Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 3, 2017, 221–231

Asymptotic properties of some functions related to regular integers modulo n

by

Brăduţ Apostol

Abstract

Let $\varrho(n)$ denote the number of positive regular integers (mod n) less than or equal to n and let $\varrho_r(n)$ $(r \geq 1)$ be the multidimensional generalization of the arithmetic function $\varrho(n)$. We study the behaviour of the sequence $(\varrho_r(n+1) - \varrho_r(n))_{n\geq 1}$. We also investigate the average orders of the functions $\dfrac{\varrho_r(n)}{\psi_r(n)}$, $\dfrac{\varrho_r(n)}{\sigma_r(n)}$ and $\dfrac{\varrho_r(n)}{\sigma_r^*(n)}$. Here the functions $\psi_r(n)$, $\sigma_r(n)$, $\sigma_r^*(n)$ generalize the Dedekind function, the sum of the divisors of n and the sum of the unitary divisors of n, respectively. Finally, we give the extremal orders of some compositions involving the functions mentioned previously and the functions $\phi_r(n)$ and $\phi_r^*(n)$ which generalize $\phi(n)$, the Euler function and the unitary function corresponding to $\phi(n)$.

Key Words: arithmetical function, composition, regular integers (mod n), average orders, extremal orders.

2010 Mathematics Subject Classification: Primary 11A25, Secondary 11N37

1 Introduction

Let $n > 1$ be a positive integer. An integer a is called regular (mod n) if there exists an integer x such that $a^2x \equiv a \pmod{n}$. Properties of regular integers (mod n) were studied by many authors. Several statements were proved elementary by Morgado [9], [10]. One of them tells us that $a > 1$ is regular (mod n) if and only if gcd (a, n) is a unitary divisor of n. We recall that d is said to be a unitary divisor of n if $d \mid n$ and $\gcd(d, n/d) = 1$, notation $d \parallel n$. Using ring theoretic considerations, Alkam and Osba [1] rediscovered some of these results, while Tóth [13] gave direct proofs of them.

Let us consider the set $\mathrm{Reg}_n = \{a : 1 \leq a \leq n \text{ and a is regular } \pmod{n}\}$, and $\varrho(n) = \#\mathrm{Reg}_n$.

The function ϱ is multiplicative and $\varrho(p^\alpha) = \phi(p^\alpha) + 1 = p^\alpha - p^{\alpha-1} + 1$ for every prime power p^α, where $\phi(n)$ is the Euler function. Consequently, $\varrho(n) = \displaystyle\sum_{d\parallel n} \phi(d)$, for every $n \geq 1$.

Also $\phi(n) < \varrho(n) \leq n$, for every $n > 1$, and $\varrho(n) = n$ if and only if n is a squarefree, see [10], [13], [1].

Thus, the function $\varrho(n)$ is an analogue of the Euler function $\phi(n)$.

Apostol and Tóth [6] considered the multidimensional generalization $\varrho_r(n)$ of the function $\varrho(n)$, defined for every fixed integer $r \geq 1$ as follows: $\varrho_r(n)$ is the number of ordered

r-tuples $(a_1, \ldots, a_r) \in \{1, \ldots, n\}^r$ such that $\gcd(a_1, \ldots, a_r)$ is regular (mod n). If $r = 1$, then $\varrho_1 = \varrho$.
The function $\varrho_r(n)$ is multiplicative and $\varrho_r(p^\alpha) = \phi_r(p^\alpha) + 1 = p^{\alpha r} - p^{(\alpha-1)r} + 1$, where $\phi_r(n)$ is the Jordan function of order r. Consequently, $\varrho_r(n) = \sum_{d \| n} \phi_r(d)$, for every $n \geq 1$. Also $\phi_r(n) < \varrho_r(n) \leq n^r$ for every $n > 1$ and $\varrho_r(n) = n^r$ if and only if n is squarefree, see [6].

In Section 2 we present some notation and results involving arithmetical functions. Section 3 is devoted to the study of the sequence $(\varrho_r(n+1) - \varrho_r(n))_{n \geq 1}$. Average orders of the function $\varrho_r(n)$ in connection with $\psi_r(n)$, $\sigma_r(n)$ and $\sigma_r^*(n)$ are given in Section 4. In Section 5 we give extremal orders of compositions of arithmetical functions.

