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Equivariant Chern classes in Hopf cyclic cohomology

by
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Dedicated to Vasile Brînzănescu on the occasion of his 70*th birthday, with appreciation and friendship*

Abstract

We present a geometric approach, in the spirit of the Chern-Weil theory, for constructing cocycles representing the classes of the Hopf cyclic cohomology of the Hopf algebra $\mathcal{H}_n$ relative to GL_n. This provides an explicit description of the universal Hopf cyclic Chern classes, which complements our earlier geometric realization of the Hopf cyclic characteristic classes of foliations.

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Introduction

The Hopf algebra $\mathcal{H}_n$ originated in the investigation of the local index formula for transversely hypoelliptic operators on foliations [5], performing the role of a 'quantum structure group' for foliations of codimension n. Its Hopf cyclic cohomology relative to O_n was shown to deliver the Gelfand-Fuks cohomology classes as characteristic classes of 'spaces of leaves'. In [15] we presented a geometric method for explicitly constructing these universal Hopf cyclic cohomology classes by means of concrete cocycles, in the spirit of the Chern-Weil theory. We now supplement that construction by adapting the procedure to the case of the Hopf cyclic cohomology of $\mathcal{H}_n$ relative to $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{R})$, which corresponds to the universal equivariant Chern classes. The essential modification needed to adjust the approach in [15] consists in the replacement of the 'differentiable' variants of the standard de Rham complexes for equivariant cohomology by a more restrictive version, to be called 'regular differentiable'.

As we often defer to [15] for additional details, in order to facilitate the reading of the present paper we keep the exposition closely parallel to the former. In §1 we introduce the regular differentiable de Rham cohomology complexes and use them to prove an analogue relative to

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GL_n of the van Est-Haefliger isomorphism. The construction proper of a basis of representative cocycles for the Hopf cyclic cohomology of $\mathcal{H}_n$ relative to GL_n is carried out in §2. This provides a complete description of the universal Hopf cyclic Chern classes, which complements the geometric realization of the Hopf cyclic characteristic classes of foliations [15]. Partial representations of these classes were obtained earlier by purely algebraic methods in [13, §3.4.1] (for Hochschild cohomology) and [14, §4.3].

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1 Chern cocycles in regular differentiable cohomology

1.1 Regular differentiable de Rham complexes

Given a manifold M we denote by $\mathbf{G}$ is the group of diffeomorphisms $\mathrm{Diff}(M)$ equipped with the discrete topology, and by $\triangle_{\mathbf{G}}M$ the simplicial manifold $\{\triangle_{\mathbf{G}}M[p] := \mathbf{G}^p \times M\}_{p\geq 0}$ with its usual face maps $\partial_i : \triangle_{\mathbf{G}}M[p] \to \triangle_{\mathbf{G}}M[p-1], \quad 1 \leq i \leq p,$ and degeneracies $\sigma_i : \triangle_{\mathbf{G}}M[p] \to \triangle_{\mathbf{G}}M[p+1], \; 0 \leq i \leq p$. The equivariant cohomology $H_{\mathbf{G}}(M,\mathbb{R})$ can be computed as the cohomology of the Bott bicomplex (cf. [1, 2]) $\{C^\bullet(\mathbf{G},\Omega^\bullet(M)), \delta, d\}$, endowed with the de Rham differential d and with the group cohomology boundary δ

$$\begin{aligned}\delta c(\phi_1,\ldots,\phi_{p+1}) &= \sum_{i=0}^{p}(-1)^i c\big(\partial_i(\phi_1,\ldots,\phi_{p+1})\big)\\ &\quad + (-1)^{p+1}\phi_{p+1}^* c(\phi_1,\ldots,\phi_p).\end{aligned}$$

For our purposes it will be convenient to work with the homogeneous version of this bicomplex, $\{\bar{C}^\bullet(\mathbf{G},\Omega^\bullet(M)), \bar{\delta}, d\}$, whose (p,q)-cochains $\bar{c}(\rho_0,\ldots,\rho_p) \in \Omega^q(M), \; \rho_0,\ldots,\rho_p \in \mathbf{G}$ satisfy the covariance condition

$$(\rho^{-1})^*\big(\bar{c}(\rho_0\rho,\ldots,\rho_p\rho)\big) = \bar{c}(\rho_0,\ldots,\rho_p), \quad \forall\, \rho,\rho_i \in \mathbf{G}; \tag{1.1}$$

the group cohomology boundary is given by

$$\bar{\delta}\bar{c}(\rho_0,\ldots,\rho_p) = \sum_{i=0}^{p}(-1)^i \bar{c}(\rho_0,\ldots,\check{\rho}_i,\ldots,\rho_p),$$

where the ‘check’ mark signifies omission of the element.

The two bicomplexes are isomorphic via the identifications

$$\begin{aligned} c(\phi_1, \ldots, \phi_p) &= \bar{c}(\phi_1 \cdots \phi_p,\, \phi_2 \cdots \phi_p,\, \ldots, \phi_p, e)\,, \\ \text{resp.} \quad \bar{c}(\rho_0, \ldots, \rho_p) &= \rho_p^* c(\rho_0\rho_1^{-1}, \rho_1\rho_2^{-1}, \ldots, \rho_{p-1}\rho_p^{-1}). \end{aligned} \tag{1.2}$$

Dupont's [8] de Rham complex of compatible forms $\{\Omega^\bullet(|\triangle_{\mathbf{G}} M|), d\}$ on the geometric realization $|\triangle_{\mathbf{G}} M| = \coprod_{p=0}^{\infty} \Delta^p \times \triangle_{\mathbf{G}} M[p]$ provides an alternative way of computing $H^\bullet_{\mathbf{G}}(M, \mathbb{R})$. By definition, such a form consists of sequences $\omega = \{\omega_p\}_{p\geq 0}$, with $\omega_p \in \Omega^\bullet(\Delta^p \times \triangle_{\mathbf{G}} M[p])$, such that for all morphisms $\mu \in \Delta(p, q)$ in the simplicial category,

$$(\mu_\bullet \times \mathrm{Id})^* \omega_q \;=\; (\mathrm{Id} \times \mu^\bullet)^* \omega_p \;\in \Omega^\bullet\left(\Delta^p \times \triangle_{\mathbf{G}} M[q]\right); \tag{1.3}$$

here $\Delta^p = \{\mathbf{t} = (t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \;\mid\; t_i \geq 0, \quad t_0 + \ldots + t_p = 1\}$, $\mu_\bullet : \Delta^p \to \Delta^q$, resp. $\mu^\bullet : \triangle_{\mathbf{G}} M[q] \to \triangle_{\mathbf{G}} M[p]$, stands for the induced cosimplicial, resp. simplicial, map, and $\Omega^k(\Delta^p \times \triangle_{\mathbf{G}} M[q]$ denotes the k-forms on $\Delta^p \times \triangle_{\mathbf{G}} M[q]$ which are extendable to smooth forms on $V^p \times \triangle_{\mathbf{G}} M[q]$, where $V^p = \{\mathbf{t} = (t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \;\mid\; t_0 + \ldots + t_p = 1\}$. As in the case of the previous complex, there is a homogeneous description of the simplicial de Rham complex, $\{\Omega^\bullet(|\bar{\triangle}_{\mathbf{G}} M|), d\}$, consisting of the $\mathbf{G}$-invariant compatible forms on the geometric realization $|\bar{\triangle}_{\mathbf{G}} M|$. The simplicial manifold $\bar{\triangle}_{\mathbf{G}} M$ is defined as follows:

