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Cohen-Macaulayness of triangular graphs

by
Hernán de Alba[1], Walter Carballosa[2], Daniel Duarte[3],
Luis Manuel Rivera[4]

Abstract

We study the Cohen-Macaulay property of triangular graphs T_n. We show that T_2, T_3 and T_5 are Cohen-Macaulay graphs, and that T_4, T_6, T_8 and T_n are not Cohen-Macaulay graphs, for $n \geq 10$. Finally, we prove that over fields of characteristic zero T_7 and T_9 are Cohen-Macaulay.

Key Words: Triangular graphs, edge ideals, Cohen-Macaulay graphs
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1 Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_N]$ be the polynomial ring over $\mathbb{K}$, where $\mathbb{K}$ is any field. Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_N\}$ and edge set $E(G)$. We identify the vertex v_i with the variable x_i. The *edge ideal* $I(G)$ of G is the ideal $\langle x_i x_j : \{v_i, v_j\} \in E(G) \rangle$. The graph G is called *Cohen-Macaulay* over $\mathbb{K}$ if $R/I(G)$ is a Cohen-Macaulay ring. According to [11], it is unlikely to have a general classification of Cohen-Macaulay graphs. This situation has led to an extensive study of the Cohen-Macaulay property of particular families of graphs (see, for instance, [5, 7, 9, 10, 11, 15, 17]).

In this note we study the Cohen-Macaulayness of triangular graphs. The *triangular graph* T_n is the simple graph whose vertices are the 2-subsets of an n-set, $n \geq 2$, and two vertices are adjacent if and only if their intersection is nonempty. It is known that T_n is isomorphic to the Johnson graph $J(n, 2)$, which is in turn the 2-token graph of the complete graph K_n (see, for instance, [2, 6, 12]). In addition, the complement of T_n is isomorphic to the Kneser graph $K(n, 2)$ and the complement of T_5 is isomorphic to the Petersen graph.

Our main theorem (Theorem 3) states that T_2, T_3 and T_5 are Cohen-Macaulay, and that T_4, T_6, T_8 and T_n are not Cohen-Macaulay graphs, for $n \geq 10$. In addition, it is proved that over fields of characteristic zero T_7 and T_9 are Cohen-Macaulay.

This note is organized as follows. We start by recalling the basic definitions and results regarding Cohen-Macaulay graphs that we need. Next, in Section 3, we first prove that T_n is unmixed for every $n \in \mathbb{N}$. Later, we give a characterization for the Cohen-Macaulay property of T_n that follows from Reisner criterion (Proposition 2). In Section 4, we first prove that T_3 and T_5 are Cohen-Macaulay. Next, using a computer algebra system, we compute explicit regular sequences to show that T_7 and T_9 are Cohen-Macaulay over fields of characteristic zero. Finally, in Section 5, we show that T_4, T_6, T_8 and T_n are not Cohen-Macaulay graphs, for $n \geq 10$.

When investigating about the Cohen-Macaulayness of T_n, we computed several regular sequences using symmetric polynomials and we noticed that there was a certain pattern on how these sequences behave as n increases. We later realized that those patterns also appeared for the edge subring of any simple graph. We conclude this note with an appendix in which we present an explicit regular sequence of a particularly nice shape for Cohen-Macaulay graphs. To that end, we first prove the existence of an explicit homogeneous system of parameters using elementary symmetric polynomials.

2 Cohen-Macaulay graphs and Cohen-Macaulay simplicial complexes

Let $R = \mathbb{K}[x_1, \ldots, x_N]$ be the polynomial ring over the field $\mathbb{K}$. Let $\mathfrak{m} = \langle x_1, \ldots, x_N \rangle$ and let I be a graded ideal of R. The *depth* of R/I is defined as the largest integer r such that there is a homogeneous sequence $\{h_1, \ldots, h_r\} \subset \mathfrak{m}$, such that h_1 is not a zero divisor of R/I and h_i is not a zero divisor of $R/\langle I, h_1, \ldots, h_{i-1} \rangle$, for every $i \geq 2$.

