem 1. *Let* (N, f) *be a slant submanifold of an* α*-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$ *such that* $sla(N) = \theta \neq \frac{\pi}{2}$. *Assume that* $\dim N = 2r + 1 \geq 5$. *Then the a.c.m. manifold* $(N, \varphi' = \frac{1}{\cos\theta}P, \xi, \eta, g)$ *belongs to the class* $C_2 \oplus C_5$. *Furthermore, the manifold* N *is, locally, almost contact isometric to a warped product manifold* $]-\varepsilon, \varepsilon[\times_\lambda \mathcal{F}$, $\varepsilon > 0$, $\mathcal{F}$ *being an almost Kähler manifold and* $\lambda\colon]-\varepsilon, \varepsilon[\to \mathbb{R}$ *a smooth function,* $\lambda > 0$.

Proof: Firstly, we remark that $(N, \varphi', \xi, \eta, g)$ is a $C_2 \oplus C_5$-manifold if and only if

$$d\eta = 0, \quad \mathcal{L}_\xi \varphi' = 0, \quad d\Phi' = \beta\eta \wedge \Phi', \tag{4.8}$$

where Φ' is the fundamental 2-form, namely $\Phi'(X, Y) = g(X, \varphi' Y)$, $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ, and β is a smooth function. Moreover, (4.8) implies that $\beta = -\frac{1}{r}\delta\eta$ ([10]).

Obviously, η is closed and (4.2) gives

$$(\nabla'_X \varphi')Y = \frac{1}{\cos\theta}(A_{FY}X + th(X, Y)) + \alpha(g(\varphi' X, Y)\xi - \eta(Y)\varphi' X).$$

Hence, given $X, Y, Z \in \Gamma(TN)$, we have

$$3d\Phi'(X, Y, Z) = -\underset{(X,Y,Z)}{\sigma} g((\nabla'_X \varphi')Y, Z) = 2\alpha \underset{(X,Y,Z)}{\sigma} \eta(X)g(Y, \varphi' Z),$$

where σ represents the cyclic sum on X, Y, Z. It follows that $d\Phi' = 2\alpha\eta \wedge \Phi'$. By (4.2), we also get

$$(\mathcal{L}_\xi \varphi')X = (\nabla'_\xi \varphi')X - \nabla'_{\varphi' X}\xi + \varphi'(\nabla'_X \xi) = 0.$$

So, $(N, \varphi', \xi, \eta, g)$ is a $C_2 \oplus C_5$-manifold. The last part of the statement follows by Theorem 3.1 and Proposition 3.2 [10]. □

Remark 2. *Applying* (4.2), *it is easy to check that a slant submanifold* (N, f) *as in* Theorem 1 *is a* C_5*-manifold if and only if, for any* $X, Y \in TN$, $A_{FX}Y = A_{FY}X$. *This agrees with a result given in* [13].
In particular, if $sla(N) = \theta = 0$, *then* F *vanishes and* N *is in the class* C_5.

Proposition 2. *Let* (N, f) *be a slant submanifold of an* α*-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$ *such that* $sla(N) = \theta \neq \frac{\pi}{2}$ *and* $\dim N = 2r+1 \geq 5$. *If, for any* X *tangent to* N, $A_{FX} \circ P = P \circ A_{FX}$, *then* $A_{FX} = 0$, *for any* X, *and* $(N, \varphi' = \frac{1}{\cos\theta}P, \xi, \eta, g)$ *is a* C_5*-manifold.*

Proof: Let D' be the integrable distribution on N associated with the subbundle $\ker\eta'$ of TN, $\eta' = f^*\eta$. By (4.2) it follows that D' defines a spheric foliation.

Let N' be a leaf of D' and consider the a.H. structure $(J = \varphi'|_{TN'}, g')$ induced on N' by the a.c.m. structure on N. By Theorem 1, we know that (N', J, g') is an almost Kähler manifold and, using (4.2), we obtain

$$(\nabla''_X J)Y = \frac{1}{\cos\theta}(A_{FY}X + th(X,Y)), \quad X, Y \in \Gamma(TN') \tag{4.9}$$

∇'' denoting the Levi-Civita connection of (N', g').