$$\bar{\triangle}_{\mathbf{G}} M = \{\bar{\triangle}_{\mathbf{G}} M[p] := \mathbf{G}^{p+1} \times M\}_{p\geq 0}\,,$$

with face maps $\bar{\partial}_i : \bar{\triangle}_{\mathbf{G}} M[p] \to \bar{\triangle}_{\mathbf{G}} M[p-1], \quad 1 \leq i \leq p, \quad$ given by

$$\bar{\partial}_i(\rho_0, \ldots, \rho_p,\, x) \;=\; (\rho_0, \ldots, \check{\rho}_i, \ldots, \rho_p), \quad 0 \leq i \leq p,$$

and degeneracies

$$\bar{\sigma}_i(\rho_0, \ldots, \rho_p,\, x) = (\rho_0, \ldots, \rho_i, \rho_i, \ldots, \rho_p, x)\,, \quad 0 \leq i \leq p.$$

The compatible forms $\omega = \{\omega_p\}_{p\geq 0} \in \Omega^\bullet(|\bar{\triangle}_{\mathbf{G}} M|$ satisfy the invariance condition

$$(\rho^{-1})^* \omega(\rho_0\rho, \ldots, \rho_p\rho) = \omega(\rho_0, \ldots, \rho_p), \quad \forall\, \rho, \rho_i \in \mathbf{G}. \tag{1.4}$$

By [8, Thm 2.3], the operation of integration along the fibers

$$\oint_{\Delta^p} : \Omega^\bullet(\Delta^p \times \triangle_{\mathbf{G}} M[p]) \to \Omega^{\bullet - p}(\triangle_{\mathbf{G}} M[p]) \tag{1.5}$$

establishes a quasi-isomorphism between the complexes $\{\Omega^\bullet(|\triangle_{\mathbf{G}} M|), d\}$ and $\{C^{\mathrm{tot}}\left(\mathbf{G}, \Omega^*(M)\right),\, \delta \pm d\}$.

Instead of the *differentiable* variants of the above complexes utilized in [15], we shall employ here their *regular* versions, defined as follows.

A cochain $\omega \in \bar{C}^p\left(\mathbf{G}, \Omega^q(M)\right)$ will be called *regular differentiable* if for any local chart $U \subset M$ with coordinates $(x^1, \ldots, x^n)$,

$$\omega(\rho_0, \ldots, \rho_p, x) = \sum P_I\left(x, j^k_x(\rho_0), \ldots, j^k_x(\rho_p)\right) dx^I, \tag{1.6}$$

with the functions P_I depending polynomially of a finite number of jet components of ρ_a, $1 \le a \le p$ and of $\left(\det \rho_a'(x)\right)^{-1}$, where $\rho_a'(x)$ denotes the Jacobian matrix $\left(\partial_i \rho_a^j(x)\right)$. As usual, $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_q}$, with $I = (i_1 < \ldots < i_q)$ running through the set of strictly increasing q-indices. The cohomology of the total complex $\{\bar{C}_{\mathrm{rd}}^{\mathrm{tot}}(\mathbf{G}, \Omega^*(M)), \delta + d\}$ thus obtained will be denoted $H^\bullet_{\mathrm{rd},\mathbf{G}}(M, \mathbb{R})$.

Similarly, the *regular differentiable* simplicial de Rham complex is defined as the subcomplex $\{\Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|), d\}$ of $\{\Omega^\bullet(|\bar{\triangle}_{\mathbf{G}} M|), d\}$ consisting of the $\mathbf{G}$-invariant compatible forms $\{\omega_p\}_{p\ge 0}$ whose components satisfy the analogous condition:

$$\omega_p(\mathbf{t}; \rho_0, \ldots, \rho_p, x) = \sum P_{I,J}\left(\mathbf{t}; x, j_x^k(\rho_0), \ldots, j_x^k(\rho_p)\right) dt^I \wedge dx^J, \tag{1.7}$$

with $P_{I,J}$ of the same form as in (1.6). We denote by $H^\bullet_{\mathrm{rd}}(|\triangle_{\mathbf{G}} M|, \mathbb{R})$ the cohomology of the complex $\{\Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|), d\}$.

Theorem 1.1. *The chain map* $\oint_{\Delta^\bullet} : \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|) \to \bar{C}^\bullet_{\mathrm{rd}}(\mathbf{G}, \Omega^*(M))$ *induces an isomorphism* $H^\bullet_{\mathrm{rd}}(|\triangle_{\mathbf{G}} M|, \mathbb{R}) \cong H^\bullet_{\mathrm{rd},\mathbf{G}}(M, \mathbb{R})$.

Proof: The operation of integration along the fibers obviously maps $\Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|)$ to $\bar{C}^\bullet_{\mathrm{rd}}(\mathbf{G}, \Omega^*(M))$. The justification of the parallel result in [8, Theorem 2.3] applies here too, since the natural chain maps in both directions and the chain homotopies relating them preserve the regular differentiable subcomplexes. ∎

1.2 Van Est-Haefliger isomorphism relative to GL_n

For $k \in \mathbb{N} \cup \{\infty\}$ we let $F^k M$ denote the frame bundle of order k, formed of k-jets $j_0^k(\phi)$ at 0 of local diffeomorphisms ϕ from a neighborhood of $0 \in \mathbb{R}^n$ to a neighborhood of $\phi(0) \in M$. In particular $F^1 M = FM$ is the usual principal frame bundle over M with structure group $\mathcal{G}^1 = \mathrm{GL}_n$. Each $F^k M$ is a principal bundle over M with structure group $\mathcal{G}^k$ formed of k-jets at 0 of local diffeomorphisms of $\mathbb{R}^n$ preserving 0. The group $\mathbf{G} = \mathrm{Diff}(M)$ operates naturally on the left on $F^k M$ by left translations.

Let $\mathfrak{a}_n$ be the Lie algebra of formal vector fields on $\mathbb{R}^n$ and denote by $C^*(\mathfrak{a}_n)$ its Gelfand-Fuks cohomology complex [9]. Each $\omega \in C^m(\mathfrak{a}_n)$ gives rise to a $\mathbf{G}$-invariant form $\tilde{\omega} \in \Omega^m(F^\infty M)$, and the assignment $\omega \in C^\bullet(\mathfrak{a}_n) \mapsto \tilde{\omega} \in \Omega^\bullet(F^\infty M)^{\mathbf{G}}$ is a DGA-isomorphism, by means of which we shall identify the two DG-algebras.

After fixing a torsion-free affine connection ∇ on M, we define a cross-section $\sigma_\nabla : FM \to F^\infty M$ of the natural projection $\pi_1 : F^\infty M \to FM$ by the formula

$$\sigma_\nabla(u) = j_0^\infty(\exp_x^\nabla \circ u), \qquad u \in F_x M. \tag{1.8}$$

Clearly, σ_∇ is GL_n-equivariant and Diff-equivariant:

$$\sigma_{\nabla^\phi} = \phi^{-1} \circ \sigma_\nabla \circ \phi, \qquad \forall\, \phi \in \mathbf{G}; \tag{1.9}$$

here $\nabla^\phi = \phi_*^{-1} \circ \nabla \circ \phi_*$, with connection form $\phi^* \omega$.