For other properties concerning regular integers modulo n and compositions of arithmetic functions see [8], [11] and [12].

2 Preliminaries

In what follows let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} > 1$ be a positive integer. We will use throughout the paper the following notation:

- $p_1, p_2, \ldots$ - the sequence of the primes;
- $d \| n$ - d is a unitary divisor of n, that is $d \mid n$ and $\gcd(d, \frac{n}{d}) = 1$;
- $\sigma_r(n)$ - the generalization of $\sigma(n)$, defined by $\sigma_r(n) = \displaystyle\prod_{i=1}^{k} \frac{p_i^{(\alpha_i+1)r} - 1}{p_i^r - 1}$;
- $\psi_r(n)$ - the generalization of $\psi(n)$, defined by $\psi_r(n) = n^r \displaystyle\prod_{p|n} \left(1 + \frac{1}{p^r}\right)$;
- $\phi_r(n)$ - the Jordan function of order r, $\phi_r(n) = n^r \displaystyle\prod_{p|n} \left(1 - \frac{1}{p^r}\right)$;
- $\zeta(s)$ - the Riemann zeta function, $\zeta(s) = \displaystyle\prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}$, $s = \sigma + it \in \mathbb{C}$ and $\sigma > 1$;
- γ - the Euler constant, $\gamma = \displaystyle\lim_{n\to\infty} (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n)$;

Now we consider the functions $\sigma^*(n)$ and $\phi^*(n)$, representing the sum of the unitary divisors of n and the unitary Euler function, respectively.
The functions $\sigma^*(n)$ and $\phi^*(n)$ are multiplicative. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorisation of $n > 1$, then

$$\phi^*(n) = (p_1^{\alpha_1} - 1) \cdots (p_k^{\alpha_k} - 1), \qquad \sigma^*(n) = (p_1^{\alpha_1} + 1) \cdots (p_k^{\alpha_k} + 1)$$

Note that $\sigma^*(n) = \sigma(n)$, $\phi^*(n) = \phi(n)$ for all squarefree n, and for every $n \geq 1$

$$\phi(n) \leq \phi^*(n) \leq n \leq \sigma^*(n) \leq \sigma(n).$$

Moreover, let $\sigma_r^*(n)$ and $\phi_r^*(n)$ be the functions representing the generalizations for the sum of the unitary divisors of n and the unitary analogue Euler function, respectively.

If $r \geq 1$ is a fixed integer, then we have $\sigma_r^*(n) = \sum_{d \| n} d^r$ and $\sigma_r^*(p^\alpha) = p^{\alpha r} + 1$. Also, $\phi_r^*(n) := \sum_{\substack{(a_1,\ldots,a_r)\in\{1,2,\ldots,n\}^r \\ \gcd(\gcd(a_1,a_2,\ldots,a_r),n)_*=1}} 1 = \sum_{d\|n} d^r \mu^*(\frac{n}{d})$, hence $\phi_r^*(p^\alpha) = p^{\alpha r} - 1$. Here $\gcd(a,b)_* = \max\{d : d|a, d \| b\}$ and $\mu^*(n)$ is the unitary analogue of the Möbius function, given by $\mu^*(n) = (-1)^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime factors of n. The functions $\sigma_r^*(n)$ and $\phi_r^*(n)$ are multiplicative. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorisation of $n > 1$. We obtain

$$\phi_r^*(n) = (p_1^{\alpha_1 r} - 1) \cdots (p_k^{\alpha_k r} - 1) \quad \text{and} \quad \sigma_r^*(n) = (p_1^{\alpha_1 r} + 1) \cdots (p_k^{\alpha_k r} + 1).$$

Observe that $\sigma_r^*(n) = \sigma_r(n)$ and $\phi_r^*(n) = \phi_r(n)$ for all squarefree n. Furthermore, for every $n \geq 1$,

$$\phi_r(n) \leq \phi_r^*(n) \leq n^r \leq \sigma_r^*(n) \leq \sigma_r(n).$$

3 The sequence $(\varrho_r(n+1) - \varrho_r(n))_{n\geq 1}$

Studying the convexity and concavity of the sequence $(p_n)_{n\geq 1}$, Erdős and Turán [7] proved that the inequality

$$p_{n+1} - 2p_n + p_{n-1} > 0$$

holds for infinitely many indices and the inequality

$$p_{n+1} - 2p_n + p_{n-1} < 0$$

also holds for infinitely many indices.