Let X, Y be vector fields on N'. Since (N', J, g') is almost Kähler, the covariant derivative $\nabla'' J$ satisfies $(\nabla''_X J)Y + (\nabla''_{JX} J)JY = 0$ and, applying (4.9), we have

$$A_{FY}X + A_{FJY}JX + th(X,Y) + th(JX, JY) = 0.$$

Taking the skew-symmetric component, we obtain

$$A_{FY}X - A_{FX}Y + A_{FJY}JX - A_{FJX}JY = 0.$$

Hence, using the hypothesis, we have

$$\begin{aligned} A_{FY}JX - A_{FX}JY - A_{FJY}X + A_{FJX}Y = J(A_{FY}X - A_{FX}Y \\ + A_{FJY}JX - A_{FJX}JY) = 0. \end{aligned}$$

By (4.7) it follows $A_{FX}JY = A_{FJX}Y$. Thus, for any $X, Y, Z \in TN'$, one has $g'(Jth(Y,Z), X) = g(h(Y,Z), FJX) = g'(A_{FJX}Y, Z) = g'(A_{FX}JY, Z) = -g'(th(JY,Z), X)$, so that

$$th(JY, Z) = -Jth(Y, Z) = th(JZ, Y), \quad Y, Z \in TN'. \tag{4.10}$$

On the other hand, the hypothesis implies, for any X, Y, Z, $g'(th(JY,Z), X) = -g'(A_{FX}JY, Z) = g'(A_{FX}Y, JZ) = -g'(th(Y, JZ), X)$, and using (4.10) we get $th(JY, Z) = 0$, for any $Y, Z \in TN'$.

Therefore, considering $X \in TN'$, the Weingarten operator A_{FX} vanishes, since $g'(A_{FX}Y, Z) = -g'(th(Y,Z), X) = 0$. □

Remark 3. *Let (N, f) be a slant submanifold as in* Proposition 2 *and assume that N is contact totally umbilical, namely $h(X,Y) = \frac{2r+1}{2r}(g(X,Y) - \eta(X)\eta(Y))H$, where* $\dim N = 2r+1$ *and H is the mean curvature vector field. Since all the Weingarten operators commute with P, N is a C_5-manifold and $tH = 0$. Indeed, for any $X \in TN$, $g(tH, X) = -\frac{1}{2r+1} traceA_{FX} = 0$.*

Theorem 2. *Let (N, f) be a slant submanifold of an α-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ such that $sla(N) = \theta \neq \frac{\pi}{2}$ and* $\dim N = 3$. *Then the a.c.m. manifold $(N, \varphi' = \frac{1}{\cos\theta}P, \xi, \eta, g)$, which is in the class C_5, is, locally, almost contact isometric to a warped product manifold $]-\varepsilon, \varepsilon[\times_\lambda \mathcal{F}$, $\varepsilon > 0$, $\mathcal{F}$ being a Kähler manifold and $\lambda\colon\,]-\varepsilon, \varepsilon[\to \mathbb{R}$ a smooth function, $\lambda > 0$.*

Proof: We prove that, for any X, Y tangent to N, one has $A_{FX}Y = A_{FY}X$.
In fact, by (4.7) we have $A_{FX}PX = A_{FPX}X$, $X \in TN$. Thus, with respect to a local orthonormal frame $\{e_1, e_2 = \varphi' e_1, \xi\}$ on N, we have $A_{Fe_1}e_2 = A_{Fe_2}e_1$.

Since $A_{Fe_i}\xi = 0$, $i = 1, 2$, we obtain $A_{FX}Y = A_{FY}X$ for any X, Y tangent to N. Therefore (4.2) gives

$$(\nabla'_X\varphi')Y = (\alpha \circ f)(g(\varphi' X, Y)\xi - \eta(Y)\varphi' X).$$

Since the function $\alpha \circ f$ satisfies $d(\alpha \circ f) \wedge f^*\eta = 0$, the C_5-manifold N is locally realized as a warped product manifold $]-\varepsilon, \varepsilon[\times_\lambda \mathcal{F}$, $\mathcal{F}$ being a Kähler manifold ([10]). □