For each $p \in \mathbb{N}$, we define $\sigma_p : \Delta^p \times \bar{\triangle}_{\mathbf{G}} FM[p] \to F^\infty M$ by

$$\begin{aligned} \sigma_p(\mathbf{t}; \rho_0, \dots, \rho_p, u) &= \sigma_{\nabla(\mathbf{t};\rho_0,\dots,\rho_p)}(u), \\ \text{where} \quad \nabla(\mathbf{t}; \rho_0, \dots, \rho_p) &= \sum_0^p t_i \nabla^{\rho_i}, \qquad \mathbf{t} \in \Delta^p. \end{aligned} \tag{1.10}$$

The collection $\hat{\sigma} = \{\sigma_p\}_{p \geq 0}$ descends to the geometric realization of $\bar{\triangle}_{\mathbf{G}} FM$, giving a map $\hat{\sigma} : |\bar{\triangle}_{\mathbf{G}} FM| \to F^\infty M$. By construction, $\hat{\sigma}$ is GL_n-equivariant and therefore it also induces a map $\hat{\sigma}^{\mathrm{GL}_n} : |\bar{\triangle}_{\mathbf{G}} M| \to F^\infty M / \mathrm{GL}_n$.

Lemma 1.2. *If $\omega \in C^\bullet(\mathfrak{a}_n)$ then $\hat{\sigma}^*(\tilde{\omega}) \in \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} FM|$.*

Proof: First we note that, because $\tilde{\omega}$ is $\mathbf{G}$-invariant, $\hat{\sigma}^*(\tilde{\omega})$ is easily seen to be a compatible form. It remains to check that for any $\phi \in \mathbf{G}$ and any local chart U, with the notation as in (1.6), one has

$$\sigma^*_{\nabla^\phi}(\tilde{\omega})(x) = \sum P_I\left(x, j^k_x(\phi)\right) dx^I, \qquad x \in U.$$

Using normal coordinates with respect to ∇, this follows from the explicit expression for σ_{∇^ϕ} in the proof of Lemma 3.5 in [15]. ∎

In view of the above lemma, it makes sense to define $\mathcal{C}_\nabla : C^\bullet(\mathfrak{a}_n) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} FM|)$ by

$$\mathcal{C}_\nabla(\omega) = \hat{\sigma}^*(\tilde{\omega}) \in \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} FM|). \tag{1.11}$$

The map $\mathcal{C}_\nabla$ is a homomorphism of DG-algebras. which in turn induces a DGA-homomorphism at the level of GL_n-basic forms,

$$\mathcal{C}^{\mathrm{GL}_n}_\nabla : C^\bullet(\mathfrak{a}_n, \mathrm{GL}_n) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|). \tag{1.12}$$

Theorem 1.3. *The map $\mathcal{C}^{\mathrm{GL}_n}_\nabla$ is a quasi-isomorphism of DG-algebras.*

Proof: The proof follows along the same lines as that of [15, Theorem 1.2]. For any connection $\tilde{\nabla}$, one has

$$(\pi_1 \circ \sigma_{\tilde{\nabla}})(u) = j^1_0(\exp^{\tilde{\nabla}}_x \circ u) = u, \quad u \in F_x M\,.$$

After upgrading π_1 and $\hat{\sigma}$ to simplicial maps $\mathrm{Id} \times \pi_1 : |\bar{\triangle}_{\mathbf{G}} F^\infty M| \to |\bar{\triangle}_{\mathbf{G}} FM|$ and $\mathrm{Id} \times \hat{\sigma} : |\bar{\triangle}_{\mathbf{G}} FM| \to |\bar{\triangle}_{\mathbf{G}} F^\infty M|$, one obtains

$$(\mathrm{Id} \times \pi_1) \circ (\mathrm{Id} \times \hat{\sigma}) = \mathrm{Id}\,.$$

Hence $(\mathrm{Id} \times \hat{\sigma}) * : \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} F^\infty M|) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} FM|)$ is a left inverse for $(\mathrm{Id} \times \pi_1)^* : \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} FM|) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} F^\infty M|)$. Both maps are GL_n-equivariant and thus descend to maps

$$\begin{aligned} (\mathrm{Id} \times \hat{\sigma})^*_{\mathrm{GL}_n} &: \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} F^\infty M / \mathrm{GL}_n\,|) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|), \\ \text{resp.} \quad (\mathrm{Id} \times \pi_1)^*_{\mathrm{GL}_n} &: \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|) \to \Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} F^\infty M / \mathrm{GL}_n\,|). \end{aligned}$$

The typical fiber $\mathcal{G}^k/\operatorname{GL}_n$ of $F^\infty M/\operatorname{GL}_n \to M$ can be canonically identified to the pronilpotent group $\mathcal{G}_1^k$ of ∞-jets at 0 of local diffeomorphisms of $\mathbb{R}^n$ preserving 0 to order 1. As such, it is *algebraically* contractible, hence $(\operatorname{Id}\times\pi_1)^*_{\operatorname{GL}_n}$ induces an isomorphism in regular differentiable cohomology. Therefore so does its inverse $(\operatorname{Id}\times\hat\sigma)^*_{\operatorname{GL}_n}$.
On the other hand, identifying the GL_n-basic forms on $F^\infty M$ with forms on $P^\infty M = F^\infty M/\operatorname{GL}_n$, one defines a horizontal homotopy as in [12, Lemma 2.3], by the formula

$$(H\alpha)_{p-1}(\mathbf{t};\rho_0,\dots,\rho_{p-1},\, j_0^\infty(\rho)\operatorname{GL}_n) = \\ \pi_{\operatorname{GL}_n}[k\in\operatorname{GL}_n \mapsto \alpha_p(\mathbf{t};(\rho k)^{-1},\rho_0,\dots,\rho_{p-1},\, j_0^\infty(\rho)\operatorname{GL}_n)],$$

where $\pi_{\operatorname{GL}_n}$ stands for the projection on the GL_n-invariant (constant) part with respect to the decomposition into isotypical components of the right regular representation of GL_n on its ring of regular functions tensored by the fiber.
Therefore the natural inclusion of $C^\bullet(\mathfrak{a}_n,\operatorname{GL}_n)\equiv\Omega^\bullet(F^\infty M/\operatorname{GL}_n)^{\mathbf{G}}$ into $\Omega^\bullet_{\mathrm{rd}}(|\bar\triangle_{\mathbf{G}}F^\infty M/\operatorname{GL}_n|)$ is also quasi-isomorphism. To complete the proof it remains to observe that when restricted to GL_n-basic forms the map $(\operatorname{Id}\times\triangle\sigma)^*$ coincides with $\mathcal{C}_\nabla^{\operatorname{GL}_n}$. ∎

Combining the Theorems 1.1 and 1.3 one obtains the 'relative to GL_n' version of the van Est-Haefliger isomorphism [11, §IV.4].

