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Infinitesimal extensions of rank two vector bundles on submanifolds of small codimension

by
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To Vasile Brînzănescu on the occasion of his seventieth birthday anniversary

Abstract

Let X be a submanifold of dimension n of the complex projective space $\mathbb{P}^N$ $(n < N)$, and let E be a vector bundle of rank two on X . If $n \geq \frac{N+3}{2} \geq 4$ we prove a geometric criterion for the existence of an extension of E to a vector bundle on the first order infinitesimal neighborhood of X in $\mathbb{P}^N$ in terms of the splitting of the normal bundle sequence of $Y \subset X \subset \mathbb{P}^N$, where Y is the zero locus of a general section of a high twist of E. In the last section we show that the universal quotient vector bundle on the Grassmann variety $\mathbb{G}(k, m)$ of k-dimensional linear subspaces of $\mathbb{P}^m$, with $m \geq 3$ and $1 \leq k \leq m-2$ (i.e. with $\mathbb{G}(k, m)$ not a projective space), embedded in any projective space $\mathbb{P}^N$, does not extend to the first infinitesimal neighborhood of $\mathbb{G}(k, m)$ in $\mathbb{P}^N$ as a vector bundle.

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Introduction

Let X be a submanifold of dimension n of a complex projective manifold P of dimension N, with $n < N$. For every $i \geq 0$ denote by $X(i)$ the i-th infinitesimal neighborhood of X in P, i.e. the subscheme of P defined by the sheaf of ideals $\mathcal{I}_X^{i+1}$, where $\mathcal{I}_X$ is the sheaf of ideals of X in $\mathcal{O}_P$. Note that $X(0) = X$. Fix an $i \geq 0$; if E is a vector bundle of rank r on $X(i)$, a natural problem is to give criteria for the extendability of E to the next infinitesimal neighborhood $X(i+1)$ as a vector bundle. The following general fundamental result was proved by Grothendieck in 1960 (see [11, Exposé III, Proposition 7.1, Page 85]):

Theorem (Grothendieck) *Under the above hypotheses and notation, assume that*

$$H^2(X, E \otimes E^\vee \otimes \mathbf{S}^{i+1}(N_{X|P}^\vee)) = 0, \tag{1}$$

where for every $j \geq 1$, $\mathbf{S}^j(N^\vee_{X|P}) = \mathcal{I}^j_X/\mathcal{I}^{j+1}_X$ is the j-th symmetric power of the conormal bundle $N^\vee_{X|P} = \mathcal{I}_X/\mathcal{I}^2_X$ of X in P. Then E can be extended to a vector bundle $\mathcal{E}$ on $X(i+1)$. If moreover $H^1(X, E \otimes E^\vee \otimes \mathbf{S}^{i+1}(N^\vee_{X|P})) = 0$ then this extension is also unique up to isomorphism.

If in Grothendieck's theorem above X is a curve and E a vector bundle on X then the vanishing (1) is automatically fulfilled, so that E can be extended to a vector bundle $\mathcal{E}_i$ on $X(i)$ for every $i \geq 1$. Note also that the vanishing (1) is only a sufficient condition for the extendability of the vector bundle E in Grothendieck's theorem.

The aim of this paper is twofold. First we prove, in the spirit of the paper [5] of Ellingsrud, Gruson, Peskine and Strømme, a necessary and sufficient geometric criterion for extending a vector bundle E of rank two on X to a vector bundle $\mathcal{E}$ on the first infinitesimal neighborhood $X(1)$ of X in P, when P is the N-dimensional complex projective space $\mathbb{P}^N$ and X is a submanifold of small codimension in $P = \mathbb{P}^N$, but without assuming the vanishing (1) for $i = 0$. In this paper "small codimension" will mean that the inequalities $n \geq \frac{N+3}{2} \geq 4$ are satisfied. For example if $n = 4$, X is a smooth hypersurface in $\mathbb{P}^5$, and if $n = 5$, X is either a smooth hypersurface in $\mathbb{P}^6$, or a 2-codimensional submanifold in $\mathbb{P}^7$, and so on. We prove that a vector bundle E of rank two on X can be extended to a vector bundle on $X(1)$ if and only if E satisfies the condition $(\mathbf{P}^2_E)$ stated at the beginning of Section 2 (see Theorem 2.4 below for the precise formulation). This condition involves the splitting of the canonical exact sequence of normal bundles

$$0 \to N_{Y|X} = E(l)|_Y \to N_{Y|\mathbb{P}^N} \to N_{X|\mathbb{P}^N}|_Y \to 0,$$