5 Particular types of slant immersions

We are going to state some results on slant submanifolds (N, f) of an α-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ involving the behavior of the second fundamental form. To this aim, one considers the vector subbundle μ of $T^\perp N$ whose fibre at any $x \in N$ is the orthogonal complement to $F(T_xN)$ in $T_x^\perp N$, namely $T_xM = T_xN \oplus F(T_xN) \oplus \mu_x$. Note that μ is φ-invariant and any normal vector field V is a section of μ if and only if $tV = 0$. If $sla(N) = \theta \neq 0$, putting $\dim M = 2m + 1$, $\dim N = 2r + 1$, one has $\mathrm{rank} F(TN) = 2r$ and $\mathrm{rank}\mu = 2(m - 2r)$. Hence, μ is trivial if and only if $m = 2r$.

Theorem 3. *Let (N, f) be a slant submanifold of an α-Kenmotsu manifold such that $\dim N = 2r + 1$ and $sla(N) = \theta \neq \frac{\pi}{2}$. Assume that the mean curvature vector field $H \in \Gamma(\mu)$ and, for any $X \in \Gamma(TN)$, $\nabla^\perp_X H \in \Gamma(\mu)$. Then (N, f) is minimal.*

Proof: We observe that the hypothesis implies $nH = \varphi H \in \Gamma(\mu)$. By (4.3) and (4.4), for any vector fields X, Y on N, we have

$$g((\overline{\nabla}_X F)Y, nH) = g(h(X, Y), H) + g(nh(X, PY), H),$$
$$g((\nabla^\perp_X n)H, FY) = (-\sin^2\theta) g(h(X, Y), H).$$

Moreover, by direct calculus and the hypothesis, we obtain

$$g((\overline{\nabla}_X F)Y, nH) + g((\nabla^\perp_X n)H, FY) = 0.$$

It follows that

$$g(H, (\cos^2\theta)h(X, Y) + nh(X, PY)) = 0, \quad X, Y \in \Gamma(TN). \tag{5.1}$$

Let $\{e_1, \ldots, e_r, \varphi' e_1, \ldots, \varphi' e_r, \xi\}$ be a local orthonormal frame on N. By (5.1) and (4.2) we get

$$0 = \sum_{i=1}^r \{g(H, (\cos^2\theta)h(e_i, e_i) + nh(e_i, Pe_i)) + g(H, (\cos^2\theta)h(\varphi' e_i, \varphi' e_i) + nh(\varphi' e_i, \varphi' P e_i))\} = (2r + 1)(\cos^2\theta) g(H, H).$$

Since $\theta \neq \frac{\pi}{2}$, we obtain $H \equiv 0$. □

Corollary 1. *Let (N, f) be a slant submanifold as in* Theorem 3. *Assume that $H \in \Gamma(\mu)$ and $A_{nH} = P \circ A_H$. Then (N, f) is minimal.*

Proof: If $H \in \Gamma(\mu)$, applying (4.3), for any $X, Y \in \Gamma(TN)$ we obtain

$$\begin{aligned} g(t(\nabla^{\perp}_X H), Y) &= -g(\nabla^{\perp}_X H, FY) = g((\overline{\nabla}_X F)Y, H) \\ &= g(P(A_H X) - A_{nH}X, Y). \end{aligned}$$

Hence, we have $t(\nabla^{\perp}_X H) = P(A_H X) - A_{nH}X = 0$, so the statement follows by Theorem 3 □

By Proposition 2 and Corollary 1 we obtain the next result.

Corollary 2. *Let (N, f) be a slant submanifold as in* Theorem 3. *Assume that* $\dim N \geq 5$ *and that the Weingarten operators satisfy:*

$$A_{nH} = P \circ A_H, \quad A_{FX} \circ P = P \circ A_{FX},$$

for any $X \in TN$. Then (N, f) is minimal.

Remark 4. Corollary 2 *is not true when* $\dim N = 3$. *Indeed, by* (4.4)*, the condition* $A_{FX} \circ P = P \circ A_{FX}$ *gives*

$$th(PX, PY) = (\cos^2\theta) th(X, Y), \quad X, Y \in TN.$$

Moreover, considering a local orthonormal frame $\{e_1, e_2 = \varphi' e_1, e_3 = \xi\}$ on N and using (4.2)*, we have $tH = \frac{2}{3} th(e_1, e_1)$. Hence, if $th(e_1, e_1) \neq 0$, we have that $H \notin \Gamma(\mu)$ and* Corollary 1 *cannot be applied .*

Corollary 3. *Let (N, f) be a slant submanifold as in* Theorem 3. *Assume that the Weingarten operators satisfy:*

$$A_{nH} = P \circ A_H, \quad A_{FX} \circ P = -P \circ A_{FX},$$

for any $X \in TN$. Then (N, f) is minimal.

