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On a class of Krasner analytic functions and applications

by
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Abstract

In this paper we provide some conditions such that by integrating rational functions, with coefficients in the class of functions of Lipschitz type defined on compact subsets of the Tate field, against unbounded distributions one obtains Krasner analytic functions. Also, we study the problem of p-adic analytic continuation to larger domains of p-adic Mellin and Laplace transforms.

Key Words: Krasner analytic functions, local fields, distributions, p-adic transforms.
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1 Introduction

Fix a prime number p. Let $\mathbb{Q}_p$ be the field of p-adic numbers, $\mathbb{Z}_p$ its ring of integers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$, which is called the Tate field, the completion of $\overline{\mathbb{Q}}_p$ with respect to the usual absolute value $|\cdot|$ normalized such that $|p| = \frac{1}{p}$, see [2,3]. The concept of analytic continuation in p-adic domains is a basic ideea of Krasner and gives a p-adic analog of the classical theorem of Runge from complex analysis, see [6,7]. In fact a Krasner analytic function defined on an open subset D of $\mathbb{C}_p$ is a uniform limit of rational functions having no poles in D.

Now, let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ without isolated points. A distribution on $\mathcal{X}$ in the sense of Mazur and Swinnerton-Dyer, see [8], is an additive function on the set of open-compact subsets of $\mathcal{X}$. In this paper we are interested to study a special class of Krasner anlytic functions defined on $\mathbb{P}^1(\mathbb{C}_p)$, eventually by removing a compact subset of $\mathbb{C}_p$, obtained by integrating rational functions against Mazur distributions defined on $\mathcal{X}$ on one hand and to give some applications to the problem of p-adic analytic continuation on the other hand.

The paper consists of four sections. The first section is an introduction in the framework of the paper. The second section contains notation and definitions. In the third section we point out a class of Krasner analytic functions, see Theorem 1. The theorem gives conditions such that by integrating rational functions, with coefficients in the class of functions of Lipschitz type, against Mazur distributions one obtains Krasner analytic functions. In the last section we apply Theorem 1 to study the problem of p-adic analytic continuation of p-adic Mellin and Laplace transforms to larger domains, via a linear operator called the modified Laplace transform, see Theorem 2. Both results, Theorems 1 and 2, improve some results of [1,13,14] to the case of unbounded distributions.

2 Notation and definitions

Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$. For any real number $\varepsilon > 0$, let $B(x,\varepsilon) = \{y \in \mathbb{C}_p : |y-x| < \varepsilon\}$, respectively $B[x,\varepsilon] = \{y \in \mathbb{C}_p : |y-x| \leq \varepsilon\}$, be the open, respectively the closed, ball of radius ε centered at x. Denote by $\mathcal{X}(\varepsilon) = \{y \in \mathbb{C}_p : |y-t| < \varepsilon, \text{ for some } t \in \mathcal{X}\}$ the ε-neighborhood of $\mathcal{X}$ in $\mathbb{C}_p$. Both sets $\mathcal{X}(\varepsilon)$ and $\mathbb{P}^1(\mathbb{C}_p)\backslash\mathcal{X}(\varepsilon)$ are open and closed, and one has $\cap_{\varepsilon>0}\mathcal{X}(\varepsilon) = \mathcal{X}$.

By an open, respectively closed, ball in $\mathcal{X}$ we mean a subset of the form $B^*(x,\varepsilon) = B(x,\varepsilon) \cap \mathcal{X}$, respectively $B^*[x,\varepsilon] = B[x,\varepsilon] \cap \mathcal{X}$, where $x \in \mathcal{X}$ and $\varepsilon > 0$. Let us denote by $\Omega(\mathcal{X})$ the set of subsets of $\mathcal{X}$ which are open and compact. It is easy to see that any $D \in \Omega(\mathcal{X})$ can be written as a finite union of pairwise disjoint balls in $\mathcal{X}$.

Definition 1. *([8]) By a distribution on $\mathcal{X}$ with values in $\mathbb{C}_p$ we mean a map $\mu : \Omega(\mathcal{X}) \to \mathbb{C}_p$ which is finitely additive, that is, if $D = \cup_{i=1}^n D_i$ with $D_i \in \Omega(\mathcal{X})$ for $1 \leq i \leq n$ and $D_i \cap D_j = \emptyset$ for $1 \leq i \neq j \leq n$, then $\mu(D) = \sum_{i=1}^n \mu(D_i)$.*

The norm of μ is defined by $\|\mu\| := \sup\{|\mu(D)| : D \in \Omega(\mathcal{X})\}$ and, if $\|\mu\| < \infty$ we say that μ is a measure on $\mathcal{X}$.