Definition 1. *We say that R/I is a Cohen-Macaulay ring (CM ring for short) if depth$(R/I) = \dim(R/I)$, where $\dim(R/I)$ denotes the Krull dimension of R/I.*

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_N\}$ and edge set $E(G)$. We identify each vertex v_i with the variable x_i in R. The *edge ideal* $I(G)$ of G is the ideal $\langle x_i x_j \colon \{v_i, v_j\} \in E(G) \rangle$. The ring $R/I(G)$ is called the *edge subring* of G. We say that G is a *Cohen-Macaulay graph over* $\mathbb{K}$ if $R/I(G)$ is CM. We say that G is a *Cohen-Macaulay graph* if G is CM over any field.

A set U of vertices in a graph G is an *independent set* of vertices if no two vertices in U are adjacent; a *maximal independent set* is an independent set which is not a proper subset of any independent set in G. The *independence number* of G is the number of vertices in a largest independent set in G. It is well known that the Krull dimension of $R/I(G)$ is equal to the independence number of G (see [8, 16]).

Let Δ be a simplicial complex on the vertex set $V = \{v_1, \ldots, v_N\}$, i.e., Δ is a family of subsets of V closed under taking subsets and such that $\{v_i\} \in \Delta$, for every i. The elements of Δ are called faces of Δ. The dimension of a face $F \in \Delta$ is $|F| - 1$. The dimension of Δ is the largest dimension of its faces. As before, we identify v_i with x_i. The *Stanley-Reisner ideal* I_Δ of Δ is the ideal generated by all monomials $x_{i_1} \cdots x_{i_r}$ such that $\{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta$. We say that Δ is a *Cohen-Macaulay simplicial complex* over $\mathbb{K}$ if R/I_Δ is a CM ring. We say that Δ is a *Cohen-Macaulay simplicial complex* if Δ is CM over any field.

Remark 1. *Let G be a simple graph.*

1. *Let Δ_G be the simplicial complex formed by the independent sets of G (this is a simplicial complex since every subset of an independent set is also independent). Hence $I(G) = I_{\Delta_G}$. Therefore, G is a CM graph if and only if Δ_G is a CM simplicial complex.*

2. *A clique of a graph G is a subset $S \subseteq V(G)$ such that the graph induced by S is a complete graph. Let $\Delta(G)$ be the simplicial complex formed by all cliques of G and let*

$\overline{G}$ be the complement graph of G. Notice that $\Delta(\overline{G}) = \Delta_G$: every clique of $\overline{G}$ is an independent set of G and vice versa.

Definition 2. *Let Δ be a simplicial complex and $F \in \Delta$. The link of F in Δ, denoted $lk_\Delta(F)$, is the simplicial complex $\{H \in \Delta\colon\ H \cap F = \emptyset \text{ and } H \cup F \in \Delta\}$. We will often denote the link of F in Δ just as $lk(F)$ if there is no risk of confusion.*

The CM property of a graph can be determined by the following homological criterion (see [13]).

Theorem 1. *(Reisner's criterion) Let Δ be a simplicial complex. The following conditions are equivalent:*

(a) Δ is Cohen-Macaulay over $\mathbb{K}$.

(b) $\widetilde{H_i}(lk(F); \mathbb{K}) = 0$, with $F \in \Delta$ and $i < \dim lk(F)$.

Corollary 1. *If Δ is a 1-dimensional simplicial complex, then Δ is CM if and only if Δ is connected.*

We will also need a result relating the CM property of a simplicial complex to some property of the $h-$vector of the simplicial complex.

Definition 3. *Let Δ be a simplicial complex of dimension d.*

i. The $f-$vector of Δ is defined as $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$, where $f_{-1} = 1$ and f_i denotes the number of faces of dimension i of Δ, for $i \geq 0$.

ii. The $h-$vector of Δ is defined as $h(\Delta) = (h_0, \ldots, h_{d+1})$, where

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1},$$

and $0 \leq k \leq d+1$.

Theorem 2. *[14, Chapter II, Corollary 3.2] Let Δ be a simplicial complex of dimension d. If Δ is CM, then $h_i(\Delta) \geq 0$, for $0 \leq i \leq d+1$.*

3 A characterizacion for the CM property of T_n

The *triangular graph* T_n is the simple graph having as vertices the 2-subsets of a n-set, $n \geq 2$, and two vertices are adjacent if and only if their intersection is nonempty. The triangular graph T_4 is shown in Figure 1.