Proof: Given $X, Y, Z \in TN$, by the hypothesis and (4.4), we have

$$\begin{aligned} g(th(PX, PY) + (\cos^2\theta) th(X, Y), Z) &= -g(A_{FZ}PX, PY) - (\cos^2\theta) g(A_{FZ}X, Y) \\ &= g(P(A_{FZ}X), PY) - (\cos^2\theta) g(A_{FZ}X, Y) \\ &= 0. \end{aligned}$$

Hence, for any $X, Y \in TN$, we obtain $th(PX, PY) + (cos^2\theta) th(X, Y) = 0$.

Considering a local orthonormal frame $\{e_i, \varphi' e_i, \xi\}_{1 \leq i \leq r}$ on N, we get $(2r+1) tH = \sum_{i=1}^{r} \{th(e_i, e_i) + (\cos^2\theta) th(Pe_i, Pe_i)\} = 0$.
Hence $H \in \Gamma(\mu)$ and the statement follows by Corollary 1. □

Theorem 4. *Let (N, f) be a proper slant submanifold of an α-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$, such that ξ is tangent to N and $(\overline{\nabla}_X F)Y = -\alpha\eta(Y) FX$, for any $X, Y \in \Gamma(TN)$. Then, one has:*

i) N is an austere submanifold.

ii) For any $V \in \Gamma(\mu)$, $A_V = 0$.

iii) $F(TN)$ is a parallel subbundle of $T^\perp N$.

iv) $(N, \varphi' = \frac{1}{\cos\theta} P, \xi, \eta, g)$ is a C_5-manifold if and only if $\nabla^\perp n = 0$.

Proof: By the hypothesis and (4.3) we get

$$h(X, PY) = nh(X, Y) = h(PX, Y), \quad X, Y \in \Gamma(TN). \tag{5.2}$$

It follows that all the Weingarten operators A_V, $V \in T^\perp N$, anti-commute with P. In particular, given $V \in T^\perp N$, the set of eigenvalues of A_V is invariant under multiplication by -1. Thus, *i)* holds.

Moreover, by (5.2), one has $n^2 h(X, Y) = nh(X, PY) = -(\cos^2\theta) h(X, Y)$ and, applying (4.1), we get

$$h(X, Y) = -\frac{1}{\sin^2\theta} Fth(X, Y), \quad X, Y \in \Gamma(TN). \tag{5.3}$$

This gives *ii)*. Also, for any $V \in \Gamma(\mu)$, $X, Y \in \Gamma(TN)$, we have $g(\nabla^\perp_X FY, V) = g((\overline{\nabla}_X F)Y, V) = 0$, so that $\nabla^\perp_X FY \in \Gamma(F(TN))$. Hence, *iii)* holds.
Moreover, by (4.3) and *ii)*, for any $X \in \Gamma(TN)$, $V \in \Gamma(\mu)$ one has $(\nabla^\perp_X n)V = 0$. By (4.1), (4.2), (4.3), (5.3) we also obtain

$$(\nabla^\perp_X n)FY = (\sin^2\theta) h(X, Y) - F(A_{FY}X) = -F(th(X, Y) + A_{FY}X).$$

Since $th(X, Y) + A_{FY}X$ is orthogonal to ξ, one has $(\nabla^\perp_X n)FY = 0$ if and only if $th(X, Y) + A_{FY}X = 0$. Then, *iv)* follows also applying (4.2). □

Remark 5. *It is easy to see that statement i) in* Theorem 4 *is also satisfied when the submanifold N is invariant.*

To get examples of submanifolds as in Theorem 4 it is enough to consider odd-dimensional totally geodesic proper slant submanifolds of an α-Kenmotsu manifold. As explained in Section 3, any totally geodesic proper slant immersion into a Kähler manifold gives rise to a whole family of such submanifolds.