where Y is the zero locus of a general section of $E(l)$ for $l \gg 0$. This is done by first interpreting the splitting of the above exact sequence of normal bundles (via a generalization of a key lemma of [5] given in [21]), and then by using a generalized form of the Hartshorne–Serre correspondence (Theorem 2.2 below, whose proof was written jointly with E. Arrondo). The second aim of this paper is to prove Theorem 3.1 below, which asserts that the universal quotient vector bundle of the Grassmann variety $\mathbb{G}(k, m)$ of linear subspaces of dimension k in $\mathbb{P}^m$ (with $1 \leq k \leq m-2$) never extends as a vector bundle to the first infinitesimal neighborhood of $\mathbb{G}(k, m)$ with respect to any projective embedding of $\mathbb{G}(k, m)$.

The paper is organized as follows. In Section 1 we recall some known results needed in the next sections. In Section 2 we prove Theorem 2.4 and in Section 3, Theorem 3.1.

As a motivation of this paper, let me first recall the following beautiful result:

Theorem (Griffiths-Harris [9], page 252, cf. also [14] and [5]) *Let X be a smooth projective complex surface embedded in $\mathbb{P}^n$ $(n \geq 3)$ as a complete intersection. Let Y be a smooth connected curve in X such that the canonical exact sequence of normal bundles*

$$0 \to N_{Y|X} \to N_{Y|\mathbb{P}^n} \to N_{X|\mathbb{P}^n}|_Y \to 0$$

splits. Then there is a hypersurface H of $\mathbb{P}^n$ such that $Y = X \cap H$ (scheme-theoretically).

The crucial step of the (short and very elegant) proof of this result given in [5] is to show that the normal bundle $N_{Y|X}$ of Y on X can be extended to a line bundle of the first infinitesimal $X(1)$ of X in $\mathbb{P}^n$. Instead, the proofs of [9] and [14] make use of the theory of infinitesimal variation of Hodge structures. The proof of Theorem 2.4 below (which also involves the splitting of certain canonical exact sequences of normal bundles) makes use of Grothendieck-Lefschetz theory plus a generalized form of Hartshorne-Serre correspondence (Theorem 2.2 below) in order to extend certain rank two vector bundles on a small-codimensional submanifold X of $\mathbb{P}^n$ to rank two vector bundles on the first infinitesimal neighborhood $X(1)$ of X in $\mathbb{P}^n$.

Unless otherwise stated, throughout this paper we shall use the standard terminology and notation. All the algebraic varieties or schemes considered are defined over the field $\mathbb{C}$ of complex numbers.

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1 Background material

In this section we recall some known results that will be used in the next two sections.

Proposition 1.1 (Bertini–Serre, see [7], Appendix B9) *Let E be a vector bundle of rank r on an algebraic variety X over k. Assume that V is a finite dimensional k-vector subspace of $H^0(X,E)$ whose sections generate E. Then there is a non-empty Zariski open subset V_0 of V such that $\operatorname{codim}_X Z(s) \geq \min\{r, \dim X+1\}$ for every $s \in V_0$, where $Z(s)$ denotes the zero locus of s (in particular, $Z(s)=\varnothing$ if $r > \dim X$).*

Theorem 1.2 (Kodaira–Le Potier vanishing theorem [17]) *Let E be an ample vector bundle of rank r on a smooth projective n-dimensional variety. Then $H^i(X, E^\vee) = 0$ for every $i \leq n-r$, where $E^\vee$ is the dual of E.*

Theorem 1.3 (Sommese [22]) *Let E be an ample vector bundle of rank r on a smooth projective n-dimensional variety such that $n-r \geq 2$. Let $s \in H^0(X,E)$ be a global section. Then the zero locus $Y := Z(s)$ is connected and nonempty of dimension $\geq n-r$. Assume moreover that Y is smooth and $\dim Y = n-r$. Then the canonical restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ of Picard groups is an isomorphism if $n-r \geq 3$, and injective with torsion-free cokernel if $n-r=2$.*

Theorem 1.4 (Barth–Larsen [16]) *Let X be a smooth closed subvariety of dimension n of $\mathbb{P}^N$. Then the canonical restriction map $\operatorname{Pic}(\mathbb{P}^N) \to \operatorname{Pic}(X)$ is an isomorphism if $n \geq \frac{N+2}{2}$ and is injective with torsion-free cokernel if $n = \frac{N+1}{2}$.*

Theorem 1.5 (Van de Ven [23]) *Let X be a smooth closed subvariety of dimension ≥ 1 of $\mathbb{P}^N$. Then the canonical exact sequence of tangent bundles*

$$0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_{X|\mathbb{P}^N} \to 0$$

splits if and only if X is a linear subspace of $\mathbb{P}^N$.