We denote by (ij) the vertices of T_n, where $1 \leq i < j \leq n$, and by $\Delta(n)$ the simplicial complex of independent sets of T_n. If $n < 2$ we define $\Delta(n) = \emptyset$.

A graph G is *unmixed* if any two maximal independent sets of G have the same cardinality. Since every CM graph is unmixed, the following proposition is relevant.

Proposition 1. *Every triangular graph T_n is unmixed.*

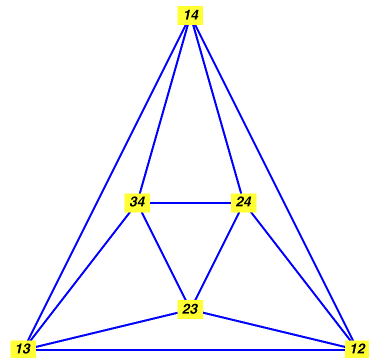

Figure 1: T_4.

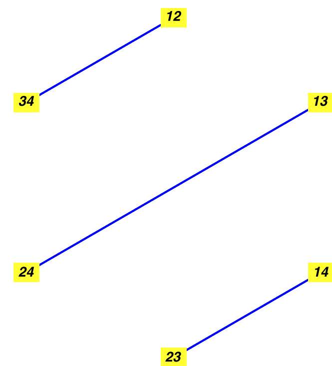

Figure 2: $\overline{T_4}$.

Proof: It is well known that the independence number of T_n is $\lfloor n/2 \rfloor$. We prove, by contradiction, that any maximal independent set in T_n has $\lfloor n/2 \rfloor$ vertices. Let A be any maximal independent set of T_n and suppose that $|A| < \lfloor n/2 \rfloor$. Notice that there are $n-2|A|$ elements in $\{1, \ldots, n\} \setminus \cup A$, with $n - 2|A| > 1$. Therefore, we can take a 2-set, say z, from $\{1, \ldots, n\} \setminus \cup A$ to construct the independent set $A' = A \cup \{z\}$, which is a contradiction. $\square$

We need the following lemma to give a characterization for the Cohen-Macaulay property of the triangular graph T_n.

Lemma 1. *Let $F \in \Delta(n)$ be any face such that $|F| = m$, where $n \geq 2$ and $m \geq 0$. Then we have the following identification of simplicial complexes:*

$$lk_{\Delta(n)}(F) \cong \Delta(n - 2m).$$

Proof: If $F = \emptyset$ the statement holds by definition of $lk_{\Delta(n)}(F)$. If $m = \lfloor n/2 \rfloor$, then $n = 2m$ or $n = 2m + 1$ which implies that $lk_{\Delta(n)}(F) = \emptyset = \Delta(n - 2m)$ in both cases. Suppose $1 \leq m < \lfloor n/2 \rfloor$. Assume $F = \{(i_1 j_1), \ldots, (i_m j_m)\}$. Let $A = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m, j_1, \ldots, j_m\}$. Since F is an independent set of T_n we have that $|A| = n - 2m \geq 2$. Notice that $lk_{\Delta(n)}(F)$ consists of every independent set of T_n formed by elements (ij) such that $i, j \in A$, and $i \neq j$. Now observe that the set of independent sets formed with couples (ij) with $i, j \in A$, $i \neq j$ can be identified with $\Delta(n - 2m)$. $\square$

Proposition 2. *Let $n \geq 2$. Assume n is odd (resp. even). The simplicial complex $\Delta(n)$ is CM if and only if $\widetilde{H_i}(\Delta(l); \mathbb{K}) = 0$ for every $l \leq n$, with l odd (resp. even), and for every $i < \dim(\Delta(l))$.*

Proof: Assume that n is odd. Suppose that $\Delta(n)$ is CM. Choose any odd number l such that $3 \leq l \leq n$. By Lemma 1, $\Delta(l) \cong lk_{\Delta(n)}(F)$ for any face $F \in \Delta(n)$ such that

$|F| = (n-l)/2$. Thus, $\widetilde{H}_i(\Delta(l);\mathbb{K}) = \widetilde{H}_i(lk_{\Delta(n)}(F);\mathbb{K}) = 0$, for every $i < \dim lk_{\Delta(n)}(F) = \dim(\Delta(l))$, according to Reisner's criterion (Theorem 1).