Theorem 1.4. *The map*

$$\mathcal{D}_\nabla^{\operatorname{GL}_n} = \oint_{\triangle^\bullet}\mathcal{C}_\nabla^{\operatorname{GL}_n} : C^\bullet(\mathfrak{a}_n,\operatorname{GL}_n)\to\bar C_{\mathrm{rd}}^{\mathrm{tot}\,\bullet}(\mathbf{G},\Omega^*(M))$$

is a quasi-isomorphism of complexes.

1.3 Equivariant Chern cocycles

Let $W(\mathfrak{gl}_n)=\wedge^\bullet\mathfrak{gl}_n^*\otimes S(\mathfrak{gl}_n)$ be the Weil algebra of $\mathfrak{gl}_n$ with its usual grading, and let $\hat W(\mathfrak{gl}_n) = W(\mathfrak{gl}_n)/\mathcal{I}_{2n}$ be its truncation by the ideal generated by the elements of $S(\mathfrak{gl}_n)$ of degree $>2n$. The universal connection and curvature forms $\vartheta=(\vartheta^i_j)$ and $R=(R^i_j)$, defined as in [1, §2], generate a DG-subalgebra $CW^\bullet(\mathfrak{a}_n)$ of $C^\bullet(\mathfrak{a}_n)$, which can be identified with $\hat W(\mathfrak{gl}_n)$. Let $CW^\bullet(\mathfrak{a}_n,\operatorname{GL}_n)$, resp. $\hat W(\mathfrak{gl}_n,\operatorname{GL}_n)$, denote their subalgebras consisting of GL_n-basic elements, also identified as above. It follows from Gelfand-Fuks [9] (cf. also [10]) that the inclusion of the latter into $C^\bullet(\mathfrak{a}_n,\operatorname{GL}_n)$ is a quasi-isomorphism. Thus, by Theorems 1.3 and 1.4,

$$\mathcal{D}_\nabla^{\operatorname{GL}_n} : \hat W(\mathfrak{gl}_n,\operatorname{GL}_n)\equiv CW^\bullet(\mathfrak{a}_n,\operatorname{GL}_n)\to\Omega^\bullet_{\mathrm{rd}}(|\bar\triangle_{\mathbf{G}}M|) \tag{1.13}$$

is a DGA quasi-isomorphism and

$$\mathcal{D}_\nabla^{\operatorname{GL}_n} : \hat W(\mathfrak{gl}_n,\operatorname{GL}_n)\equiv CW^\bullet(\mathfrak{a}_n,\operatorname{GL}_n)\to\bar C_{\mathrm{rd}}^{\mathrm{tot}\,\bullet}(\mathbf{G},\Omega^*(M)) \tag{1.14}$$

is a quasi-isomorphism of complexes.

The cohomology of $\hat{W}(\mathfrak{gl}_n, \mathrm{GL}_n)$ is well-known to be isomorphic to the truncated polynomial ring generated by the universal Chern classes $P_{2n}[c_1, \dots, c_n]$, with $c_1, \dots, c_n$ given by the invariant polynomials

$$c_q(A) = \sum_{1 \le i_1 < \dots < i_q \le n} \sum_{\mu \in S_q} (-1)^\mu A^{i_1}_{\mu(i_1)} \cdots A^{i_q}_{\mu(i_q)}, \quad A \in \mathfrak{gl}_n. \tag{1.15}$$

The above quasi-isomorphisms allow to transport the standard basis of $P_{2n}[c_1, \dots, c_n]$ to a basis of $H^\bullet_{\mathrm{rd}}(\mathbf{G}, \Omega^*(M))$, as follows.
Let $\omega_\nabla = (\omega^i_j)$, resp. $\Omega_\nabla = (\Omega^i_j)$, denote the matrix-valued connection form, resp. curvature form, corresponding to ∇. One has the naturality relation (cf. [7, Lemma 18]),

$$\sigma^*_\nabla(\widetilde{\vartheta}^i_j) = \omega^i_j \qquad \text{hence} \qquad \sigma^*_\nabla(\widetilde{R}^i_j) = \Omega^i_j. \tag{1.16}$$

In homogeneous group coordinates (cf. (1.2)), the simplicial connection form-valued matrix $\hat{\omega}_\nabla = \{\hat{\omega}_p\}_{p \in \mathbb{N}}$ associated to ∇ has components

$$\hat{\omega}_p(\mathbf{t}; \rho_0, \dots, \rho_p) := \sum_{i=0}^p t_i \rho^*_i(\omega_\nabla), \tag{1.17}$$

and the simplicial curvature form-valued matrix $\hat{\Omega}_\nabla := d\hat{\omega}_\nabla + \hat{\omega}_\nabla \wedge \hat{\omega}_\nabla$ has components $\hat{\Omega}_p = \hat{\Omega}^{(1,1)}_p + \hat{\Omega}^{(0,2)}_p$, given by

$$\begin{aligned}\hat{\Omega}_p(\mathbf{t}; \rho_0, \dots, \rho_p) \;=\; &\sum_{i=0}^p dt_i \wedge \rho^*_i(\omega_\nabla) + \\ \sum_{i=0}^p t_i\big(\rho^*_i(\Omega_\nabla) - \rho^*_i(\omega_\nabla) \wedge \rho^*_i(\omega_\nabla)\big) &+ \sum_{i,j=0}^p t_i t_j\, \rho^*_i(\omega_\nabla) \wedge \rho^*_j(\omega_\nabla).\end{aligned} \tag{1.18}$$

The forms $\hat{\omega}^i_j$ and $\hat{\Omega}^i_j$ clearly belong to the regular differentiable de Rham complex $\Omega^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} FM|)$. In addition, the Chern forms $c_k(\hat{\Omega}_\nabla)$ are GL_n-basic and therefore descend to $\Omega^{2k}_{\mathrm{rd}}(|\triangle_{\mathbf{G}} M|)$, and we denote by the same symbols the corresponding cohomology classes. In view of the DGA quasi-isomorphism (1.13) the cohomology ring $H^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|, \mathbb{C})$ is isomorphic to $P_{2n}[c_1, \dots, c_n]$. Therefore the collection of forms

$$c_J(\hat{\Omega}_\nabla) \;=\; c_{j_1}(\hat{\Omega}_\nabla) \wedge \dots \wedge c_{j_q}(\hat{\Omega}_\nabla) \in \Omega^{2|J|}_{\mathrm{rd}}(|\triangle_{\mathbf{G}} M|, \tag{1.19}$$

with $J = (j_1 \le \dots \le j_q)$ and $|J| := j_1 + \dots + j_q \le n$, represents a linear basis of $H^\bullet_{\mathrm{rd}}(|\bar{\triangle}_{\mathbf{G}} M|, \mathbb{C})$. Applying now the quasi-isomorphism (1.14) (which is linear, but not multiplicative) one obtains representative cocycles for a linear basis of $H^\bullet_{\mathrm{rd},\mathbf{G}}(M, \mathbb{C})$, namely

$$C_J(\hat{\Omega}_\nabla) \;:=\; \oint_{\Delta^\bullet} c_J(\hat{\Omega}_\nabla), \quad J = (j_1 \le \dots \le j_q),\ |J| \le n\}. \tag{1.20}$$

2 Hopf cyclic universal Chern classes

2.1 Hopf algebra $\mathcal{H}_n$ and its Hopf cyclic complex

The Hopf algebra $\mathcal{H}_n$ arises quite naturally as the symmetry structure of the convolution algebra $C_c^\infty(\bar{\Gamma}_n)$ of the étale groupoid $\bar{\Gamma}_n$ of germs of local diffeomorphisms of $\mathbb{R}^n$ acting by prolongation on the frame bundle $F\mathbb{R}^n$, identified with the affine group $G = \mathbb{R}^n \ltimes \mathrm{GL}_n$. Equivalently, it acts naturally on the crossed product algebras $\mathcal{A}_\Gamma = C_c^\infty(F\mathbb{R}^n) \rtimes \Gamma$, with Γ a discrete subgroup of $\mathbf{G} = \mathrm{Diff}\,\mathbb{R}^n$. We briefly review below its operational construction, and refer the reader to [15] for a more detailed account.