Definition 2. *For a positive real number s, a distribution μ on $\mathcal{X}$ is of type s, or a s-distribution, if*

$$\lim_{\varepsilon \to 0} \varepsilon^s \sup_{a \in \mathcal{X}} |\mu(B^*(a,\varepsilon))| = 0.$$

The set of s-distributions on $\mathcal{X}$ with values in $\mathbb{C}_p$ is denoted by $\mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$. In the case $s = 1$ such a distribution is called Lipschitz distribution.

Remark 1. *Any measure on $\mathcal{X}$ is a distribution of type s, for any positive real number s.*

Definition 3. *([2,4,9]) Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ and, for any $\varepsilon > 0$ let $\mathcal{X}(\varepsilon)$ denote the ε-neighborhood of $\mathcal{X}$ in $\mathbb{C}_p$. A function $f : \mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{X} \to \mathbb{C}_p$ is said to be Krasner analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{X}$ provided that for any $\varepsilon > 0$ there is a sequence of rational functions with all their poles in $\mathcal{X}(\varepsilon)$ that converges uniformly to f on $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{X}(\varepsilon)$. We denote by $\mathcal{A}(\mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{X}, \mathbb{C}_p)$ the set of all Krasner analytic functions defined on $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{X}$ with values in $\mathbb{C}_p$.*

Definition 4. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ and let r be a positive real number. A function $f : \mathcal{X} \to \mathbb{C}_p$ is called of type r, or r-Lipschitz function, iff there exists a positive constant c such that*

$$|f(x) - f(y)| \leq c|x - y|^r, \tag{1}$$

for any $x, y \in \mathcal{X}$. In the case $r = 1$ such a function is called Lipschitz function.

Denote by $Lip_r(\mathcal{X}, \mathbb{C}_p)$ the set of r-Lipschitz functions defined on $\mathcal{X}$ with values in $\mathbb{C}_p$. The set of Lipschitz functions is denoted by $Lip(\mathcal{X}, \mathbb{C}_p)$. For a r-Lipschitz function f as above, the best constant in (1) is

$$c_f = \sup_{\substack{x,y\in\mathcal{X} \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^r}. \tag{2}$$

Denote $\|f\|_r := \max\{c_f, \|f\|\}$, where $\|f\| = \sup_{x\in\mathcal{X}} |f(x)|$.

The set $Lip_r(\mathcal{X}, \mathbb{C}_p)$ becomes naturally a $\mathbb{C}_p$-Banach algebra with the norm $\|\cdot\|_r$ defined above.

Remark 2. *Let $r \geq s > 0$ be positive real numbers. Then any r-Lipschitz function of $Lip_r(\mathcal{X}, \mathbb{C}_p)$ is Riemann integrable against any s-distribution of $\mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$, see [12, Theorem 11].*

3 A class of Krasner analytic functions

Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ without isolated points, $f, g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$ with $r \geq s > 0$. It is clear that $t \mapsto \frac{f(t)}{(z-g(t))^k}$ is a function of type r for any $z \in \mathbb{P}^1(\mathbb{C}_p) \setminus g(\mathcal{X})$ and any $k \geq 0$. By [12, Theorem 11] the above function is integrable with respect to μ. Let $\{\varepsilon_n\}_{n\geq 1}$ be a decreasing sequence that converges to zero and such that the sequence $\left\{\frac{\varepsilon_n}{\varepsilon_{n+1}}\right\}_{n\geq 1}$ is upper bounded by a positive constant M. One considers $N(\varepsilon_n)$ the number of the open balls of radius ε_n that cover $\mathcal{X}$ and $\{B^*(a_i^{(n)}, \varepsilon_n)\}_{1\leq i\leq N(\varepsilon_n)}$ an ε_n-covering of $\mathcal{X}$ with open balls of radius ε_n centered at $a_i^{(n)} \in \mathcal{X}$. We can consider the following disjoint decomposition $B^*(a_i^{(n)}, \varepsilon_n) = \cup_{j\in J_i} B^*(a_j^{(n+1)}, \varepsilon_{n+1})$ of the open ball $B^*(a_i^{(n)}, \varepsilon_n)$ into open balls of radius ε_{n+1}, for any $1 \leq i \leq N(\varepsilon_n)$. One has $\sum_{i=1}^{N(\varepsilon_n)} \mathrm{card}(J_i) = N(\varepsilon_{n+1})$. Let us denote by

$$A_n(z) = \sum_{i=1}^{N(\varepsilon_n)} \frac{f(a_i^{(n)})}{(z - g(a_i^{(n)}))^k} \mu(B^*(a_i^{(n)}, \varepsilon_n)) \tag{3}$$

the corresponding Riemann sum such that

$$\lim_{n\to\infty} A_n(z) = \int_{\mathcal{X}} \frac{f(t)}{(z - g(t))^k} \mathrm{d}\mu(t), \tag{4}$$

where $z \in \mathbb{P}^1(\mathbb{C}_p) \setminus g(\mathcal{X})$.