To prove the other implication, let $F \in \Delta(n)$ be such that $|F| = m$, with $m \geq 0$. By Lemma 1, $lk_{\Delta(n)}(F) \cong \Delta(n-2m)$. Since n is odd, $n-2m$ is also odd and $n-2m \leq n$. By the hypothesis, $\widetilde{H}_i(lk_{\Delta(n)}(F);\mathbb{K}) = \widetilde{H}_i(\Delta(n-2m);\mathbb{K}) = 0$, for every $i < \dim(\Delta(n-2m)) = \dim lk_{\Delta(n)}(F)$. By Reisner's criterion, $\Delta(n)$ is CM. The proof is completely analogous for n even. □

Corollary 2. *Suppose that there exists an odd (resp. even) integer n_0 such that T_{n_0} is not CM. Then T_n is not CM for every odd (resp. even) $n \geq n_0$.*

4 The Cohen-Macaulay property of T_3, T_5, T_7 and T_9

Proposition 3. *T_3 and T_5 are CM graphs.*

Proof: Since T_3 is a complete graph, by Example 1 below, T_3 is CM. Now consider the following path in $\Delta(5)$:

$$(12), (34), (25), (14), (35), (24), (13), (45), (23), (15).$$

This path passes through all vertices in $\Delta(5)$, hence it is connected. Since the independence number of T_5 is 2, the simplicial complex $\Delta(5)$ is 1-dimensional. By Corollary 1 we conclude that $\Delta(5)$ is CM, that is, T_5 is CM. □

To verify that T_7 and T_9 are CM we used the computer algebra system `SINGULAR 4-0-2` [3]. One minor difficulty here is to effectively compute the edge ideal of T_n. To that end we use the following remark.

Remark 2. *The graph T_n can be obtained from T_{n-1} and the complete graph K_{n-1} on the vertices $(1\ n), (2\ n), \ldots, (n-1\ n)$ by joining the vertex $(i\ j) \in V(T_{n-1})$ with the vertices $(i\ n)$ and $(j\ n)$ of K_{n-1}. Then we can compute recursively the edge ideal $I(T_n)$: if the edge ideal $I(T_{n-1})$ has been computed, we only need to add the monomials corresponding to $(i\ j) \sim (i\ n)$, $(i\ j) \sim (j\ n)$, and all the monomials coming from K_{n-1}.*

Using the previous procedure we computed $I(T_7) \subset R_1 = \mathbb{Q}[z_1, z_2, \ldots, z_{21}]$ and $I(T_9) \subset R_2 = \mathbb{Q}[z_1, z_2, \ldots, z_{36}]$. Using the library `primdec.lib` [4], we compute primary decomposition of ideals and we found that the sequence

$$\Big\{ \sum_{i=1}^{21} z_i, \sum_{i=1}^{21} z_i^2, \sum_{i=1}^{21} z_i^3 \Big\},$$

is a regular sequence of $R_1/I(T_7)$. Similarly, the sequence

$$\Big\{ \sum_{i=1}^{36} z_i, \sum_{i=1}^{36} z_i^2, \sum_{i=1}^{36} z_i^3, \sum_{i=1}^{36} z_i^4 \Big\},$$

is a regular sequence of $R_2/I(T_9)$ (see the appendix for a discussion on homogeneous system of parameters for edge ideals using symmetric polynomials). Since the independence number of T_7 and T_9 are 3 and 4, respectively, we conclude that

Proposition 4. *T_7 and T_9 are CM graphs over any field of characteristic zero.*

5 Non-Cohen-Macaulayness of T_4, T_6, T_8 and T_n for $n \geq 10$

In this section we show that T_n is not CM for n even, $n \geq 4$. We also show that T_n is not CM for n odd, $n \geq 11$.

Proposition 5. *The triangular graph T_n is not CM if n is even, except for $n = 2$.*

Proof: If $n = 2$, T_n is a single vertex and so it is CM. Let $n = 4$. The simplicial complex $\Delta(4)$ is 1-dimensional and non-connected, actually $\Delta(4) = \overline{T_4}$ (see Figure 2). By Corollary 1, $\Delta(4)$ is not CM. Now, Corollary 2 implies that T_n is not CM for every $n \geq 4$ with n even. $\square$

Now we turn our attention to T_n for $n \geq 11$, n odd.