The primary generators of $\mathcal{H}_n$ are the (horizontal, resp. vertical) left-invariant vector fields $\{X_k, Y_i^j \mid i,j,k = 1,\ldots,n\}$, that form the standard basis of the Lie algebra $\mathfrak{g} = \mathbb{R}^n \ltimes \mathfrak{gl}_n$ of G. The vector fields $Z \in \mathfrak{g}$ are made to act on the algebra $\mathcal{A} := C_c^\infty(F\mathbb{R}^n) \rtimes \mathbf{G}$ by

$$Z(f\,U_\varphi) \,=\, Z(f)\,U_\varphi, \qquad f\,U_\varphi^* \in \mathcal{A}\,,$$

the resulting linear operators on $\mathcal{A}$ satisfy generalized Leibnitz rules, which in the Sweedler notation take the form

$$Z(a\,b) \,=\, Z_{(1)}(a)\,Z_{(2)}(b), \quad a,b \in \mathcal{A}.$$

In particular,

$$X_k(a\,b) \,=\, X_k(a)\,b \,+\, a\,X_k(b) \,+\, \delta_{jk}^i(a)\,Y_i^j(b)\,,$$

where

$$\begin{aligned} &\delta_{jk}^i(f\,U_{\varphi^{-1}}) \,=\, \gamma_{jk}^i(\phi)\,f\,U_{\varphi^{-1}}, \quad \text{with} \\ &\gamma_{jk}^i(\phi)(x,\mathbf{y}) \,=\, \left(\mathbf{y}^{-1}\cdot\phi'(x)^{-1}\cdot\partial_\mu\phi'(x)\cdot\mathbf{y}\right)_j^i\,\mathbf{y}_k^\mu\,. \end{aligned} \tag{2.1}$$

The operators δ_{jk}^i are derivations, but their successors $\delta_{jk\,\ell_1\ldots\ell_r}^i = [X_{\ell_r},\ldots[X_{\ell_1},\delta_{jk}^i]\ldots]$,

$$\begin{aligned} &\delta_{jk\,\ell_1\ldots\ell_r}^i\,(f\,U_{\varphi^{-1}}) = \gamma_{jk\,\ell_1\ldots\ell_r}^i(\phi)\,f\,U_{\varphi^{-1}} \qquad \text{where} \\ &\gamma_{jk\,\ell_1\ldots\ell_r}^i(\phi) = X_{\ell_r}\cdots X_{\ell_1}\left(\gamma_{jk}^i(\phi)\right), \quad \phi \in \mathbf{G}, \end{aligned} \tag{2.2}$$

obey progressively more elaborated Leibnitz rules. The subspace $\mathfrak{h}_n$ of linear operators on $\mathcal{A}$ generated by the operators X_k, Y_j^i, and $\delta_{jk\,\ell_1\ldots\ell_r}^i$ forms a Lie algebra $\mathfrak{h}_n$.

By definition, $\mathcal{H}_n$ is the algebra of linear operators on $\mathcal{A}$ generated by $\mathfrak{h}_n$ and the scalars. For $n > 1$ the operators $\delta_{jk\,\ell_1\ldots\ell_r}^i$ are not all distinct. They satisfy the "structure identities"

$$\delta_{j\ell\,k}^i \,-\, \delta_{jk\,\ell}^i \,=\, \delta_{jk}^s\,\delta_{s\ell}^i \,-\, \delta_{j\ell}^s\,\delta_{sk}^i\,,$$

reflecting the flatness of the standard connection. The algebra $\mathcal{H}_n$ is isomorphic to the quotient $\mathfrak{A}(\mathfrak{h}_n)/\mathcal{I}$ of the universal enveloping algebra $\mathfrak{A}(\mathfrak{h}_n)$ by the ideal $\mathcal{I}$ generated by the above identities. It has a distinguished character $\delta : \mathcal{H}_n \to \mathbb{C}$, which extends the modular character of $\mathfrak{gl}_n(\mathbb{R})$, and is induced from the character of $\mathfrak{h}_n$ defined by

$$\delta(Y_i^j) = \delta_i^j, \quad \delta(X_k) = 0, \quad \delta(\delta_{jk\,\ell_1\ldots\ell_r}^i) = 0.$$

The coproduct of $\mathcal{H}_n$ stems from the interaction of $\mathcal{H}_n$ with the product of $\mathcal{A}$. More precisely, any $h \in \mathcal{H}_n$ satisfy an identity of the form

$$h(ab) \;=\; \sum_{(h)} h_{(1)}(a)\, h_{(2)}(b)\,, \quad h_{(1)},\, h_{(2)} \in \mathcal{H}_n, \quad a, b \in \mathcal{A}\,,$$

and this uniquely determines a *coproduct* $\Delta : \mathcal{H} \to \mathcal{H}_n \otimes \mathcal{H}_n$, by setting (using Sweedler's notation)

$$\Delta(h) \;=\; \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

The *counit* is $\varepsilon(h) \;=\; h(1)$, while the antipode S is uniquely determined by its very definition

$$\sum_{(h)} S(h_{(1)})h_{(2)} \;=\; \varepsilon(h)\cdot 1 \;=\; \sum_{(h)} h_{(1)}S(h_{(2)}).$$

Although the antipode S fails to be involutive, its twisted version

$$S_\delta(h) \;=\; \sum_{(h)} \delta(h_{(1)})S(h_{(2)})$$

does satisfy the property

$$S_\delta^2 \;=\; \mathrm{Id}\,. \tag{2.3}$$

The algebra $\mathcal{A}$ has a canonical trace, namely

$$\tau\left(f\, U_\varphi\right) \;=\; \begin{cases} \displaystyle\int_{F\mathbb{R}^n} f\,\varpi\,, & \text{if} \quad \varphi = \mathrm{Id}\,, \\[2ex] 0\,, & \text{otherwise}\,; \end{cases} \tag{2.4}$$

here ϖ is the volume form determined by the dual to the canonical basis of $\mathfrak{g}$. This trace satisfies

$$\tau(h(a)) \;=\; \delta(h)\,\tau(a), \qquad h \in \mathcal{H}_n,\ a \in \mathcal{A}\,. \tag{2.5}$$