Theorem 1.6 ([5], [4] if $\operatorname{codim}_X Y = 1$ and [21] if $\operatorname{codim}_X Y > 1$) *Let P, X and Y be three smooth projective irreducible varieties such that $Y \subsetneq X \subsetneq P$ and $\dim Y \geq 1$. Set $r := \operatorname{codim}_X Y$. Then the canonical exact sequence of normal bundles*

$$0 \to N_{Y|X} \to N_{Y|P} \to N_{X|P}|_Y \to 0$$

splits if and only if there exists a closed subscheme Y' of the first infinitesimal neighborhood $X(1)$ of X in P such that Y' is a local complete intersection of codimension r in $X(1)$ and $Y' \cap X = Y$ (scheme theoretically in $X(1)$, i.e. $\mathcal{I}_{Y'} + \mathcal{I}_X = \mathcal{I}_Y$, where $\mathcal{I}_{Y'}$, $\mathcal{I}_X$ and $\mathcal{I}_Y$ are the ideal sheaves of Y', X and Y in $\mathcal{O}_{X(1)}$ respectively).

Note that the fact that Y' is a local complete intersection in $X(1)$ of codimension r is the essential part of the conclusion in Theorem 1.6.

2 Infinitesimal extensions of rank two vector bundles

In this section we shall prove a geometric criterion for the extendability of a vector bundle E of rank 2 on a small-codimensional submanifold X of $\mathbb{P}^N$ to a vector bundle $\mathcal{E}$ on $X(1)$ (Theorem 2.4 below). We start (more generally) with a submanifold X of $\mathbb{P}^N$ of dimension n and with a vector bundle E a rank r on X, with $1 \leq r \leq n-1$. Then consider the following condition on the triple $(\mathbb{P}^N, X, E)$:

($\mathbf{P}_E^r$) *There exists an integer $l_0 > 0$ such that for every $l \geq l_0$ there exists a section $s = s_l \in H^0(E(l))$ whose zero locus $Y := Z(s)$ is a smooth r-codimensional subvariety of X such that the following canonical exact sequence of normal bundles*

$$0 \to N_{Y|X} = E(l)|_Y \to N_{Y|\mathbb{P}^N} \to N_{X|\mathbb{P}^N}|_Y \to 0 \qquad (2)$$

splits.

Proposition 2.1 *With the above notation, let E be a vector bundle of rank r, with $1 \leq r \leq n-1$, on an n-dimensional submanifold $X \subset \mathbb{P}^N$. If there exists a vector bundle $\mathcal{E}$ on $X(1)$ which extends E then there exists an integer $l_0 > 0$ such that for every $l \geq l_0$ and for every section $s \in H^0(E(l))$ whose zero locus Y is smooth r-codimensional in X, the exact sequence (2) splits. In particular, condition ($\mathbf{P}_E^r$) above holds true.*

Proof: Consider the exact sequence

$$0 \to F \to \mathcal{E} \to \mathcal{E}|_X = E \to 0,$$

where $F := \operatorname{Ker}(\mathcal{E} \to E)$. Since by a well-known theorem of Serre $H^1(X(1), F(l)) = 0$ for $l \gg 0$, the map $H^0(X(1), \mathcal{E}(l)) \to H^0(X, E(l))$ is surjective for $l \gg 0$. Moreover, enlarging l enough, we can also assume that the vector bundle $E(l)$ is ample and generated by its global sections. Let $s \in H^0(X, E(l))$ be a global section whose zero locus $Y := Z(s)$ is smooth and $(n-r)$-dimensional (indeed, since $E(l)$ is generated by its global sections, by Proposition 1.1 a general section of $E(l)$ satisfies this condition). Moreover, by Theorem 1.3, Y is also connected, and hence irreducible because Y is smooth. Then the section s lifts to a global section $s' \in H^0(X(1), \mathcal{E}(l))$. If Y' denotes the zero locus of s' it follows that $Y' \cap X = Y$ (scheme-theoretic intersection in $X(1)$). Moreover, Y' is a local complete intersection of codimension r in $X(1)$. Then by Theorem 1.6 above we conclude that the exact sequence (2) splits. ☐

To prove the main result of this section we need the following generalization of the so-called Hartshorne–Serre correspondence:

Theorem 2.2 (Generalized Hartshorne–Serre correspondence) *Let $\mathcal{X}$ be an arbitrary irreducible algebraic scheme (not necessarily reduced) over a field k, and let $\mathcal{Y} \subset \mathcal{X}$ be a local complete intersection subscheme of $\mathcal{X}$ of codimension two. Assume that the determinant of the normal bundle $N_{\mathcal{Y}|\mathcal{X}}$ of $\mathcal{Y}$ in $\mathcal{X}$ extends to a line bundle L on $\mathcal{X}$ such that $H^2(\mathcal{X}, L^{-1}) = 0$. Then there exists a vector bundle $\mathcal{E}$ of rank two on $\mathcal{X}$ and a global section $t \in H^0(\mathcal{X}, \mathcal{E})$ such that $\det(\mathcal{E}) = L$ and $Z(t) = \mathcal{Y}$, i.e. the zero locus of t is $\mathcal{Y}$ (scheme-theoretically). If moreover $H^1(\mathcal{X}, L^{-1}) = 0$ then the pair $(\mathcal{E}, t)$ is also unique up to isomorphism.*

Proof: In the case when $\mathcal{X}$ is smooth the result is well-known (see e.g. [2]). For the lack of an appropriate reference we sketch a proof in this generality. Let $\mathcal{I}_{\mathcal{Y}}$ denote the sheaf of ideals of $\mathcal{Y}$ in $O_{\mathcal{X}}$ and consider the spectral sequence (see e.g. [1, Proposition (IV, 2.4])

$$E_2^{p,q} := H^p(\mathcal{X}, \mathcal{E}xt^q_{\mathcal{O}_\mathcal{X}}(\mathcal{I}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X})) \Longrightarrow E^n := \operatorname{Ext}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{I}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X}),$$

which yields the exact sequence in low degrees:

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0}. \tag{3}$$

Since $\mathcal{Y}$ is a local complete intersection in $\mathcal{X}$ of codimension two, $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{I}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X}) \cong L^{-1}$, so that our hypothesis that $H^2(\mathcal{X}, L^{-1}) = 0$ implies $E_2^{2,0} = 0$. Thus (3) yields a canonical surjection

$$\operatorname{Ext}^1_{\mathcal{O}_\mathcal{X}}(\mathcal{I}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X}) \to H^0(\mathcal{X}, \mathcal{E}xt^1_{\mathcal{O}_\mathcal{X}}(\mathcal{I}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X})). \tag{4}$$

On the other hand, the long exact cohomology sequence obtained by applying $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(-, \mathcal{O}_\mathcal{X})$ to the short exact sequence $0 \to \mathcal{I}_\mathcal{Y} \otimes L \to L \to \mathcal{O}_\mathcal{Y} \otimes L \to 0$ immediately yields

$$\mathcal{E}xt^1_{\mathcal{O}_\mathcal{X}}(\mathcal{I}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X}) \cong \mathcal{E}xt^2_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X}). \tag{5}$$

Since $\mathcal{Y}$ is a local complete intersection in $\mathcal{X}$, by [1], Theorem (I, 4.5) we infer that there is an isomorphism

$$\mathcal{E}xt^2_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{Y} \otimes L, \mathcal{O}_\mathcal{X}) \cong \det(N_{\mathcal{Y}|\mathcal{X}}) \otimes L^{-1}|\mathcal{Y} \cong \mathcal{O}_\mathcal{Y}, \tag{6}$$

because by assumption, $L|_\mathcal{Y} = \det(N_{\mathcal{Y}|\mathcal{X}})$. Therefore the target of the surjection (4) becomes $H^0(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) = \operatorname{Hom}_{\mathcal{O}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}, \mathcal{O}_\mathcal{Y})$. Hence the identity map lifts to an extension

$$0 \longrightarrow \mathcal{O}_\mathcal{X} \xrightarrow{\ t\ } \mathcal{E} \longrightarrow \mathcal{I}_\mathcal{Y} \otimes L \longrightarrow 0, \tag{7}$$

which produces a rank-two coherent sheaf $\mathcal{E}$ on $\mathcal{X}$. We shall prove that $\mathcal{E}$ is actually locally free. To show this it is enough to prove that

$$\mathcal{E}xt^1_{\mathcal{O}_\mathcal{X}}(\mathcal{E}, \mathcal{O}_\mathcal{X}) = 0. \tag{8}$$

Indeed, the problem being local this follows from [20, Lemma 5.1.2 and its proof, pages 98–99]. To prove (8), observe that (7) yields the following exact sequence

$$\mathcal{O}_{\mathcal{X}} \cong \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{\ \varphi\ } \mathcal{E}xt^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{I}_{\mathcal{Y}} \otimes L, \mathcal{O}_{\mathcal{X}}) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{O}_{\mathcal{X}}) \longrightarrow 0.$$

Thus (8) is equivalent to the surjectivity of the map φ. But by (5) and (6) we get an isomorphism $\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{I}_{\mathcal{Y}} \otimes L, \mathcal{O}_{\mathcal{X}}) \cong \mathcal{O}_{\mathcal{Y}}$. It follows that the map φ is identified with the canonical surjection $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$. This finishes the existence part of the theorem.