Several consequences can be obtained by Theorem 4. Firstly, we give a short proof of a similar result stated in [13].

Proposition 3. *Let (N, f) be a minimal proper slant submanifold of an α-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ such that* $\dim N = 3$ *and* $\dim M = 5$*. Then N is immersed in M as an austere submanifold.*

Proof: We claim that the Weingarten operators satisfy

$$A_V \circ P + A_{nV} = 0, \quad V \in T^\perp N. \tag{5.4}$$

Indeed, putting $sla(N) = \theta$, we consider a point $x \in N$ and an adapted slant frame $\{e_1, e_2 = \frac{1}{\cos\theta} Pe_1, e_3 = \xi, e_4 = \frac{1}{\sin\theta} Fe_1, e_5 = \frac{1}{\sin\theta} Fe_2\}$ defined in a neighborhood of x. Applying (4.1) we have

$$ne_4 = -(\cos\theta)e_5, \quad ne_5 = (\cos\theta)e_4. \tag{5.5}$$

Moreover, by Theorem 2, we have $A_{Fe_1}e_2 = A_{Fe_2}e_1$ and, since f is minimal and $A_V\xi = 0$ for any $V \in T^\perp N$, we also get $A_{Fe_1}e_1 + A_{Fe_2}e_2 = 0$. Using (5.5), a direct calculus entails

$$A_{e_k}Pe_i + A_{ne_k}e_i = 0, \quad i = 1, 2, 3, \quad k = 4, 5.$$

Hence, (5.4) holds and by (4.3), for any $X, Y \in \Gamma(TN)$, we have $(\overline{\nabla}_X F)Y = -\alpha\eta(Y)FX$. So, the statement follows by Theorem 4. □

6 Slant immersions and curvature

In this section we establish some results on slant submanifolds involving suitable restrictions on the curvature of the ambient space or of the submanifold.
Firstly, also applying Theorem 4, we are going to prove a reduction theorem for submanifolds of a space-form.

Let $M^{2m+1}(c)$, $m \geq 2$, be an α-Kenmotsu manifold with constant sectional curvature c. Then $c = -\alpha^2$, so α is constant and either M is cosymplectic and flat, or $c < 0$ ([21]). It follows that, for any $\alpha \neq 0$, the hyperbolic space $\mathbb{H}^{2m+1}(-\alpha^2)$ is the local model of space-forms carrying a non-cosymplectic α-Kenmotsu structure.

Given $\alpha \in \mathbb{R}$, $\alpha > 0$, on $\mathbb{H}^{2m+1} = \{(x^1, \ldots, x^{2m+1}) \in \mathbb{R}^{2m+1}; x^1 > 0\}$ one considers the metric $g_\alpha = \frac{1}{(\alpha x^1)^2}\sum_{i=1}^{2m+1} dx^i \otimes dx^i$ and puts $E_i = \alpha x^1 \frac{\partial}{\partial x^i}$, $i = 1 \ldots 2m+1$. Let $(\varphi, \xi, \eta, g_\alpha)$ be any a.c.m. structure such that φ has constant components with respect to the orthonormal frame $\{E_i\}_{1\leq i\leq 2m+1}$, $\xi = E_1$ and $\eta = \xi^b = \frac{1}{\alpha x^1}dx^1$. Then $(\mathbb{H}^{2m+1}, \varphi, \xi, \eta, g_\alpha)$ is an α-Kenmotsu manifold, simply denoted by $\mathbb{H}^{2m+1}(-\alpha^2)$ ([8]).