Lemma 2. *[1, Theorem 6.9.1] The number of faces of dimension i of $\Delta(n)$ is given by the following formula:*

$$f_i = \frac{1}{2^{i+1}} \cdot \frac{n!}{(i+1)!(n-2(i+1))!}.$$

Proposition 6. *T_n is not CM for every $n \geq 11$, n odd.*

Proof: Using the formula of lemma 2, we find that

$$f(\Delta(11)) = (1, 55, 990, 6930, 17325, 10395)$$
$$h(\Delta(11)) = (1, 50, 780, 4280, 6220, -936)$$

Since there is a negative entry in $h(\Delta(11))$, Theorem 2 implies that T_{11} is not CM. Corollary 2 implies that T_n is not CM for every odd n, $n \geq 11$. $\square$

Putting together the results of the previous sections we obtain the following classification of T_n in terms of the CM property:

Theorem 3. *For triangular graphs T_n, the following holds:*

(i) T_2, T_3 *and* T_5 *are CM graphs.*

(ii) T_7 *and* T_9 *are CM graphs over any field of characteristic zero.*

(iii) T_4, T_6, T_8 *and* T_n, *for* $n \geq 10$, *are not CM graphs.*

Proof: The theorem follows from propositions 3, 4, 5, and 6. □

Remark 3. *Using* `SINGULAR 4-0-2`*, we verified that T_7 is CM over some fields of positive characteristic, such as $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_5$. In every case, we found explicit regular sequences using symmetric polynomials (see appendix). This fact suggests that T_7 and T_9 might be CM over any field, giving a complete classification of T_n in terms of the CM property.*

A An explicit regular sequence for CM graphs

It is well known that for any graded ideal $I \subset \mathbb{K}[x_1, \dots, x_N]$, there exists a homogeneous system of parameters (h.s.o.p. for short) for $\mathbb{K}[x_1, \dots, x_N]/I$ (see, for instance, [16, Proposition 2.2.10]). In this appendix we revisit this result for edge ideals by showing the existence of an explicit h.s.o.p. of a particularly nice shape.

This study was motivated by the following fact. When investigating about the Cohen-Macaulayness of T_n, experimental computation showed that for small odd values of n, the sequence

$$\left\{ \sum_{\{x_i\} \in A_1} x_i, \sum_{\{x_{i_1}, x_{i_2}\} \in A_2} x_{i_1} x_{i_2}, \dots, \sum_{\{x_{i_1}, \dots, x_{i_d}\} \in A_d} x_{i_1} \cdots x_{i_d} \right\}$$

is a regular sequence of the edge subring of T_n, where A_j is the set of independent sets of size j in T_n and d is the Krull dimension of the edge subring. Inspired by this fact, we show that for every simple graph there is a h.s.o.p. for its edge subring having this shape.

Let G be a simple graph on the set of vertices $\{x_1, \dots, x_N\}$ and let $I(G) \subset \mathbb{K}[x_1, \dots, x_N]$ be its edge ideal. Let us recall the correspondence between minimal vertex covers of G, i.e., complements of maximal independent sets of $V(G)$, and minimal primes of $I(G)$.

Proposition 7. *[16, Proposition 6.1.16] If $\mathfrak{p}$ is an ideal of $\mathbb{K}[x_1, \dots, x_N]$ generated by $C = \{x_{i_1}, \dots, x_{i_r}\}$, then $\mathfrak{p}$ is a minimal prime of $I(G)$ if and only if C is a minimal vertex cover of G.*

Example 1. *Let K_N be the complete graph on the vertices $\{x_1, \dots, x_N\}$. Every minimal vertex cover of K_N has the form $K_N \setminus \{x_i\}$, for some i. By the previous correspondence, every minimal prime of $I(K_N)$ is generated by all variables except x_i. It follows that $\sum_{i=1}^N x_i$ is not a zero divisor in $R/I(K_N)$, that is, $1 \leq depth(R/I(K_N)) \leq \dim(R/I(K_N)) = 1$. Thus, K_N is CM.*