Now the condition that $\det(\mathcal{E}) \cong L$ follows immediately. Indeed, restricting the exact sequence (7) to $\mathcal{X} \setminus \mathcal{Y}$ and taking the determinant we get that $\det(\mathcal{E})|_{\mathcal{X}\setminus\mathcal{Y}} \cong L|_{\mathcal{X}\setminus\mathcal{Y}}$. Since the ideal of $\mathcal{Y}$ in $\mathcal{X}$ is locally generated by a regular sequence of length 2, a standard argument based on Local Cohomology [10] implies that $\det(\mathcal{E}) \cong L$. Moreover, the condition that $Z(t) = Y$ follows directly from (7) and from the definition of the zero locus.

Finally assume that $H^1(\mathcal{X}, L^{-1}) = 0$. This means that $E_2^{1,0} = 0$ in the exact sequence (3), hence the surjective map (4) is also injective. This yields the uniqueness of $\mathcal{E}$ (up to isomorphism), concluding the proof of the theorem. □

Remark 2.3 The above proof of Theorem 2.2 came out from a discussion with Enrique Arrondo.

The main result of this section is a sort of converse of Proposition 2.1 for rank two vector bundles on small-codimensional submanifolds in $\mathbb{P}^N$. Precisely, we prove the following:

Theorem 2.4 *Let $X \subset \mathbb{P}^N$ be a smooth n-dimensional subvariety, with $n \geq \frac{N+3}{2} \geq 4$. Let E be a rank two vector bundle on X which satisfies condition $(\mathbf{P}^2_E)$ above. Then E can be extended to a rank two vector bundle $\mathcal{E}$ on the first infinitesimal neighborhood $X(1)$ of X in $\mathbb{P}^N$.*

Proof: Assume first $n \geq 5$. It is clear that E extends to a (rank-two) vector bundle on $X(1)$ if and only if $E(l)$ does, so that we can replace E by a sufficiently high twist $E(l)$; in particular we may assume that E is ample and generated by its global sections. Since X is in the range of Barth-Larsen theorem (Theorem 1.4), its Picard group is generated by the class of $\mathcal{O}_X(1)$. In particular, since E is ample, there exists an $m > 0$ such that $\det(E) \cong \mathcal{O}_X(m)$. Replacing again E by $E(l)$ with $l \gg 0$ if necessary, we may also assume that $m \gg 0$. Then the exact sequence

$$0 \to N^\vee_{X|\mathbb{P}^N} \to \mathcal{O}_{X(1)} \to \mathcal{O}_X \to 0$$

yields the cohomology sequences $(i = 1, 2)$

$$H^i(N^\vee_{X|\mathbb{P}^N}(-m)) \to H^i(\mathcal{O}_{X(1)}(-m)) \to H^i(\mathcal{O}_X(-m)).$$

By [12, Éxposé XII, Corollaire 1.4], the first and the last vector space are zero for $i = 1, 2$ because X is smooth of dimension ≥ 3, $N_{X|\mathbb{P}^N}$ is a vector bundle, and $m \gg 0$. Therefore we get:

$$H^i(\mathcal{O}_{X(1)}(-m)) = 0 \text{ for } i = 1, 2 \text{ and } m \gg 0. \tag{9}$$

(Alternatively, a standard small argument shows that the projective scheme $X(1)$ is locally Cohen-Macaulay, and then (9) follows directly from [12, Éxposé XII, Corollaire 1.4].)

Replacing E by $E(l)$ with $l \gg 0$, condition $(\mathbf{P}^2_E)$ implies that there is a section $s \in H^0(X, E)$ whose zero locus $Y := Z(s)$ is a smooth 2-codimensional subvariety of X, and the canonical exact sequence (2) splits. Moreover, by Theorem 1.3 Y is also connected (cf. also a subsequent more general connectivity theorem of Fulton-Lazarsfeld [8]).

Now by condition $(\mathbf{P}^2_E)$ again and Theorem 1.6, there exists a 2-codimensional local complete intersection subscheme Y' of $X(1)$ such that $Y' \cap X = Y$ scheme-theoretically (i.e. $\mathcal{I}_{Y'} + \mathcal{I}_X = \mathcal{I}_Y$). We will show that Y' is the zero locus of a section of a a rank two vector bundle on $X(1)$ by applying Theorem 2.2, with $\mathcal{X} := X(1)$ and $\mathcal{Y} := Y'$.