Theorem 5. *Given $\alpha \in \mathbb{R}$, $\alpha > 0$, let (N, f) be a proper slant submanifold of $\mathbb{H}^{2m+1}(-\alpha^2)$ such that $\dim N = 2r + 1$ and, for any $X, Y \in \Gamma(TN)$, $(\overline{\nabla}_X F)Y = -\alpha\eta(Y)FX$. Then N is contained in a $(4r+1)$-dimensional totally geodesic submanifold of $\mathbb{H}^{2m+1}(-\alpha^2)$ as an austere submanifold.*

Proof: By Theorem 4 and formula (5.3), it follows that the normal subbundle $F(TN)$, of rank $2r$, is parallel in $T^\perp N$ and, for any $x \in N$, the first normal space N^1_x, spanned by $\{h(X, Y); X, Y \in T_xN\}$, is a subspace of $F(T_xN)$.
Thus, the statement follows applying the reduction theorem of Erbacher ([9]). □

Now, we focus on slant submanifolds which are generalized Sasakian space-forms with respect to the a.c.m. structure considered in Section 4.
A generalized Sasakian space-form (g.S. space-form) $M(f_1, f_2, f_3)$ is an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ which admits three smooth functions f_1, f_2, f_3 such that the curvature tensor R satisfies

$$R = f_1\pi_1 + f_2S + f_3T, \tag{6.1}$$

π_1, S, T being the algebraic curvature tensor fields defined by

$$\pi_1(X,Y,Z) = g(Y,Z)X - g(X,Z)Y,$$
$$S(X,Y,Z) = 2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X,$$
$$T(X,Y,Z) = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi.$$

In the context of contact Geometry this concept, introduced in [1], resembles the one of generalized complex space-form (g.c. space-form), arising in Hermitian Geometry. As in [22], a g.c. space-form $M(F_1,F_2)$ is an a.H. manifold (M,J,g) admitting two smooth functions F_1, F_2 such that the curvature tensor satisfies

$$R = F_1\pi_1 + F_2\pi_2, \tag{6.2}$$

π_2 being defined by

$$\pi_2(X,Y,Z) = 2g(X,JY)JZ + g(X,JZ)JY - g(Y,JZ)JX.$$

Many examples of g.S. space-forms can be obtained. For instance, taking into account Theorems 3.3, 5.3 in [20], one can see that every α-Kenmotsu manifold (M,φ,ξ,η,g) with pointwise constant φ-sectional curvature C and $\dim M \geq 5$, is a g.S. space-form with functions $f_1 = \frac{C-3\alpha^2}{4}$, $f_2 = \frac{C+\alpha^2}{4}$, $f_3 = \frac{C+\alpha^2}{4} + \xi(\alpha)$. Furthermore, if α is a constant function, then any 3-dimensional α-Kenmotsu manifold is a g.S. space-form with functions $f_1 = \frac{C-3\alpha^2}{4}$, $f_2 = f_3 = \frac{C+\alpha^2}{4}$.
Moreover, in [1], the authors obtained a wide range of g.S. space-forms using warped products. More precisely, given a g.c. space-form $M(F_1,F_2)$, the a.c.m. warped product manifold $\mathbb{R}\times_\lambda M$, where $\lambda > 0$ is a smooth function on $\mathbb{R}$, is a g.S. space-form with functions $f_1 = \frac{(F_1\circ\sigma)-\lambda'^2}{\lambda^2}$, $f_2 = \frac{F_2\circ\sigma}{\lambda^2}$, $f_3 = \frac{(F_1\circ\sigma)-\lambda'^2}{\lambda^2} + \frac{\lambda''}{\lambda}$, where $\sigma\colon \mathbb{R}\times M \to M$ is the canonical projection and λ', λ'' are the first and second derivatives of λ.
Finally, we point out that the second author obtained local classification results for suitable g.S. space-forms ([10]).

Proposition 4. *Let (N,f) be a slant submanifold of an α-Kenmotsu manifold (M,φ,ξ,η,g) such that $sla(N) = \theta \neq \frac{\pi}{2}$. Assume that $\dim N = 2r+1 \geq 5$ and $N(f_1,f_2,f_3)$ is a g.S. space-form. Then, one has:*

i) $df_1 \wedge \eta = 0, \quad df_2 \wedge \eta = 0.$

ii) Any leaf $(N', J = \frac{1}{\cos\theta}P|_{TN'}, g')$ of the distribution D' on N orthogonal to ξ is a g.c. space-form.

Proof: By Theorem 1, we know that $(N, \varphi' = \frac{1}{\cos\theta}P, \xi, \eta, g)$ is a $C_2 \oplus C_5$-manifold, in particular it is a C_{1-5}-manifold with Lee form $\omega = \frac{\delta\eta}{2r}\eta = -(\alpha\circ f)\eta$. So, applying Lemma 4.2 [10], for any unit section X of D', one has $X(f_1) = -X(f_2) = -3f_2\omega(X) = 0$. This proves *i)*.

