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On the exponential map on Riemannian polyhedra
by

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Abstract

We prove that Riemannian polyhedra admit explicit exponential maps at points in codimension–one strata, that behave similarly to the classical case.

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1 Introduction

Riemannian polyhedra are canonical generalizations of Riemannian differentiable manifolds. It is natural to believe that many of the geometric properties of manifolds extend suitably to polyhedra.

One major geometric ingredient in the geometry of Riemannian manifolds is the exponential map. It is known that it establish a connection between the tangent space of a manifold, at origin, and the manifold itself, by carrying the straight lines through origin of the tangent space to geodesics of the manifold. Moreover, the exponential map defined around a given point, gives a local diffeomorphism between the tangent space at that point and the manifold (see for example [8]). The exponential map has numerous applications in the theory of Riemmannian manifolds, e.g. it is used in the definition of the sectional curvature.

The exponential map can also be defined for manifolds with boundary (see [7], [6], [10]). Starting from this fact, the aim of the paper is to prove that Riemannian polyhedra also admit explicit exponential maps at points in codimension–one strata, that behave similarly to the classical case. The main difficulty is to replace the tangent space, which is of little use at singular points, with the tangent cone. We need a reinterpretation of the tangent cone, originally defined in [1, pag. 179] to a definition that suits better our purposes. Then locally, around a point in a codimension–one stratum, the tangent cone looks like a union of *closed* semi–spaces and the exponential map can be defined on each of these components. The exponential map is a first necessary step towards finding a good definition for the sectional curvature for polyhedra, which is still an open project.

The outline of the paper is as follows. Section 2 is an overview of the Riemannian polyhedra and exponential maps on manifolds with boundary. In Section 3, we reformulate the definition of the tangent cone, we define the exponential map on an admissible Riemannian polyhedron and we prove that the exponential map is a local homeomorphism.

2 Preliminaries.

In this section we recall some basic notions and results which will be used throughout the paper.

2.1 Riemannian admissible polyhedra. [1], [3], [4], [5], [9]

Let K be a locally finite simplicial complex. Considering the set of all formal finite linear combinations $\alpha = \sum_{v \in K} \alpha(v)v$ of vertices v of K, such that $0 \leqslant \alpha(v) \leqslant 1$, $\sum_{v \in K} \alpha(v) = 1$ and $\{v; \alpha(v) > 0\}$ is a simplex of K, we obtain the space $|K|$ which is a subset in $\mathrm{lin}(K)$, the linear space of all formal finite linear combinations of vertices of K. Note that $|K|$ is endowed with a natural distance, called the *barycentric distance*, [5, page 43].

The complex K is *admissible*, if it is dimensionally homogeneous, and for every connected open subset U of K, the open set $U \setminus \{U \cap \{(n-2)-$ skeleton $\}\}$ is connected, where n is the dimension of K (i.e. K is $(n-1)$-chainable).

We understand by *polyhedron,* a connected locally compact separable Hausdorff space X, for which there exists a pair (K, θ), where K is a simplicial complex and $\theta : |K| \to X$ is a homeomorphism. The pair (K, θ) is called a *triangulation* of K. The complex K is necessarily countable and locally finite (see [9, page 120] or [5, page 44]) and the space X is path connected and locally contractible. The *dimension* of X is by definition the dimension of K and it is independent of the triangulation.

The vertices, simplexes, i-skeletons (the set of simplexes of dimensions lower or equal to i) of a polyhedron X with specified triangulation (K, θ) are the images under θ of vertices, simplexes, i-skeletons of K. Thus our simplexes become compact subsets of X.

A *Lip polyhedron*, [5, page 46], is a metric space X which is the image of $|K|$ under a Lip homeomorphism $\theta : |K| \to X$ where $|K|$ is endowed with the barycentric distance, as above. Note that *every* polyhedron X with a triangulation (K, θ) can be considered to be a Lip polyhedron if X is given the metric corresponding to the barycentric distance of $|K|$ via θ.

Using the property mentioned in [5, Lemma 4.1], every Lip polyhedron is mapped Lip homemorphically and simplexwise affine onto a closed subset of an Euclidean space.

A *Riemannian polyhedron* (X, g) (see [5]) is a Lip polyhedron X with a specific Lip triangulation (K, θ) and a covariant bounded measurable Riemannian metric tensor g_S on each maximal simplex S of X. It is also required, [5, page 47] that an ellipticity condition, given as follows, is satisfied. Suppose that X has homogeneous dimension n. We are given a measurable Riemannian metric $\widehat{g}_S$ on the open Euclidean n-simplex $\theta^{-1}(\overset{o}{S})$ of $|K|$, the pullback via θ of g_S.