So, the sequence $(p_n)_{n\geq 1}$ is neither convex nor concave.
We will prove that for each $r \geq 1$ the sequence $(\varrho_r(n))_{n\geq 1}$ has the same property.
We begin with:

Proposition 1. *If $r \geq 1$, then*

$$\limsup_{n\to\infty}(\varrho_r(n+1) - \varrho_r(n)) = \infty \text{ and } \liminf_{n\to\infty}(\varrho_r(n+1) - \varrho_r(n)) = -\infty.$$

Proof: Let $n = 2^t m$, ($t \geq 1$, m odd) be an even number. Since $\varrho_r(n)$ is multiplicative,

$$\varrho_r(n) = \varrho_r(2^t)\varrho_r(m) \leq m^r(2^{rt} - 2^{r(t-1)} + 1) = m^r 2^{rt}\left(1 - \frac{1}{2^r}\right) + \frac{n^r}{(2^t)^r}.$$

So

$$\varrho_r(n) \leq n^r\left(1 - \frac{1}{2^r} + \frac{1}{4^r}\right),$$

for every n which is a multiple of 4.

Let p be a prime number of the form $p = 4t + 1$. Then $\varrho_r(p) = p^r$, so by the above inequality we have

$$\varrho_r(p) - \varrho_r(p-1) \geq p^r - \left(1 - \frac{1}{2^r} + \frac{1}{4^r}\right)(p-1)^r.$$

Since (according to Dirichlet's theorem of arithmetic progressions) we may take p as large as we please, the first assertion is proved.

Now take p a prime number of the form $p = 4t + 3$. Then $4 \mid p + 1$ and deduce

$$\varrho_r(p) - \varrho_r(p+1) \geq p^r - \left(1 - \frac{1}{2^r} + \frac{1}{4^r}\right)(p+1)^r.$$

Since p in the above relation may be taken arbitrarily large, the second assertion is proved. □

As an immediate consequence of Proposition 1 we obtain

Proposition 2. *For $r \geq 1$, the function $\varrho_r(n)$ is neither convex nor concave.*

Proof: Assume that the sequence of positive integers $(\varrho_r(n))_{n\geq 1}$ is convex (concave). Then the sequence $(\varrho_r(n+1) - \varrho_r(n))_{n\geq 1}$ is increasing (decreasing), wich contradicts Proposition 1. □

4 Average orders

In [13] there are given average orders for the functions $\frac{\varrho(n)}{\phi(n)}, \frac{\phi(n)}{\varrho(n)}, \frac{1}{\varrho(n)}$. Apostol and Petrescu [5] considered average orders for $\frac{\varrho(n)}{\psi(n)}, \frac{\varrho(n)}{\sigma(n)}$ and $\frac{\varrho(n)}{\sigma^*(n)}$. We prove similar results involving $\frac{\varrho_r(n)}{\psi_r(n)}, \frac{\varrho_r(n)}{\sigma_r(n)}$ and $\frac{\varrho_r(n)}{\sigma_r^*(n)}$. If k is a nonnegative integer, we define the function id_k by $id_k(n) = n^k$; let $\mathbf{1} = id_0$. It is well-known that $\mathbf{1}$ is the inverse of the Möbius function μ with respect to the Dirichlet convolution.

Proposition 3. *For every fixed $r \geq 2$ we have*

$$\sum_{n\leq x} \frac{\varrho_r(n)}{\psi_r(n)} = K_r x + O(1),$$

where $K_r = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \cdot \frac{p^{\alpha r} - p^{(\alpha-1)r} + 1}{p^{\alpha r} + p^{(\alpha-1)r}}\right).$

Proof: Consider the quotient $f_r(n) = \frac{\varrho_r(n)}{\psi_r(n)}$. Writing $g_r = \mu * f_r$, where " $*$ " is the Dirichlet convolution and μ the Möbius function, we have $g_r(p) = -\frac{1}{p^r+1}$ and for every prime power p^α, $\alpha \geq 2$,

$$g_r(p^\alpha) = \frac{p^{(\alpha-2)r} - p^{\alpha r}}{(p^{\alpha r} + p^{(\alpha-1)r})(p^{(\alpha-1)r} + p^{(\alpha-2)r})}.$$