Let (N',g') be a leaf of the distribution D'. We remark that it is a totally umbilical submanifold of N with mean curvature vector field $H' = -(\alpha\circ f)\xi|_{N'}$.
Indeed, by (4.2) the second fundamental form h' of N' acts as $h'(X,Y) = -g(\nabla'_X\xi, Y)\xi = -(\alpha\circ f)g'(X,Y)\xi$.

Let R, R' denote the curvature tensors of N, N', respectively. For the corresponding Riemannian curvatures we adopt the convention $R(X, Y, Z, W) = g(R(Z, W, Y), X) = -g(R(X, Y, Z), W)$.
Applying the Gauss equation and (6.1), for any $X, Y, Z, W \in \Gamma(TN')$ we have

$$\begin{aligned} R'(X, Y, Z, W) =& R(X, Y, Z, W) \\ & + (\alpha \circ f)^2 \{g'(X, Z)g'(Y, W) - g'(Y, Z)g'(X, W)\} \\ =& (f_1 + (\alpha \circ f)^2)\{g'(X, Z)g'(Y, W) - g'(Y, Z)g'(X, W)\} \\ & - f_2\{2g'(X, JY)g'(JZ, W) + g'(X, JZ)g'(JY, W) \\ & \quad - g'(Y, JZ)g'(JX, W)\}. \end{aligned}$$

It follows

$$R' = (f_1 + (\alpha \circ f)^2)_{|_{N'}} \pi_1 + f_{2|_{N'}} \pi_2. \tag{6.3}$$

Hence, N' is a g.c. space-form and both functions $F_1 = (f_1 + (\alpha \circ f)^2)_{|_{N'}}$, $F_2 = f_{2|_{N'}}$ are constant. □

Theorem 6. *Let (N, f) be a slant submanifold of an α-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ such that $sla(N) = \theta \neq \frac{\pi}{2}$ and $\dim N = 2r + 1 \geq 5$. Assume that $N(f_1, f_2, f_3)$ is a g.S. space-form. Then $(N, \varphi' = \frac{1}{\cos\theta} P, \xi, \eta, g)$, which falls in the class C_5, is, locally, almost contact isometric to a warped product manifold $]-\varepsilon, \varepsilon[\times_\lambda \mathcal{F}$, $\varepsilon > 0$, $\lambda \colon]-\varepsilon, \varepsilon[\to \mathbb{R}$ being a smooth function, $\lambda > 0$, and $\mathcal{F}$ a Kähler manifold with constant holomorphic sectional curvature.*

Proof: By Theorem 1, we know that $(N, \varphi', \xi, \eta, g)$ is, locally, almost contact isometric to a warped product manifold $]-\varepsilon, \varepsilon[\times_\lambda \mathcal{F}$, $\mathcal{F}$ being an almost Kähler manifold biholomorphic to a leaf $(N', J = \varphi'|_{TN'}, g')$ of the distribution D' on N orthogonal to ξ.

We claim that each leaf of D' is a Kähler manifold and has constant holomorphic sectional (c.h.s.) curvature.
To this aim, fixed a point $x_0 \in N$, we consider the leaf (N', J, g') of D' through x_0. Note that $\dim N' \geq 4$ and, by (6.3), the curvature of N' is a combination of the tensor fields π_1, π_2 by means of constant functions. So, if $f_2(x_0) \neq 0$, applying the theory developed in [22], namely Theorem 12.7 and the corresponding remark, N' is a Kähler manifold with c.h.s. curvature $C = 4(f_1(x_0) + \alpha(f(x_0))^2) = 4f_2(x_0)$.