In terms of Euclidean coordinates $\{x_1, \ldots, x_n\}$ on $\theta^{-1}(\overset{o}{S})$, $\widehat{g}_S$ assigns to almost every point $x \in \theta^{-1}(\overset{o}{S})$, an $n \times n$ symmetric positive definite matrix $(g^S_{ij}(x))_{i,j=1,\ldots,n}$, with measurable real entries and there is a constant $\Lambda_S > 0$ such that (ellipticity condition):

$$\Lambda_S^{-2} \sum_{i=0}^{n} (\xi^i)^2 \leq \sum_{i,j} g^S_{ij}(x) \xi^i \xi^j \leq \Lambda_S^2 \sum_{i=0}^{n} (\xi^i)^2$$

for *a.e.* $x \in \theta^{-1}(\overset{o}{S})$ and every $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$. This condition is independent not only of the choice of the Euclidean frame on $\theta^{-1}(\overset{o}{S})$ but also of the chosen triangulation.

Note that *'almost everywhere'* (a.e.) means everywhere except in some null set. A *null set* in a Lip polyhedron X is a set $Y \subset X$ such that Y meets every maximal simplex S, relative to a triangulation (K, θ) (hence any) in a set whose pre-image under θ has n-dimensional Lebesgue measure 0, with $n = \dim S$.

A Riemannian polyhedron X is said to be *admissible* if for a fixed triangulation (K, θ) (hence any) the Riemannian simplicial complex K is admissible.

We underline that, for simplicity, the given definition of a Riemannian polyhedron (X, g) contains already the fact (because of the definition above of the Riemannian admissible complex) that the metric g is *continuous* relative to some (hence any) triangulation (i.e. for every maximal simplex S the metric g_S is continuous up to the boundary). This fact is sometimes omitted in the literature. The polyhedron is said to be *simplexwise smooth* if relative to some triangulation (K, θ) (and hence any), the complex K is simplexwise smooth. Both continuity and simplexwise smoothness are preserved under subdivision.

2.2 Boundary exponential map, boundary normal coordinates. [6], [10], [7]

Let M be a Riemannian manifold with non-empty boundary ∂M. Similarly to the exponential mapping on a Riemannian manifold without boundary, we can define (see [6]) the *boundary exponential mapping* as follows. Let $x_0 \in \partial M$ be a point and W a small open chart neighborhood of x_0 in ∂M. Consider $\mathcal{U}_\rho = W \times [0, \rho)$ a collar neighbourhood of $W \times \{0\}$ in the boundary cylinder $\partial M \times \mathbb{R}_+$. For ρ sufficiently small, we define

$$\exp_{\partial M} : \mathcal{U}_\rho \to M,$$

by: $\exp_{\partial M}(z, t) = \gamma_{z,\nu}(t)$, where $\gamma_{z,\nu}$ denotes the normal geodesic to ∂M whose derivative at zero equals ν, the unitary normal vector to ∂M at the point z. It is clear that the definition does not depend on the choice of W nor of ρ. Moreover, if M is compact, the boundary exponential map is well-defined on $\partial M \times \mathbb{R}_+$, see [6, Chapter 2].

Using the boundary exponential mapping, one introduced (see for example [6] or [10]) the *boundary normal* (or *semi-geodesic*) *coordinates*, analogously to the Riemann normal coordinates. Compared to the classical case of empty boundary, instead of a set of geodesics starting from a point one considers the set of geodesics normal to ∂M.

Denote by $\mathcal{V}_\rho = \exp_{\partial M}(\mathcal{U}_\rho)$ and define $(\mathcal{V}_\rho, x_1, \ldots, x_n)$ local coordinates in M (the boundary normal coordinates) in the following way:

for $x \in \mathcal{V}_\rho$, $x_n := d(x, \partial M)$, where $z \in W \subset \partial M$ is the unique boundary point such that $d(x, z) = d(x, \partial M)$ and $(x_1, \ldots, x_{n-1})$ on are local coordinates around z on W. ρ is chosen small enough such that $\gamma_{z,\nu}(t)$ is the unique shortest geodesic to ∂M for $t < \rho$ and hence $\mathcal{V}_\rho = \{x \in M; d(x, \partial M) < \rho\}$.

3 The exponential map on Riemannian polyhedra

3.1 The tangent cone.

Let (X, g) be an n-dimensional admissible Riemannian polyhedron and p a point in the $((n-1)$-skeleton$)\setminus((n-2)$-skeleton$)$.

We shall slightly reformulate the *definition of the tangent cone* previously introduced in [1].

Suppose that p is in $\overset{o}{\widehat{S_{n-1}}}$, the topological interior of the $(n-1)$-simplex S_{n-1}. Let $S_n^1, S_n^2, \dots, S_n^k$, $k \geq 2$, denote the n-simplexes adjacent to S_{n-1}. Then each S_n^ℓ, for $\ell = 1, \dots, k$, can be viewed as an affine simplex in $\mathbb{R}^n$, that is $S_n^\ell = \bigcap_{i=0}^n H_i$ where H_i are closed half spaces in $\mathbb{R}^n$. The Riemannian metric $g_{S_n^\ell}$ is the restriction to S_n^ℓ of a smooth Riemannian metric defined in an open neighbourhood of S_n^ℓ in $\mathbb{R}^n$.