In order to do this we will first show that $\det(N_{Y'|X(1)})$ extends to a line bundle L on $X(1)$ such that $H^i(X(1), L^{-1}) = 0$ for $i = 1, 2$. In this sense, consider the following commutative square of restriction maps

$$\begin{array}{ccc} \operatorname{Pic}(X(1)) & \xrightarrow{\alpha} & \operatorname{Pic}(X) \\ \beta\Big\downarrow & & \Big\downarrow\gamma \\ \operatorname{Pic}(Y') & \xrightarrow{\delta} & \operatorname{Pic}(Y). \end{array} \tag{10}$$

Claim. If $n \geq 5$, all the maps in diagram (10) are isomorphisms, and if $n = 4 = \frac{N+3}{2}$ (i.e. X is a smooth hypersurface in $\mathbb{P}^5$), α is an isomorphism and γ, β and δ are injective.

Let us prove the claim. The fact that α is an isomorphism if $n \geq 4$ follows immediately from the truncated exponential exact sequence

$$0 \to N^\vee_{X|\mathbb{P}^N} \to \mathcal{O}^*_{X(1)} \to \mathcal{O}^*_X \to 1$$

and from Theorem 1.2 (which implies in particular that $H^i(X, N^\vee_{X|\mathbb{P}^N}) = 0$, $i = 1, 2$, because $n \geq \frac{N+3}{2} \geq 4$). Here $\mathcal{O}^*_Z$ denotes the sheaf of multiplicative groups of regular nowhere vanishing functions on a scheme Z. Theorem 1.3 implies that the map γ is an isomorphism because $n - \operatorname{rank}(E) = n - 2 \geq 4 - 2 = 2$.

Assume first $n \geq 5$. At this point, since α and γ are isomorphisms, the commutative diagram (10) implies that β is injective and δ surjective. Therefore to finish the proof of the claim it is enough to show that the map δ is injective. To do this, since the ideal sheaf of Y in Y' is square-zero, one still has the following truncated exponential sequence

$$0 \to N^\vee_{Y|Y'} \to \mathcal{O}^*_{Y'} \to \mathcal{O}^*_Y \to 1.$$

On the other hand, by [21, Remark 1.2 ii)], $N_{Y|Y'} \cong N_{X|\mathbb{P}^N}|_Y$, and in particular, $N_{Y|Y'}$ is an ample vector bundle because $N_{X|\mathbb{P}^N}$ is so. Therefore by Theorem 1.2 we get

$$H^1(Y, N^\vee_{Y|Y'}) \cong H^1(Y, N^\vee_{X|\mathbb{P}^N}|_Y) = 0,$$

because $N_{X|\mathbb{P}^N}$ is ample, $\dim Y = n - 2$, $\operatorname{rank}(N_{Y|Y'}) = N - n$ and $N \leq 2n - 3$. Then the exact sequence

$$0 = H^1(Y, N^\vee_{Y|Y'}) \to \operatorname{Pic}(Y') \to \operatorname{Pic}(Y)$$

implies that δ is injective.

If instead $n = 4 = \frac{N+3}{2}$ the injectivity of γ follows from the last part of Theorem 1.3. The proof of the injectivity of δ when $n \geq 5$ works also if $n = 4$. The claim is proved.

Now, as $\dim X \geq \frac{N+3}{2}$, by Theorem 1.4 the map $\operatorname{Pic}(\mathbb{P}^N) \to \operatorname{Pic}(X)$ is an isomorphism. Since $\mathcal{I}_{Y'} \subset \mathcal{I}_Y$ and $N^\vee_{Y'|X(1)} = \mathcal{I}_{Y'}/\mathcal{I}^2_{Y'}$ and $N^\vee_{Y|X} = \mathcal{I}_Y/\mathcal{I}^2_Y$ are vector bundles of the same rank, $N_{Y'|X(1)}|_Y \cong N_{Y|X} = E|_Y$, hence $\det(N_{Y'|X(1)})|_Y \cong \det(E)|_Y = \mathcal{O}_Y(m)$. Since by the above claim the map δ is injective (even an isomorphism if $n \geq 5$), the we get $\det(N_{Y'|X(1)}) \cong \mathcal{O}_{Y'}(m)$, hence $L := \mathcal{O}_{X(1)}(m)$ is the unique extension (up to isomorphism) of $\det(N_{Y'|X(1)})$ on $X(1)$. Then by (9) we have $H^i(X(1), L^{-1}) = H^1(X(1), \mathcal{O}_{X(1)}(-m)) = 0$ for $i = 1, 2$ and for $m \gg 0$.