The standard Hopf cyclic model for $\mathcal{H}_n$ is imported from the standard cyclic model of the algebra $\mathcal{A}$, by means of the characteristic map

$$\begin{aligned} &h^1 \otimes \ldots \otimes h^q \in \mathcal{H}_n^{\otimes^q} \longmapsto \chi_\tau(h^1 \otimes \ldots \otimes h^q) \in C^q(\mathcal{A})\,, \\ &\chi_\tau(h^1 \otimes \ldots \otimes h^q)(a^0, \ldots, a^q) = \tau(a^0 h^1(a^1)\ldots h^q(a^q)), \quad a^j \in \mathcal{A}, \end{aligned} \tag{2.6}$$

It gives rise to a cyclic structure [3] on $\{C^q(\mathcal{H}_n;\delta) := \mathcal{H}_n^{\otimes^q}\}_{q\geq 0}$, with faces, degeneracies and cyclic operator given by

$$\begin{aligned} \delta_0(h^1 \otimes \ldots \otimes h^{q-1}) &= 1 \otimes h^1 \otimes \ldots \otimes h^{q-1}, \\ \delta_j(h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{q-1}, \quad 1 \leq j \leq q-1, \\ \delta_n(h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes h^{q-1} \otimes 1; \\ \sigma_i(h^1 \otimes \ldots \otimes h^{q+1}) &= h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{q+1}, \quad 0 \leq i \leq q\,; \\ \tau_q(h^1 \otimes \ldots \otimes h^q) &= S_\delta(h^1)\cdot(h^2 \otimes \ldots \otimes h^q \otimes 1)\,. \end{aligned}$$

The identity $\tau_q^{q+1} = \mathrm{Id}$ is satisfied precisely because of the involutive property (2.3), to which is actually equivalent.
The periodic Hopf cyclic cohomology $HP^\bullet(\mathcal{H}_n; \mathbb{C}_\delta)$ of $\mathcal{H}_n$ with coefficients in the modular pair $(\delta, 1)$ is, by definition (cf. [5, 6]), the $\mathbb{Z}_2$-graded cohomology of the total complex $CC^{\mathrm{tot}}(\mathcal{H}_n; \mathbb{C}_\delta)$ associated to the bicomplex $\{CC^{*,*}(\mathcal{H}_n; \mathbb{C}_\delta),\, b,\, B\}$, where

$$b = \sum_{k=0}^{q+1} (-1)^k \delta_k, \qquad B = (\sum_{k=0}^{q} (-1)^{q\,k} \tau_q^k) \sigma_{q-1} \tau_q\,.$$

The periodic Hopf cyclic cohomology of $\mathcal{H}_n$ relative to GL_n, denoted $HP^\bullet(\mathcal{H}_n, \mathrm{GL}_n; \mathbb{C}_\delta)$, is the cohomology of the cyclic complex defined as follows. One considers the quotient $\mathcal{Q}_n := \mathcal{H}_n \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathbb{C} \equiv \mathcal{H}_n / \mathcal{H}_n \mathcal{U}^+(\mathfrak{gl}_n)$, which is an $\mathcal{H}_n$-module coalgebra with respect to the coproduct and counit inherited from $\mathcal{H}_n$. One then forms the cochain complex

$$C^q(\mathcal{H}_n, \mathrm{GL}_n; \mathbb{C}_\delta) := \mathbb{C}_\delta \otimes_{\mathcal{U}(\mathfrak{gl}_n)} \mathcal{Q}_n^{\otimes q} \equiv \left(\mathcal{Q}_n^{\otimes q}\right)^{\mathrm{GL}_n}, \qquad q \geq 0,$$

endowed with the cyclic structure given by restricting to GL_n-invariants the operators

$$\begin{aligned}
\delta_0(c^1 \otimes \ldots \otimes c^{q-1}) &= \dot{1} \otimes c^1 \otimes \ldots \otimes \ldots \otimes c^{q-1}, \\
\delta_i(c^1 \otimes \ldots \otimes c^{q-1}) &= c^1 \otimes \ldots \otimes \Delta c^i \otimes \ldots \otimes c^{q-1}, \quad 1 \leq i \leq q-1; \\
\delta_n(c^1 \otimes \ldots \otimes c^{q-1}) &= c^1 \otimes \ldots \otimes c^{q-1} \otimes \dot{1}\,; \\
\sigma_i(c^1 \otimes \ldots \otimes c^{q+1}) &= c^1 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{q+1}, \quad 0 \leq i \leq q\,; \\
\tau_q(\dot{h}^1 \otimes c^2 \otimes \ldots \otimes c^q) &= S_\delta(h^1) \cdot (c^2 \otimes \ldots \otimes c^q \otimes \dot{1}).
\end{aligned}$$

The corresponding characteristic map lands in the cyclic cohomology of the crossed product algebra $\mathcal{A}_{\mathrm{base}} = C_c^\infty(\mathbb{R}^n) \rtimes \mathbf{G}$, where $\mathbb{R}^n$ is identified with $G/\mathrm{GL}_n(\mathbb{R})$. It is given at the chain level by the map $c \in \left(\mathcal{Q}_n^{\otimes q}\right)^{\mathrm{GL}_n} \mapsto \chi_{\mathrm{base}}(c) \in C^q(\mathcal{A}_{\mathrm{base}})$ defined as follows:

$$\chi_{\mathrm{base}}(\dot{h}^1 \otimes \ldots \otimes \dot{h}^q)(a^0, \ldots, a^q) = \tau_{\mathrm{base}}(\tilde{a}^0 h^1(\tilde{a}^1) \ldots h^q(\tilde{a}^q) \mid_{\mathbf{y}=\mathbf{1}}), \tag{2.7}$$

where τ_{base} is the trace on $\mathcal{A}_{\mathrm{base}}$ associated to the relatively invariant measure on $G/\mathrm{GL}_n(\mathbb{R})$ (which coincides with the Lebesgue measure on $\mathbb{R}^n$), $\dot{h}$ stands for the class in $\mathcal{Q}_n$ of $h \in \mathcal{H}_n$, and for a monomial $a = f U_\phi \in \mathcal{A}_{\mathrm{base}}$ we let $\tilde{a} := \tilde{f} U_{\tilde{\phi}} \in \mathcal{A}$, with $\tilde{f} \in C^\infty(F\mathbb{R}^n)$ and $\tilde{\phi}$ denoting the natural lifts of $f \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in \mathbf{G}$ to the frame bundle $F\mathbb{R}^n$. The definition makes sense, as it can be checked that the element $\tilde{a}^0 h^1(\tilde{a}^1) \ldots h^q(\tilde{a}^q) \in \mathcal{A}$ is independent of the representatives h^i of the classes $\dot{h}^i$, and descends to $\mathcal{A}_{\mathrm{base}}$ by evaluation at $\mathbf{y} = \mathbf{1}$. Moreover, the chain map χ_{base} thus obtained is injective.