We obtain $|g_r(p^\alpha)| < \dfrac{1}{p^{(\alpha-1)r}}$, for every $\alpha \geq 2$. If $\sigma = \operatorname{Re} s$, observe that $\left|\dfrac{g_r(p^\alpha)}{(p^\alpha)^s}\right| < \dfrac{1}{p^{(\alpha-1)r+\alpha\sigma}}$ and $(\alpha-1)r + \alpha\sigma > 1$ for every $\alpha \geq 2$, if $\sigma > \dfrac{1-r}{2}$.
For $\alpha = 1$, $\left|\dfrac{g_r(p)}{p^s}\right| < \dfrac{1}{p^{r+\sigma}}$ and $r + \sigma > 1$ if $\sigma > 1 - r$. So, if we take into account that g_r is multiplicative, as a Dirichlet convolution of two multiplicative arithmetic functions, the Dirichlet series $G_r(s) = \displaystyle\sum_{n=1}^{\infty} \frac{g_r(n)}{n^s}$ converges absolutely for $\sigma > \dfrac{1-r}{2}$. This implies that $G_r(s)$ converges absolutely for $\sigma = 0$, also.

Since $g_r = \mu * f_r$ we have $f_r = \mu^{-1} * g_r = \mathbf{1} * g_r = g_r * \mathbf{1}$ and

$$\sum_{n \leq x} \frac{\varrho_r(n)}{\psi_r(n)} = \sum_{d \leq x} g_r(d) \sum_{n \leq \frac{x}{d}} 1 = \sum_{d \leq x} g_r(d) \left(\frac{x}{d} + O(1) \right),$$

so

$$\sum_{n \leq x} \frac{\varrho_r(n)}{\psi_r(n)} = G_r(1)x + O(1).$$

It follows that

$$K_r = G_r(1) = \sum_{n=1}^{\infty} \frac{g_r(n)}{n} = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{f_r(p^\alpha)}{p^\alpha}\right) =$$

$$= \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \cdot \frac{p^{\alpha r} - p^{(\alpha-1)r} + 1}{p^{\alpha r} + p^{(\alpha-1)r}}\right)$$

and the proof is complete. ∎

Corollary 1. *For all $r \geq 2$ the average order of $\dfrac{\varrho_r(n)}{\psi_r(n)}$ is K_r.*

Proposition 4. *For every fixed $r \geq 2$ and for every $\varepsilon > 0$ we have*

$$\sum_{n \leq x} \frac{\varrho_r(n)}{\sigma_r(n)} = C_r x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $C_r = \displaystyle\prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \cdot \frac{(p^r - 1)(p^{\alpha r} - p^{(\alpha-1)r} + 1)}{p^{(\alpha+1)r} - 1}\right)$.

Proof: Let $f_r(n) = \dfrac{\varrho_r(n)}{\sigma_r(n)}$ and $g_r = \mu * f_r$. Then, for every prime p, $g_r(p) = -\dfrac{1}{p^r + 1}$ and for every prime power p^α, $\alpha \geq 2$, we get

$$g_r(p^\alpha) = (p^r - 1) \cdot \frac{2p^{(\alpha-1)r} + p^{(\alpha-2)r} - p^{(\alpha+1)r}}{(p^{(\alpha+1)r} - 1)(p^{\alpha r} - 1)}$$

Observe that $|g_r(p^\alpha)| < \dfrac{1}{p^{(\alpha-2)r}}$ for every $\alpha \geq 2$. Using a similar argument as in the proof of Proposition 3, , the Dirichlet series $G_r(s) = \sum_{n=1}^{\infty} \dfrac{g_r(n)}{n^s}$ is absolutely convergent for $\sigma = \text{Re } s > \dfrac{1}{2}$.