Then by Theorem 2.2 applied to $\mathcal{X} := X(1)$ and $\mathcal{Y} = Y'$, there is a pair $(\mathcal{E}, t)$, with $\mathcal{E}$ a vector bundle of rank 2 on $X(1)$ and a global section $t \in H^0(X(1), \mathcal{E})$, uniquely determined up to isomorphism, such that:

i) $\det(\mathcal{E}) = L$, and

ii) $Z(t) = Y'$, i.e. the zero locus of t is Y' (scheme-theoretically).

Set $E' := \mathcal{E}|_X$ and $s' := t|_X \in H^0(X, E')$. Clearly, $\det(E') \cong \mathcal{O}_X(m) \cong \det(E)$. Moreover, as $Y' \cap X = Y$ (scheme-theoretically) we infer that $Z(s') = Y$. As $\det(N_{Y|X}) \cong N_{Y|X} = \det(E|_Y) \cong \mathcal{O}_X(m)$, $\det(N_{Y|X})$ extends to $\mathcal{O}_X(m)$ with $m > 0$. Then a Serre vanishing we get $H^i(X, \mathcal{O}_X(-m)) = 0$ for $i = 1, 2$ and for $m \gg 0$. In conclusion, $\det(E)$ and $\det(E')$ extend both on X to $\mathcal{O}_X(m)$ and $Z(s) = Z(s') = Y$. Then by the uniqueness part of Theorem 2.2 there is an isomorphism $\varphi\colon E \to E'$ of vector bundles such that $\varphi(s) = s'$. This implies that $\mathcal{E}|_X \cong E$, i.e. $\mathcal{E}$ is an infinitesimal extension of E. $\square$

Remark 2.5 A careful look at the proof of Theorem 2.4 shows that this result is still true if one replaces condition $(\mathbf{P}^2_E)$ above on the triple $(\mathbb{P}^N, X, E)$, with $r = \operatorname{rank}(E) = 2$, by the following (slightly) weaker one:

$(\mathbf{P}^2_E)'$ *There exists a sequence of positive integers $l_0 < l_1 < l_2 < \cdots$ such that for every $i \geq 0$ there exists a section $s_i \in H^0(E(l_i))$ whose zero locus $Y_i := Z(s_i)$ is a smooth and 2-codimensional in X such that the following canonical exact sequence*

$$0 \to N_{Y_i|X} = E(l)|_{Y_i} \to N_{Y_i|\mathbb{P}^N} \to N_{X|\mathbb{P}^N}|_{Y_i} \to 0$$

splits.

3 Examples of infinitesimally non extendable vector bundles

Consider the Grassmann variety $\mathbb{G}(k, m)$ of k-dimensional linear subspaces of $\mathbb{P}^m$, with $m \geq 3$ and $1 \leq k \leq m-2$ (hence $\mathbb{G}(k, m)$ is not a projective space). Then $\dim \mathbb{G}(k, m) = (k+1)(m-k)$. Let E denote the universal quotient bundle of $\mathcal{O}^{\oplus m+1}_{\mathbb{G}(k,m)}$ (of rank $m - k$). Fix an arbitrary projective embedding $\mathbb{G}(k, m) \hookrightarrow \mathbb{P}^N$ (for example, the Plücker embedding $i\colon X \hookrightarrow \mathbb{P}^{\binom{m+1}{k+1}-1}$), and denote by X the image of $\mathbb{G}(k, m)$ in $\mathbb{P}^N$.

In this section we prove the following result:

Theorem 3.1 *Under the above notation and hypotheses the universal quotient vector bundle E of $X \cong \mathbb{G}(k,m)$ (with $1 \leq k \leq m-2$) cannot be extended to a vector bundle on the first infinitesimal neighborhood $X(1)$ of X in $\mathbb{P}^N$.*

Proof: Assume by way of contradiction that there would exist a vector bundle $\mathcal{E}$ on $X(1)$ such that $\mathcal{E}|_X \cong E$. Tensoring by $\mathcal{E}$ the exact sequence

$$0 \to N^\vee_{X|\mathbb{P}^N} \to \mathcal{O}_{X(1)} \to \mathcal{O}_X \to 0$$

and taking into account that $\mathcal{E} \otimes N^\vee_{X|\mathbb{P}^N} \cong E \otimes N^\vee_{X|\mathbb{P}^N}$ we get the exact sequence

$$0 \to E \otimes N^\vee_{X|\mathbb{P}^N} \to \mathcal{E} \to E \to 0. \tag{11}$$