2.2 From equivariant to Hopf cyclic cohomology

We recall the definition of the map Φ of Connes [4, III.2.δ], specialized to the present context. Consider the DG-algebra, $\mathcal{B}_{\mathbf{G}}(G) = \Omega_c^*(G) \otimes \wedge \mathbb{C}[\mathbf{G}']$, where $\mathbf{G}' = \mathbf{G} \setminus \{e\}$, with the differential $d \otimes \mathrm{Id}$. One labels the generators of $\mathbb{C}[\mathbf{G}']$ as γ_ϕ, $\phi \in \mathbf{G}$, with $\gamma_e = 0$, and one forms the crossed product $\mathcal{C}_{\mathbf{G}}(G) = \mathcal{B}_{\mathbf{G}}(G) \rtimes \mathbf{G}$, with the commutation rules

$$\begin{aligned}
U_\phi^*\, \omega\, U_\phi &= \phi^*\, \omega, && \omega \in \Omega_c^*(G), \\
U_{\phi_1}^*\, \gamma_{\phi_2}\, U_{\phi_1} &= \gamma_{\phi_2 \circ \phi_1} - \gamma_{\phi_1}, && \phi_1, \phi_2 \in \mathbf{G}\,.
\end{aligned}$$

$\mathcal{C}_{\mathbf{G}}(G)$ is also a DG-algebra, equipped with the differential

$$\mathbf{d}(b\,U_\phi^*) = db\,U_\phi^* - (-1)^{\partial b}\, b\,\gamma_\phi\, U_\phi^*, \qquad b \in \mathcal{B}_{\mathbf{G}}(G), \quad \phi \in \mathbf{G}, \tag{2.8}$$

A cochain $\lambda \in \bar{C}^q(\mathbf{G}, \Omega^p(G))$ determines a linear form $\widetilde{\lambda}$ on $\mathcal{C}_{\mathbf{G}}(G)$ as follows:

$$\begin{aligned} &\widetilde{\lambda}(b\,U_\phi^*) = 0 \qquad \text{for} \quad \phi \neq \mathbf{1}; \\ &\text{if} \quad \phi = \mathbf{1} \quad \text{and} \quad b = \omega \otimes \gamma_{\rho_1} \dots \gamma_{\rho_q} \qquad \text{then} \\ &\widetilde{\lambda}(\omega \otimes \gamma_{\rho_1} \dots \gamma_{\rho_q}) = \int_G \lambda(1, \rho_1, \dots, \rho_q) \wedge \omega. \end{aligned} \tag{2.9}$$

The map Φ from $\bar{C}^\bullet(\mathbf{G}, \Omega^\bullet(G))$ to the (b, B)-complex of the algebra $\mathcal{A} = C_c^\infty(G) \rtimes \mathbf{G}$ is now defined for $\lambda \in \bar{C}^q(\mathbf{G}, \Omega^p(G))$ by

$$\begin{aligned} \Phi(\lambda)(a^0, \dots, a^m) &= \frac{p!}{(m+1)!} \sum_{j=0}^m (-1)^{j(m-j)} \widetilde{\lambda}(\mathbf{d}a^{j+1} \cdots \mathbf{d}a^m \; a^0 \; \mathbf{d}a^1 \cdots \mathbf{d}a^j) \\ \text{where} \quad m &= \dim G - p + q, \qquad a^0, \dots, a^m \in \mathcal{A}. \end{aligned} \tag{2.10}$$

By [4, III.2.δ, Thm. 14], Φ is a chain map to the total (b, B)-complex of the algebra $\mathcal{A}$.
The relative version Φ^{GL_n} of the map Φ is obtained by first replacing $\Omega_c^*(G)$ with the GL_n-basic forms $\Omega_{c,\mathrm{basic}}^*(G)$ which are compact modulo GL_n, and so can be identified to $\Omega_c^*(\mathbb{R}^n)$, and then replacing in the definition (2.9) the integration over $G = \mathbb{R}^n \ltimes \mathrm{GL}_n$ by integration over the base $\mathbb{R}^n$. One obtains this way the induced chain map

$$\Phi^{\mathrm{GL}_n} : \bar{C}^\bullet(\mathbf{G}, \Omega^\bullet(\mathbb{R}^n)) \to C^\bullet(\mathcal{A}_{\mathrm{base}}). \tag{2.11}$$

Assume now that $\lambda \in \bar{C}^q(\mathbf{G}, \Omega^p(\mathbb{R}^n))$ is of the form $\lambda = \mathcal{D}_\nabla(\omega)$ with $\omega \in C(\mathfrak{a}_n, \mathrm{GL}_n)$, where ∇ stands for the standard flat connection. Using [15, Lemma 3.5] which identifies the map $\mathcal{D}_\nabla$ with the map $\mathcal{D}$ employed in [5], one shows as in [5, pp. 233-234]) that $\Phi^{\mathrm{GL}_n}(\lambda)$ has the expression

$$\Phi^{\mathrm{GL}_n}(\lambda)(a^0, \dots, a^q) \;=\; \sum_\alpha \tau_{\mathrm{base}}(\tilde{a}^0 h_\alpha^1(\tilde{a}^1) \dots h_\alpha^q(\tilde{a}^q)), \tag{2.12}$$

with $\sum_\alpha \dot{h}_\alpha^1 \otimes \dots \otimes \dot{h}_\alpha^q \in (\mathcal{Q}_n^{\otimes q})^{\mathrm{GL}_n}$ uniquely determined by λ. This means that $\Phi^{\mathrm{GL}_n}(\lambda)$ lands in the (b, B)-complex which defines the Hopf cyclic cohomology of $\mathcal{H}_n$ relative to GL_n. Thus, by restricting Φ^{GL_n} to the subcomplex

$$\bar{C}_{\mathcal{D}}^{\mathrm{tot}}(\mathbf{G}, \Omega^*(\mathbb{R}^n)) := \mathcal{D}_\nabla\big(C(\mathfrak{a}_n, \mathrm{GL}_n)\big) \subset \bar{C}_{\mathrm{rd}}^{\mathrm{tot}}(\mathbf{G}, \Omega^*(\mathbb{R}^n)), \tag{2.13}$$

one obtains a chain map

$$\Phi_{\mathrm{rd}}^{\mathrm{GL}_n} : \bar{C}_{\mathcal{D}}^{\mathrm{tot}}(\mathbf{G}, \Omega^*(\mathbb{R}^n)) \to CC^{\mathrm{tot}}(\mathcal{H}_n, \mathrm{GL}_n; \mathbb{C}_\delta). \tag{2.14}$$

By [5, Theorem 11], or more precisely its relative to GL_n version, the composition $\Phi_{\mathrm{rd}}^{\mathrm{GL}_n} \circ \mathcal{D}_\nabla^{\mathrm{GL}_n}$ is a quasi-isomorphism. Since, by construction (cf. §1.3) the cocycles $C_J(\hat{\Omega}_\nabla)$ are images via the map $\mathcal{D}_\nabla^{\mathrm{GL}_n}$ of representatives for a basis of $H^*(\mathfrak{a}_n, \mathrm{GL}_n)$, we can finally conclude that:

Theorem 2.1. *The collection of cocycles*

$$\{\Phi_{\mathrm{rd}}^{\mathrm{GL}_n}\big(C_J(\hat{\Omega}_\nabla)\big)\,; \qquad J=(j_1\leq \ldots \leq j_q), \quad |J|\leq n\}$$

represent a basis of $HP^\bullet(\mathcal{H}_n, \mathrm{GL}_n; \mathbb{C}_\delta)$.