Let $\varepsilon > 0$, We obtain

$$\sum_{n\leq x} \frac{\varrho_r(n)}{\sigma_r(n)} = \sum_{d\leq x} g_r(d)\left(\frac{x}{d} + O(1)\right) = xG_r(1) - x\sum_{d>x} \frac{g_r(d)}{d} + O\left(\sum_{d\leq x} |g_r(d)|\right),$$

so

$$\sum_{n\leq x} \frac{\varrho_r(n)}{\sigma_r(n)} = C_r x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $C_r = G_r(1) = \prod_p \left(1 - \dfrac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \dfrac{1}{p^\alpha} \cdot \dfrac{(p^r-1)(p^{\alpha r} - p^{(\alpha-1)r} + 1)}{p^{(\alpha+1)r} - 1}\right)$. $\square$

Corollary 2. *For all $r \geq 2$ the average order of $\dfrac{\varrho_r(n)}{\sigma_r(n)}$ is C_r.*

Proposition 5. *For every fixed $r \geq 2$ we have*

$$\sum_{n\leq x} \frac{\varrho_r(n)}{\sigma_r^*(n)} = A_r x + O(1),$$

where $A_r = \prod_p \left(1 - \dfrac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \dfrac{1}{p^\alpha} \cdot \dfrac{p^{\alpha r} - p^{(\alpha-1)r} + 1}{p^{\alpha r} + 1}\right)$.

Proof: Let $f_r(n) = \dfrac{\varrho_r(n)}{\sigma_r^*(n)}$ and $g_r = \mu * f_r$. Then, for every prime p, $g_r(p) = -\dfrac{1}{p^r+1}$ and for every prime power p^α, $\alpha \geq 2$, we get

$$g_r(p^\alpha) = \frac{p^{(\alpha-2)r} - p^{(\alpha-1)r}}{p^{(2\alpha-1)r} + p^{\alpha r} + p^{(\alpha-1)r} + 1},$$

so $|g_r(p^\alpha)| < \dfrac{1}{p^{\alpha r}}$ for every $\alpha \geq 2$. The Dirichlet series $G_r(s) = \sum_{n=1}^{\infty} \dfrac{g_r(n)}{n^s}$ is absolutely convergent for $\sigma = \text{Re } s > 1 - r$ and $\sigma = 0$ satisfies the previous condition. We obtain

$$\sum_{n\leq x} \frac{\varrho_r(n)}{\sigma_r^*(n)} = xG_r(1) + O(1)$$

and some easy computation gives

$$G_r(1) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \cdot \frac{p^{\alpha r} - p^{(\alpha-1)r} + 1}{p^{\alpha r} + 1}\right) = A_r,$$

as claimed. $\square$

Corollary 3. *For all $r \geq 2$ the average order of $\dfrac{\varrho_r(n)}{\sigma_r^*(n)}$ is A_r.*

5 Extremal Orders

We now move to the study of composite arithmetic functions. Sándor and Tóth [11] investigated the maximal order of $\phi^*(\phi(n))$. Apostol [2] gives maximal orders of $\varrho(\phi(n))$, $\varrho(\phi^*(n))$ and other compositions. Apostol and Petrescu [4] generalize some of these results and find the maximal orders of $\varrho_r(\phi_r(n))$ and $\varrho_r(\phi^*(n))$. We extend the study of exact extremal orders to other compositions of arithmetical functions, considering also the functions $\phi_r^*(n)$ and $\sigma_r^*(n)$.

Next, let $n_k = p_1 \cdots p_k$ be the product of the first k primes. Since

$$\lim_{k\to\infty} \frac{\phi^*(\varrho(n_k))}{n_k} = \lim_{k\to\infty} \left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_k}\right) = 0,$$

we get $\liminf_{n\to\infty} \dfrac{\phi^*(\varrho(n))}{n} = 0.$

For the minimal order of the composition $\phi_r^*(\varrho(n))$, where $r \geq 1$, we show

Proposition 6. *For $r > 1$,*

$$\liminf_{n\to\infty} \frac{\phi_r^*(\varrho(n))}{(\varrho(n))^r} = \frac{1}{\zeta(r)}$$

and for $r = 1$,

$$\liminf_{n\to\infty} \frac{\phi^*(\varrho(n)) \log\log n}{\varrho(n)} = e^{-\gamma}.$$