Let us fix a positive real number δ and consider $z \in \mathbb{P}^1(\mathbb{C}_p)\setminus g(\mathcal{X})_\delta$ an arbitrary element, where $g(\mathcal{X})_\delta$ is a δ-neighborhood of $g(\mathcal{X})$. One has

$$A_n(z)-A_{n+1}(z) = \sum_{i=1}^{N(\varepsilon_n)} \sum_{j\in J_i} \left[\frac{f(a_i^{(n)})}{(z-g(a_i^{(n)}))^k} - \frac{f(a_j^{(n+1)})}{(z-g(a_j^{(n+1)}))^k}\right]\cdot\mu(B^*(a_j^{(n+1)},\varepsilon_{n+1})). \tag{5}$$

Let $x, y \in \mathcal{X}$ be arbitrary elements. We have

$$\begin{aligned} \frac{f(x)}{(z-g(x))^k} - \frac{f(y)}{(z-g(y))^k} &= \frac{f(x)(z-g(y))^k - f(y)(z-g(x))^k}{(z-g(x))^k(z-g(y))^k} \\ &= \frac{\sum_{l=0}^k (-1)^l \binom{k}{l} z^{k-l}[f(x)g^l(y) - f(y)g^l(x)]}{(z-g(x))^k(z-g(y))^k}. \end{aligned} \tag{6}$$

For any $l \geq 1$ it is clear that

$$f(x)g^l(y) - f(y)g^l(x) = [f(x)-f(y)]g^l(y) - f(y)[g^l(x) - g^l(y)]$$

and because $f, g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ one derives

$$|f(x)g^l(y) - f(y)g^l(x)| \leq \max\{c_f \|g\|^l, c_g \|f\| \|g\|^{l-1}\}|x-y|^r. \tag{7}$$

By (5)-(7) one has

$$\begin{aligned} |A_{n+1}(z) - A_n(z)| \leq &\max_{1\leq i\leq N(\varepsilon_n)} \max_{j\in J_i} \max_{l\leq k} \frac{|z|^{k-l}\max\{c_f \|g\|^l, c_g \|f\| \|g\|^{l-1}\}}{|z-g(a_i^{(n)})|^k |z-g(a_j^{(n+1)})|^k} \\ &\times \left(\frac{\varepsilon_n}{\varepsilon_{n+1}}\right)^r \varepsilon_{n+1}^r \max_{a\in\mathcal{X}} |\mu(B^*(a,\varepsilon_{n+1}))|. \end{aligned} \tag{8}$$

In the right hand side of (8), when $l = 0$, we consider $\|g\|^{-1} = 0$.

By (8), for $|z| > \max\{\delta, \|g\|\}$ one has

$$|A_{n+1}(z)-A_n(z)| \leq M^r \max_{l\leq k} \frac{\max\{c_f \|g\|^l, c_g \|f\| \|g\|^{l-1}\}}{\delta^{k+l}} \times \varepsilon_{n+1}^r \max_{a\in\mathcal{X}} |\mu(B^*(a,\varepsilon_{n+1}))|. \tag{9}$$

Again, by (8), for $|z| \leq \max\{\delta, \|g\|\}$ we derive

$$\begin{aligned} |A_{n+1}(z) - A_n(z)| \leq &\frac{M^r}{\delta^{2k}} \max_{l\leq k}\{\delta, \|g\|\}^{k-l} \max_{l\leq k} \max\{c_f \|g\|^l, c_g \|f\| \|g\|^{l-1}\} \\ &\times \varepsilon_{n+1}^r \max_{a\in\mathcal{X}} |\mu(B^*(a,\varepsilon_{n+1}))|. \end{aligned} \tag{10}$$

Because $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$ we have $\lim_{n\to\infty} \varepsilon_{n+1}^r \max_{a\in\mathcal{X}} |\mu(B^*(a,\varepsilon_{n+1}))| = 0$ and by (9) and (10) one has that $A_{n+1}(z) - A_n(z)$ converges uniformly to zero with respect to $z \in \mathbb{P}^1(\mathbb{C}_p) \setminus g(\mathcal{X})_\delta$ so, by Definition 3, the map $z \mapsto \int_{\mathcal{X}} \frac{f(t)}{(z-g(t))^k} \mathrm{d}\mu(t)$ is a Krasner analytic function on $\mathbb{P}^1(\mathbb{C}_p) \setminus g(\mathcal{X})$.