To get more insight into the makeup of these cocycles, we recall that ∇ is the flat connection on $G\equiv F\mathbb{R}^n \to \mathbb{R}^n$, so its connection form is $\omega_\nabla = \big(\omega^i_j\big)$ with $\omega^i_j := (\mathbf{y}^{-1})^i_\mu\, d\mathbf{y}^\mu_j = \big(\mathbf{y}^{-1}\, d\mathbf{y}\big)^i_j$, $i,j=1,\ldots,n$. With the usual summation convention, for any $\phi \in \mathbf{G}$,

$$\phi^*(\omega^i_j) \,=\, \omega^i_j + \gamma^i_{jk}(\phi)\,\theta^k \,=\, \omega^i_j + \big(\mathbf{y}^{-1}\cdot \phi'(x)^{-1}\cdot \partial_\mu \phi'(x)\cdot \mathbf{y}\big)^i_j\, dx^\mu,$$

since

$$\gamma^i_{j\,k}(\phi)(x,\mathbf{y}) = \big(\mathbf{y}^{-1}\cdot \phi'(x)^{-1}\cdot \partial_\mu \phi'(x)\cdot \mathbf{y}\big)^i_j\ \mathbf{y}^\mu_k \quad \text{and} \quad \theta^k = \big(\mathbf{y}^{-1}\big)^k_\ell\, dx^\ell.$$

Thus, denoting

$$\tilde{\Gamma}_\mu(\phi)(x,\mathbf{y}) = \mathbf{y}^{-1}\cdot \phi'(x)^{-1}\cdot \partial_\mu \phi'(x)\cdot \mathbf{y}, \tag{2.15}$$

one has $\phi^*(\omega_\nabla) \,=\, \omega_\nabla \,+\, \tilde{\Gamma}_\mu(\phi)\, dx^\mu$. Therefore, the simplicial connection is

$$\hat{\omega}_\nabla(\mathbf{t};\phi_0,\ldots,\phi_p) = \sum_{r=0}^p t_r \phi_r^*(\omega_\nabla) = \omega_\nabla + \sum_{r=0}^p t_r\, \tilde{\Gamma}_k(\phi_r)\, dx^k.$$

Since $\phi^*(\Omega_\nabla)=0$, the simplicial curvature (1.18) takes the form

$$\begin{aligned}\hat{\Omega}_\nabla(\mathbf{t};\phi_0,\ldots,\phi_p) &= \sum_{r=0}^p dt_r \wedge \phi_r^*(\omega_\nabla) - \sum_{r=0}^p t_r\, \phi_r^*(\omega_\nabla)\wedge \phi_r^*(\omega_\nabla) \\ &\quad + \sum_{r,s=0}^p t_r t_s\, \phi_r^*(\omega_\nabla)\wedge \phi_s^*(\omega_\nabla).\end{aligned}$$

Furthermore, being given by invariant polynomials, the Chern cocycles (1.19) are built out of the pull-back of the curvature form by the cross-section $x\in\mathbb{R}^n \mapsto (x,\mathbf{1}) \in \mathbb{R}^n\times \mathrm{GL}_n$. The latter is given by the matrix-valued form

$$\begin{aligned}\hat{R}(\mathbf{t};\phi_0,\ldots,\phi_p) \,&= \sum_{r=0}^p dt_r\wedge \Gamma(\phi_r) - \sum_{r=0}^p t_r\, \Gamma(\phi_r)\wedge\Gamma(\phi_r) \\ &+ \sum_{r,s=0}^p t_r t_s\, \Gamma(\phi_r)\wedge \Gamma(\phi_s), \qquad \text{where} \quad \Gamma(\phi) := (\phi')^{-1}\cdot d\phi',\end{aligned}$$

with $\phi' = \big(\partial_j \phi^i\big)$ denoting the Jacobian matrix of $\phi\in\mathbf{G}$. This ensures that the diffeomorphisms $\phi\in\mathbf{G}$ appear in all the basic cocycles (1.20) solely through the matrix-valued 1-forms $\Gamma(\phi)\in \Omega^1(\mathbb{R}^n)\otimes \mathfrak{gl}_n$. For example, the Chern cocycle $C_q(\hat{\Omega}_\nabla)$ has components

$$\begin{aligned}&C_q^{(p)}(\hat{\Omega}_\nabla)(\phi_0,\ldots,\phi_p) \,= \\ &= (-1)^p \sum_{1\leq i_1<\ldots<i_q\leq n}\ \sum_{\mu\in S_q} (-1)^\mu \int_{\Delta^p} R^{i_1}_{\mu(i_1)}\wedge\cdots\wedge R^{i_q}_{\mu(i_q)}(\mathbf{t};\phi_0,\ldots,\phi_p).\end{aligned}$$

In particular, up to a constant factor $C_q^{(q)}(\hat{\Omega}_\nabla)(\phi_0, \ldots, \phi_q)$ equals

$$\sum_{\sigma \in S_{q+1}} (-1)^\sigma \ \mathrm{Tr}\left(\Gamma(\phi_{\sigma(1)}) \wedge \cdots \wedge \Gamma(\phi_{\sigma(q)})\right),$$

where σ runs through the permutations of $\{0, 1, \ldots, q\}$.
It is thus seen that all cohomology classes in $HP^\bullet(\mathcal{H}_n, \mathrm{GL}_n; \mathbb{C}_\delta)$ can be represented by cocycles $c \in \sum_{q\geq 0} \left(\mathcal{Q}_n^{\otimes^q}\right)^{\mathrm{GL}_n}$ whose characteristic image $\chi_{\mathrm{base}}(c) \in \sum_{q\geq 0} C^q\left(C_c^\infty(\mathbb{R}^n) \rtimes \mathbf{G}\right)$ involves only 2-jets.
The above property can be stated more intrinsically, in terms of the standard Hopf cyclic complex. Let $\mathcal{F}_n^\delta$ denote the subalgebra of $\mathcal{H}_n$ generated by the multiplication operators δ_{jk}^i of (2.1) and set

$$\mathcal{X}_n := \mathcal{F}_n^\delta + \sum_{k=1}^n \mathcal{F}_n^\delta \cdot X_k;$$

it is a GL_n-invariant subspace of $\mathcal{H}_n$, and we let $\dot{\mathcal{X}}_n$ be its image in $\mathcal{Q}_n$.

Corollary 2.2. *Every cohomology class in $HP^\bullet(\mathcal{H}_n, \mathrm{GL}_n; \mathbb{C}_\delta)$ can be represented by cocycles formed of elements in $\sum_{q\geq 0} \left(\dot{\mathcal{X}}_n^{\otimes^q}\right)^{\mathrm{GL}_n}$.*

Proof: The horizontal operators appear because of the first summand in the definition (2.8) of the differential $\mathbf{d}$, which contributes to the formula (2.12) as follows: when applied to monomials $a = f\, U_\phi^* \in C_c^\infty(\mathbb{R}^n)$, it brings in the forms $df = \sum_{k=1}^n X_k(f)\, dx^k$. ∎

Explicit representatives for the Hopf cyclic Chern classes can also be given in the cohomological models of Chevalley-Eilenberg type constructed in [13, 14], by transporting the equivariant Chern classes from the Bott complex as in [15, §3], via the partial inverse of the map Θ therein defined, only this time restricted to GL_n-basic forms.

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