Proof: With n_k from above, observe that for every $n \geq 2$ there is $k = k(n)$ such that $n_k \leq n < n_{k+1}$. We will need the following inequality:

$$\frac{\phi_r^*(\varrho(n))}{(\varrho(n))^r} \geq \frac{\phi_r^*(\varrho(n_k))}{(\varrho(n_k))^r}. \tag{5.1}$$

To show this let $\varrho(n) = q_1^{b_1} \cdots q_s^{b_s}$, where $q_1 < q_2 < \ldots < q_s$ are the prime factors of $\varrho(n)$ and $b_1, \ldots, b_s \geq 1$. Then

$$\frac{\phi_r^*(\varrho(n))}{(\varrho(n))^r} = \prod_{i=1}^{s} \left(1 - \frac{1}{q_i^{b_i r}}\right).$$

But $\varrho(n) \le n < n_k + 1$, that is $s \le k$. Since $q_i \ge p_i$ for $i = \overline{1, s}$, we obtain

$$\frac{\phi_r^*(\varrho(n_k))}{(\varrho(n_k))^r} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^r}\right) \le \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{b_i r}}\right)$$

$$\le \prod_{i=1}^{s} \left(1 - \frac{1}{p_i^{b_i r}}\right) \le \prod_{i=1}^{s} \left(1 - \frac{1}{q_i^{b_i r}}\right)$$

and (5.1) is proved. For $r > 1$ we have

$$\lim_{k\to\infty} \frac{\phi_r^*(\varrho(n_k))}{(\varrho(n_k))^r} = \lim_{k\to\infty} \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^r}\right) = \frac{1}{\zeta(r)}.$$

Hence it follows that

$$\liminf_{n\to\infty} \frac{\phi_r^*(\varrho(n))}{(\varrho(n))^r} = \frac{1}{\zeta(r)}.$$

According to the result of Mertens

$$\lim_{n\to\infty} \log n \prod_{p \le n} \left(1 - \frac{1}{p}\right) = e^{-\gamma},$$

for $r = 1$ we deduce that

$$\frac{\phi^*(\varrho(n_k))}{\varrho(n_k)} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\log p_k} \sim \frac{e^{-\gamma}}{\log\log n_k}$$

when $k \to \infty$, taking into account that $\log n_k = p_k(1 + O(1))$. By (5.1), for sufficiently large n, we have

$$\frac{\phi^*(\varrho(n)) \log\log n}{\varrho(n)} \ge \frac{\phi^*(\varrho(n_k)) \log\log n_k}{\varrho(n_k)}.$$

So

$$\liminf_{n\to\infty} \frac{\phi^*(\varrho(n)) \log\log n}{\varrho(n)} = e^{-\gamma}$$

and the proof is complete. $\square$

It is obvious that

$$\frac{\sigma^*(\varrho(n_k))}{n_k} = \left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_k}\right) \to \infty$$

as $k \to \infty$, so

$$\limsup_{n\to\infty} \frac{\sigma^*(\varrho(n))}{n} = \infty.$$

If we refer to the composion $\sigma_r^*(\varrho(n))$, using similar arguments as in the proof of Proposition 6, observe that

$$\frac{\sigma_r^*(\varrho(n))}{n^r} \leq \frac{\sigma_r^*(\varrho(n_k))}{n_k^r}.$$

Also, for $r > 1$

$$\lim_{k\to\infty} \frac{\sigma_r^*(\varrho(n_k))}{n_k^r} = \lim_{k\to\infty} \prod_{i=1}^{k} \left(1 + \frac{1}{p_i^r}\right) = \frac{\zeta(r)}{\zeta(2r)}$$

and for $r = 1$

$$\frac{\sigma^*(\varrho(n_k))}{n_k} = \prod_{i=1}^{k} \frac{\left(1 - \frac{1}{p_i^2}\right)}{\left(1 - \frac{1}{p_i}\right)} \sim \frac{e^\gamma}{\zeta(2)} \log p_k \sim \frac{e^\gamma}{\zeta(2)} \log\log n_k, \quad (k \to \infty).$$

Therefore, the maximal order of $\sigma_r^*(\varrho(n))$ is $\frac{\zeta(r)}{\zeta(2r)} n^r$ for $r > 1$ and the maximal order of the same function is $\dfrac{e^\gamma}{\zeta(2)} n \log\log n$ for $r = 1$.