On the other hand, if $f_2(x_0) = 0$, being $R' = (f_1 + (\alpha \circ f)^2)_{|_{N'}} \pi_1$, N' turns out to be an almost Kähler manifold with constant curvature. Hence, applying a theorem of Oguro ([19]), we get that N' is a flat Kähler manifold.
In any case, all the manifolds $\mathcal{F}$ occurring in the local description of N carry a Kähler structure with c.h.s. curvature and N falls in the class C_5. □

Finally, we observe that any 3-dimensional slant submanifold (N, f) of an α-Kenmotsu manifold, such that $sla(N) \neq \frac{\pi}{2}$, is a g.S. space-form. Indeed, being $d(\alpha \circ f) \wedge f^*\eta = 0$, we directly apply (3.9) in [21]. In our notation, the curvature of N is given by

$$R = (\frac{\tau}{2} + 2(\alpha \circ f)^2 + 2\xi(\alpha) \circ f)\pi_1 + (\frac{\tau}{2} + 3(\alpha \circ f)^2 + 3\xi(\alpha) \circ f)T,$$

where τ is the scalar curvature.

Acknowledgement. The authors thank Professor Anna Maria Pastore for her valuable encouragement during the development of this work.

Work partially supported by INDAM.

References

[1] P. Alegre, D. E. Blair, A. Carriazo, Generalized Sasakian-space-forms, *Israel J. Math.* **141** (2004), 157-183.

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhäuser Boston, Boston, 2010.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, M. Fernández, Slant submanifolds in Sasakian manifolds, *Glasg. Math. J.* **42** (2000), 125-138.

[4] B. Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Louvain, 1990.

[5] B. Y. Chen, Slant immersions, *Bull. Austral. Math. Soc.* **41** (1990), 135-147.

[6] B. Y. Chen, *Riemannian submanifolds*, Handbook of differential geometry **I**, North-Holland, Amsterdam, 2000, 187-418.

[7] B. Y. Chen, On warped product immersions, *J. Geom.* **82** (2005), 36-49.

[8] D. Chinea, C. Gonzalez, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.* (4) **156** (1990), 15-36.

[9] J. Erbacher, Reduction of the codimension of an isometric immersion, *J. Differential Geometry* **5** (1971), 333-340.

[10] M. Falcitelli, A class of almost contact metric manifolds and twisted products, *Balkan J. Geom. Appl.* **17** (2012), 17-29.

[11] A. Gray, L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.* (4) **123** (1980), 35-58.

[12] R. S. Gupta, Non-existence of contact totally umbilical proper slant submanifolds of a Kenmotsu manifold, *Rend. Semin. Mat. Univ. Politec. Torino*, **69** (2011), 51-55.

[13] R. S. Gupta, S. M. Khursheed Haider, A. Sharfuddin, Slant submanifolds of a trans-Sasakian manifold, *Bull. Math. Soc. Sci. Math. Roumanie* **47** (95) (2004), 45-57.

[14] R. S. Gupta, S. M. Khursheed Haider, A. Sharfuddin, Slant submanifolds in cosymplectic manifolds, *Colloq. Math.* **105** (2006), 207-219.

[15] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tôhoku Math. J.* (2) **24** (1972), 93-103.

[16] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Sc. Math. Roumanie* **39** (87) (1996), 183-198.

[17] A. Lotta, Three-dimensional slant submanifolds of K-contact manifolds, *Balkan J. Geom. Appl.* **3** (1998), 37-51.

[18] S. Nölker, Isometric immersions of warped products, *Differential Geom. Appl.* **6** (1996), 1-30.

[19] T. Oguro, On almost Kähler manifolds of constant curvature, *Tsukuba J. Math.* **21** (1997), 199-206.

[20] Z. Olszak, Locally conformal almost cosymplectic manifolds, *Colloq. Math.* **57** (1989), 73-87.

[21] Z. Olszak, R. Roşca, Normal locally conformal almost cosymplectic manifolds, *Publ. Math. Debrecen* **39** (1991), 315-323.

[22] F. Tricerri, L. Vanhecke, Curvature tensors on almost Hermitian manifolds, *Trans. Amer. Math. Soc.* **267** (1981), 365-397.

[23] K. Yano, M. Kon, *Structures on manifolds*, Series in Pure Mathematics **3**, World Scientific, Singapore, 1984.

Received:

Received: 29.02.2016
Revised: 07.11.2016
Accepted: 24.11.2016

Dipartimento di Matematica,
Universitá degli Studi di Bari "Aldo Moro",
Via E. Orabona 4, 70125 Bari-Italia.
E-mail: salvatore.decandia@uniba.it
maria.falcitelli@uniba.it