Now, let us consider $P(t, Z), Q(t, Z) \in Lip_r(\mathcal{X}, \mathbb{C}_p)[Z]$ polynomials in Z with coefficients in $Lip_r(\mathcal{X}, \mathbb{C}_p)$ with $Q(t, Z) = \prod_{i=1}^n (Z - g_i(t))^{k_i}$, where $n \geq 1$ and $k_i \geq 1$. By decomposition of $R(t, Z) := \frac{P(t,Z)}{Q(t,Z)}$ in simple fractions one has that $z \mapsto \int_{\mathcal{X}} R(t, z)\mathrm{d}\mu(t)$ is a Krasner analytic function on $\mathbb{P}^1(\mathbb{C}_p) \setminus \cup_{i=1}^n g_i(\mathcal{X})$. By summing up one obtains the following result.

Theorem 1. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ without isolated points and $r \geq s$ be positive real numbers. Let $P(t, Z), Q(t, Z) \in Lip_r(\mathcal{X}, \mathbb{C}_p)[Z]$ be polynomials in Z with coefficients in $Lip_r(\mathcal{X}, \mathbb{C}_p)$ such that $Q(t, Z) = \prod_{i=1}^n (Z - g_i(t))^{k_i}$ with $k_i \geq 1$ for any $1 \leq i \leq n$. Then, for any p-adic distribution $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$ the function given by*

$$z \mapsto \int_{\mathcal{X}} R(t, z)\mathrm{d}\mu(t),$$

where $R(t, z) := \frac{P(t,z)}{Q(t,z)}$, is Krasner analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus \cup_{i=1}^n g_i(\mathcal{X})$.

4 Applications to p-adic Mellin and Laplace transforms

We preserve the same notation and definitions as in the previous paragraphs.

Proposition 1. *Let $\{f_n\}_{n\geq 0}$ be a sequence of functions such that $f_n \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ for any $n \geq 0$. If $\lim_{n\to\infty} \|f_n\| = 0$ and $\sup_{n\geq 0} c_{f_n} < \infty$ then $f = \sum_{n\geq 0} f_n$ is well defined and $f \in Lip_r(\mathcal{X}, \mathbb{C}_p)$. Moreover, if $\lim_{n\to\infty} \|f_n\|_r = 0$ then $\int_{\mathcal{X}} f \mathrm{d}\mu = \sum_{n\geq 0} \int_{\mathcal{X}} f_n \mathrm{d}\mu$, for any $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$, where $r \geq s > 0$.*

Proof: One has

$$|f(x) - f(y)| = \left| \sum_{n\geq 0} [f_n(x) - f_n(y)] \right| \leq \sup_{n\geq 0} |f_n(x) - f_n(y)| \leq \sup_{n\geq 0} c_{f_n} |x - y|^r \quad (11)$$

so $f \in Lip_r(\mathcal{X}, \mathbb{C}_p)$. By [11, Theorem 2] there is a positive constant $A(\mu)$ such that

$$\begin{aligned} \left| \int_{\mathcal{X}} f \mathrm{d}\mu - \sum_{i=0}^n \int_{\mathcal{X}} f_i \mathrm{d}\mu \right| &= \left| \int_{\mathcal{X}} \left(f - \sum_{i=0}^n f_i \right) \mathrm{d}\mu \right| \\ &\leq A(\mu) \left\| f - \sum_{i=0}^n f_i \right\|_r = A(\mu) \left\| \sum_{i>n} f_i \right\|_r . \end{aligned} \quad (12)$$

By hypothesis, the right hand side of (12) goes to zero and this completes the proof of the proposition. ∎

The operator Λ. The modified Laplace transform Λ is defined on the space of power series which are convergent in a neighborhood of a point $z_0 \in \mathbb{C}_p$. Precisely, if

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

converges on $B(z_0, \varepsilon)$, where $\varepsilon > 0$, we set

$$\Lambda f(z) := \sum_{n \geq 0} n! a_n (z - z_0)^n,$$

which converges on a slighty larger ball $B(z_0, \varepsilon r_p^{-1})$, where $r_p = |p|^{\frac{1}{p-1}}$. It is easy to see that Λ invariates the order of vanishing of f at z_0 i.e. $\mathrm{ord}_{z=z_0} \Lambda f(z) = \mathrm{ord}_{z=z_0} f(z)$.