The extremal orders of the quotient $\dfrac{\varrho(n)}{\phi(n)}$ were investigated in [13]. In [4] there are given extremal orders concerning classical generalized arithmetic functions, e.g. $\dfrac{\sigma_r(n)}{\varrho_r(n)}$ and $\dfrac{\psi_r(n)}{\varrho_r(n)}$.

Consider now the quotient $\dfrac{\sqrt[r]{\phi_r(n)}}{\varrho(n)}$. Proposition 7 shows that the maximal order of the function $\dfrac{\sqrt[r]{\phi_r(n)}}{\varrho(n)}$ is $e^\gamma \dfrac{1}{\sqrt[r]{\zeta(r)}} \log\log n$.

Proposition 7. *For every $r \geq 2$,*

$$\limsup_{n\to\infty} \frac{\sqrt[r]{\phi_r(n)}}{\varrho(n) \log\log n} = e^\gamma \frac{1}{\sqrt[r]{\zeta(r)}}.$$

Proof: Apply the following general result, see (Tóth, Wirsing [14, Corollary 1]): If f is a nonnegative real-valued multiplicative arithmetic function such that for each prime p,

(i) $\quad \rho(p) = \sup_{\alpha \geq 0} (f(p^\alpha)) \leq \left(1 - \dfrac{1}{p}\right)^{-1}$, and

(ii) $\quad$ there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ satisfying $f(p^{e_p}) \geq 1 + \dfrac{1}{p}$,

then $\quad \displaystyle\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^\gamma \prod_p \left(1 - \frac{1}{p}\right) \rho(p).$

For $r \geq 2$, let $f_r(n) = \dfrac{\sqrt[r]{\phi_r(n)}}{\varrho(n)}$ be a nonnegative real-valued multiplicative arithmetic function.

For $\alpha \geq 0$

$$f_r(p^\alpha) = \frac{p^\alpha \sqrt[r]{1 - \frac{1}{p^r}}}{p^\alpha(1 - \frac{1}{p} + \frac{1}{p^\alpha})} \leq \frac{\sqrt[r]{1 - \frac{1}{p^r}}}{1 - \frac{1}{p}},$$

so

$$\rho_r(p) = \sup_{\alpha \geq 0} f_r(p^\alpha) = \frac{\sqrt[r]{1 - \frac{1}{p^r}}}{1 - \frac{1}{p}} \leq \big(1 - \frac{1}{p}\big)^{-1}.$$

Some easy computations lead us to choose $e_p = 4$ for $r = 2, 3, 4$ and $e_p = 3$, for $r \geq 5$. We obtain

$$\limsup_{n \to \infty} \frac{\sqrt[r]{\phi_r(n)}}{\varrho(n) \log\log n} = e^\gamma \prod_p (1 - \frac{1}{p}) \frac{\sqrt[r]{1 - \frac{1}{p^r}}}{1 - \frac{1}{p}} = e^\gamma \frac{1}{\sqrt[r]{\zeta(r)}},$$

as desired. ∎

Note that

$$\liminf_{n \to \infty} \frac{\varrho_r(\phi(n))}{n^r} = \liminf_{n \to \infty} \frac{\varrho_r(\phi^*(n))}{n^r} = 0.$$

We have

$$\frac{\varrho_r(\phi(n_k))}{n_k^r} = \frac{\varrho_r((p_1 - 1) \cdots (p_k - 1))}{p_1^r \cdots p_k^r}$$

$$\leq \frac{(p_1 - 1)^r \cdots (p_k - 1)^r}{p_1^r \cdots p_k^r} = \big((1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_k})\big)^r,$$

so

$$\lim_{k \to \infty} \frac{\varrho_r(\phi(n_k))}{n_k^r} = \lim_{k \to \infty} \big((1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_k})\big)^r = 0,$$

similarly the other relation.

Acknowledgement. The author would like to thank the referee for the valuable suggestions wich improved the presentation. The author thanks also C. Sana for helpful correspondence regarding some extremal orders.

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Received: 05.05.2015
Revised: 08.09.2016
Accepted: 12.10.2016

"Spiru Haret" Pedagogical High School,
5 Timotei Cipariu St.,
620004, Focşani, Romania
E-mail: `apo_brad@yahoo.com`