Since $p \in ((n-1) - \text{skeleton}) \setminus ((n-2) - \text{skeleton})$, each S_n^ℓ for $\ell = 1, \dots, k$, can be viewed, locally around p, as a manifold with boundary, where the boundary is S_{n-1}. Then there exists a unique hyperplane, for $i = 0, \dots, n$, containing p. Define $T_p S_n^\ell$ as the *closed* half-space H_i which contains the corresponding hyperplane.

Notice that $T_p S_n^\ell$ can be naturally embedded in $\mathrm{lin}(S_n^\ell) \subset \mathrm{lin}(K)$ and

$$T_p S_n^\ell = T_p S_{n-1} \times [0, \infty). \tag{3.1}$$

Define the *tangent cone of* K *over* p as: $T_p K = \bigcup_{\ell=1}^k T_p S_n^\ell \subset \mathrm{lin}(K)$.

The difference from the original definition of Ballmann and Brin (see [1, pag. 179]) is that we do not need to pass to subdivision of K in order to make the point p become a vertex. Since the definition of the tangent cone in [1] is practically given for vertices, and in the next section it will be essential to work at points outside the $(n-2)$–skeleton, our definition is more suitable for our purposes. On the other side, the definition of [1] makes sense for *every* point of X, which is clearly not the case with ours. Note however, that the two definitions are equivalent for points in the $((n-1) - \text{skeleton}) \setminus ((n-2) - \text{skeleton}$. Indeed, in both definitions the tangent cone is the union of the tangent spaces to faces adjacent to the point, and by passing to a subdivision, this union does not change as a set.

3.2 The exponential map.

Having defined the tangent cone, and using the boundary exponential map, we can introduce next the *exponential map* locally around a point p in the topological interior of an $(n-1)$-simplex S_{n-1}. This map will be defined in three steps.

Step 1. Take V_0 a small neighbourhood of 0 in $T_p X$. The definition of the exponential map $E_p : T_p X \to X$ on each maximal face $V_0 \bigcap T_p S_n^\ell$, $\ell = 1, \dots, k$ is based on the fact that, locally around p, each S_n^ℓ becomes a manifold with boundary, with $\partial(S_n^\ell) = S_{n-1}$. This allows us to consider the boundary exponential map (see Section 2.2):

$$\exp_{\partial S_n^\ell} : \mathcal{U}_\rho \to \mathcal{V}_\rho,$$

where, for U_p a small neighbourhood of $p, \mathcal{U}_\rho = (U_p \cap S_{n-1}) \times [0, \rho)$ is a collar neighbourhood of $(U_p \cap S_{n-1}) \times \{0\}$ in the boundary cylinder $(U_p \cap S_{n-1}) \times \mathbb{R}_+$ and

$$\mathcal{V}_\rho = \exp_{\partial S_n^\ell}(\mathcal{U}_\rho) := \{x \in (U_p \cap S_n^\ell); d(x, (U_p \cap S_{n-1})) < \rho\}.$$

Using the boundary normal coordinates, the Collar Neighbourhood Theorem (see [2]) asserts that $\exp_{\partial S_n^\ell}$ is a diffeomorphism.

Step 2. On the manifold S_{n-1} we consider the usual exponential map at p:

$$\exp_p : T_p S_{n-1} \to S_{n-1}.$$

It is a local diffeomorphism around the point p.

Step 3. We use the decomposition $T_p S_n^\ell \cong T_p S_{n-1} \times [0, \infty)$, defined by the following map. Any $w_p \in T_p S_n^\ell$ decomposes as a pair $w_p = (u_p, v_p)$ where u_p is tangent to S_{n-1} and v_p is orthogonal to S_{n-1} and inward pointing. The map under question assigns to w_p the pair $(u_p, ||v_p||)$.

Summing up, we define the exponential map

$$E_p : V_0 \cap T_p X \to X$$

in the following way. Consider w_p a tangent vector in $T_p X$. There exists an ℓ such that $w_p \in V_0 \cap T_p S_n^\ell \subset T_p X$. We decompose $w_p = (u_p, v_p)$ where $u_p \in T_p S_{n-1}$ and v_p is a normal vector to ∂S_n^ℓ. Then

$$E_p(w_p) = \exp_{\partial S_n^\ell}(\exp_p(u_p), ||v_p||).$$

Since E_p is a local diffeomorphism on any $T_p S_n^\ell$ and its restrictions to $T_p S_n^\ell$ agree on tangent space to S_{n-1}, it is continous, and its inverse is also continous. We obtain

Theorem 1. *Let X be an admissible Riemannian polyhedron of dimension n and p be a point in the topological interior of a $(n-1)$–simplex S_{n-1}. Then there exist a neighbourhood V of p in $T_p X$ and a neighbourhood W of p in X such that $E_p : V \to W$ is a homeomorphism. Moreover, $E_p|_{V \cap T_p S_n^\ell}$ is a diffeomorphism onto $W \cap S_n^\ell$, where S_n^ℓ are the maximal simplexes adjacent to S_{n-1}.*

Note that on every maximal simplex the exponential map defined above maps straight lines to geodesics, and hence the terminology is natural.

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