Theorem 2. *Let $\mathcal{X} \subset \mathbb{C}_p^\times \setminus \log^{-1}(0)$ be a compact subset of $\mathbb{C}_p$ without isolated points, $f \in Lip(\mathcal{X}, \mathbb{C}_p)$ and μ a Lipschitz distribution on $\mathcal{X}$. Then the images through Λ of the p-adic Mellin and Laplace transforms of f on $\mathcal{X}$, by integrating against μ, are Krasner analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus \left(1 + \frac{1}{\log \mathcal{X}}\right)$ and respectively on $\mathbb{P}^1(\mathbb{C}_p) \setminus \left(\frac{-1}{\mathcal{X}}\right)$.*

Proof: Let $L(s, f, \mu) = \int_{\mathcal{X}} f(t) t^{s-1} \mathrm{d}\mu(t)$ be the p-adic Mellin transform of f with respect to μ, which is well defined because $t \mapsto f(t) t^{s-1}$ is Lipschitz on $\mathcal{X}$ for any s in an open ball around 1. Indeed, t^{s-1} is defined by

$$t^{s-1} = \exp[(s-1) \log t] = \sum_{n \geq 0} \frac{(s-1)^n \log^n t}{n!}, \tag{13}$$

where log is the Iwasawa logarithm, see [5,10], $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ that is $\mathcal{C}^\infty$ on any compact subset of $\mathbb{C}_p^\times$ without isolated points. The series (13) converges for any s in the ball $B(1, \rho)$ where $\rho = \frac{r_p}{\|\log\|}$. Denote $f_{n,s}(t) = \frac{(s-1)^n \log^n t}{n!}$. It is clear that $\|f_{n,s}\| \to 0$, when $n \to \infty$, uniformly with respect to $s \in B(1, \rho)$. We have

$$f_{n,s}(t_1) - f_{n,s}(t_2) = \frac{(s-1)^n}{n!} (\log t_1 - \log t_2)(\log^{n-1} t_1 + \cdots + \log^{n-1} t_2)$$

so

$$|f_{n,s}(t_1) - f_{n,s}(t_2)| \leq \frac{|s-1|^n}{|n!|} c_{\log} |t_1 - t_2| \, \|\log\|^{n-1}.$$

One has

$$c_{f_{n,s}} \leq \frac{|s-1|^n}{|n!|} c_{\log} \, \|\log\|^{n-1}. \tag{14}$$

Let us choose $0 < \rho' < \rho$ and let s be an arbitrary element of $B(1, \rho')$. From [5, Lemma 3, page 21] we have

$$r_p^n \leq |n!| \leq n p r_p^n \tag{15}$$

so, by (14) and (15),

$$c_{f_{n,s}} \leq \left(\frac{\rho'}{\rho}\right)^n \frac{c_{\log}}{\|\log\|}, \tag{16}$$

which converges uniformly to zero for any $s \in B(1, \rho')$. Clearly $\lim_{n\to\infty} \|f_{n,s}\|_1 = 0$. By Proposition 1, $t^{s-1} = \sum_{n\geq 0} f_{n,s}(t)$ is Lipschitz and, moreover,

$$L(s, f, \mu) = \sum_{n\geq 0} a_n(f)(s-1)^n \tag{17}$$

where the coefficients $a_n(f)$ of $L(s, f, \mu)$ in (17) are given by

$$a_n(f) = \frac{1}{n!} \int_{\mathcal{X}} f(t) \log^n t \mathrm{d}\mu(t).$$

From [11, Theorem 2] there is a positive constant $A(\mu)$, which depends only on μ, such that $|a_n(f)| \leq \frac{A(\mu)\|f\|_1 \|\log\|_1^n}{|n!|}$ so the series

$$\Lambda L(s, f, \mu) = \sum_{n\geq 0} n! a_n(f)(s-1)^n$$

is convergent in a larger ball that of L. Again, by Proposition 1 we have

$$\Lambda L(s, f, \mu) = \int_{\mathcal{X}} \sum_{n\geq 0} f(t)(s-1)^n \log^n t \mathrm{d}\mu(t) = \int_{\mathcal{X}} \frac{f(t)}{1-(s-1)\log t} \mathrm{d}\mu(t).$$

From Theorem 1 in the case $R(t, s) = \frac{\frac{-f(t)}{\log t}}{s - \left(1 + \frac{1}{\log t}\right)}$, we derive that $\Lambda L(s, f, \mu)$ is Krasner analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus \left(1 + \frac{1}{\log \mathcal{X}}\right)$, which completes the proof of the first part of the theorem. The proof of the second part of the theorem, which is related to the p-adic Laplace transform of f through Λ, goes on the same lines as above and is left to the reader. $\square$

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