Example 2. *The edge ideal of T_4 (Figure 1) is given by*

$$I(T_4) = \langle x_{12}x_{13}, x_{12}x_{14}, x_{12}x_{23}, x_{12}x_{24}, x_{13}x_{34}, x_{14}x_{34}, x_{23}x_{34}, x_{24}x_{34} \rangle.$$

Let $R = \mathbb{K}[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. Let $\sigma_1, \dots, \sigma_6 \in R$ be the elementary symmetric polynomials. Since the independence number of G is 2, we have that $[\sigma_i] = [0]$ in $R/I(T_4)$, for $i \in \{3, 4, 5, 6\}$. In addition,

$$[\sigma_1] = [x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34}],$$
$$[\sigma_2] = [x_{12}x_{34} + x_{13}x_{24} + x_{14}x_{23}].$$

Using lemma 3 below, we conclude that

$$\sqrt{\langle[\sigma_1],[\sigma_2]\rangle} = \sqrt{\langle[\sigma_1],\dots,[\sigma_6]\rangle} = \langle[x_{12}],[x_{13}],[x_{14}],[x_{23}],[x_{24}],[x_{34}]\rangle.$$

Since $\dim R/I(T_4) = 2$, *we conclude that* $\{[\sigma_1],[\sigma_2]\}$ *is a h.s.o.p. for* $R/I(T_4)$.

Lemma 3. *Let* $\sigma_1,\dots,\sigma_m \in \mathbb{K}[z_1,\dots,z_m]$ *be the elementary symmetric polynomials. Then* $\sqrt{\langle\sigma_1,\dots,\sigma_m\rangle} = \langle z_1,\dots,z_m\rangle$.

Proof: It is enough to consider the following telescopic sum:

$$z_i^m = z_i^{m-1}\sigma_1 - z_i^{m-2}\sigma_2 + z_i^{m-3}\sigma_3 - \cdots + (-1)^{m+1}\sigma_m.$$

□

Proposition 8. *Let* G *be a simple graph on* N *vertices,* $S = \mathbb{K}[x_1,\dots,x_N]/I(G)$, *and* $d = \dim S$. *Let* A_j *denote the set of independent sets of size* j *in* G. *Then the sequence*

$$\Big\{ \sum_{\{x_i\}\in A_1} x_i, \sum_{\{x_{i_1},x_{i_2}\}\in A_2} x_{i_1}x_{i_2},\dots, \sum_{\{x_{i_1},\dots,x_{i_d}\}\in A_d} x_{i_1}\cdots x_{i_d}\Big\}$$

is a homogeneous system of parameters for S. *In particular, if* G *is CM then this sequence is a regular sequence for* S.

Proof: Let $F_k = \sum_{\{x_{i_1},\dots,x_{i_k}\}\in A_k} x_{i_1}\cdots x_{i_k}$, for $1 \le k \le d$. Let $\sigma_1,\dots,\sigma_N \in \mathbb{K}[x_1,\dots,x_N]$ be the elementary symmetric polynomials. Since the independence number of G is d, we have $[\sigma_k] = [0]$ in S for every $k = d+1,\dots,N$. In addition, $[\sigma_k] = [F_k]$ for every $k \in \{1,\dots,d\}$. The proposition then follows from the previous lemma:

$$\sqrt{\langle[F_1],\dots,[F_d]\rangle} = \sqrt{\langle[\sigma_1],[\sigma_2],\dots,[\sigma_N]\rangle} = \langle[x_1],\dots,[x_N]\rangle.$$

□

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[1] CONACYT-UAZ.
E-mail: `hdealbac@matematicas.reduaz.mx`

[2] CONACYT-UAZ
& Department of Mathematics and Statistics
Florida International University*
Miami FL
& Department of Mathematics
Miami Dade College*
Miami FL.
(*Current affiliations.)
Email: `waltercarb@gmail.com`

[3] CONACYT-UAZ.
Email: `adduarte@matematicas.reduaz.mx`

[4] Unidad Académica de Matemáticas, UAZ.
Calzada Solidaridad y Paseo La Bufa
Col. Hidráulica
C.P. 98060, Zacatecas, Zacatecas
Email: `luismanuel.rivera@gmail.com`