Now assume for the moment that the following condition holds true

$$H^1(X, E \otimes N^\vee_{X|\mathbb{P}^N}) = 0. \tag{12}$$

Then (11) and (12) imply that the restriction map $H^0(X(1), \mathcal{E}) \to H^0(X, E)$ is surjective. Considering the canonical surjection $\varphi\colon \mathcal{O}_X^{\oplus(m+1)} \twoheadrightarrow E$ given by $(s_0, s_1, \ldots, s_m) \in H^0(X,E)^{\oplus(m+1)}$, it follows that there exists an $(m+1)$-uple $(s'_0, s'_1, \ldots, s'_m) \in H^0(X(1), \mathcal{E})^{\oplus(m+1)}$ such that $s'_i|X = s_i$, $i = 0, 1, \ldots, m$. Since φ is surjective, the sections $s_0, s_1, \ldots, s_m$ generate E, hence by Nakayama's Lemma the sections $s'_0, s'_1, \ldots, s'_m$ generate $\mathcal{E}$. In other words, the surjection φ lifts to a surjection $\varphi'\colon \mathcal{O}_{X(1)}^{\oplus(m+1)} \twoheadrightarrow \mathcal{E}$. Then by the universal property of the Grassmann variety $X = \mathbb{G}(k,m)$ there exists a morphism of schemes $\pi\colon X(1) \to X$ such that $\pi^*(E) = \mathcal{E}$. Since $\mathcal{E}|_X = E$ it follows that π is a retraction of the canonical embedding $X \hookrightarrow X(1)$. By a well known result (see [18], or also [3, Lemma 6.2]), this latter fact is equivalent with the splitting of the canonical exact sequence of tangent and normal bundles

$$0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_{X|\mathbb{P}^N} \to 0. \tag{13}$$

By Theorem 1.6 of Van de Ven, the splitting of (13) implies that X is a linear subspace of $\mathbb{P}^N$, which is a contradiction (otherwise X would be isomorphic to a projective space).

Now we prove (12). We first claim that (12) is equivalent with the following vanishing:

$$H^0(E \otimes F) = 0, \tag{14}$$

where F is is defined in the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N^\vee_{X|\mathbb{P}^N} \arrow[r] \arrow[d, "\mathrm{id}"'] & \Omega^1_{\mathbb{P}^N}|X \arrow[r] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N^\vee_{X|\mathbb{P}^N} \arrow[r, "\varphi"] & \mathcal{O}_X(-1)^{\oplus(N+1)} \arrow[r] \arrow[d] & F := \mathrm{Coker}(\varphi) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_X \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{O}_X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}.$$

The first row of this diagram is the cotangent sequence of X and the second column is the Euler sequence restricted to X. Note that the sheaf F coincides with $\mathcal{P}^1(\mathcal{O}_X(1))(-1)$, where $\mathcal{P}^1(\mathcal{O}_X(1))$ is the sheaf of first-order principal parts of $\mathcal{O}_X(1)$. Tensoring this diagram by E we get the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E \otimes N^\vee_{X|\mathbb{P}^N} \arrow[r] \arrow[d, "\mathrm{id}"'] & E \otimes \Omega^1_{\mathbb{P}^N}|X \arrow[r] \arrow[d] & E \otimes \Omega^1_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \otimes N^\vee_{X|\mathbb{P}^N} \arrow[r] & E(-1)^{\oplus(N+1)} \arrow[r] \arrow[d] & E \otimes F \arrow[r] \arrow[d] & 0 \\
 & & E \arrow[r, "\mathrm{id}"] \arrow[d] & E \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{15}$$

The second row of (15) yields the cohomology sequence

$$H^0(X, E(-1)^{\oplus(N+1)}) \to H^0(X, E \otimes F) \to H^1(E \otimes N^\vee_{X|\mathbb{P}^N}) \to H^1(E(-1)^{\oplus(N+1)}).$$

By [6, Corollary (4.11) and Theorem (4.17)] (whose proofs are based on some vanishing results for flag manifolds of Kempf [15]) we have $H^i(E(-1)^{\oplus(N+1)}) = 0$ for $i = 0, 1$ (since $\mathrm{Pic}(X) \cong \mathbb{Z}$). Thus the canonical map $\delta\colon H^0(E \otimes F) \to H^1(X, E \otimes N^\vee_{X|\mathbb{P}^N})$ is an isomorphism, which proves the claim.

Therefore it will be sufficient to prove (14). But, as Giorgio Ottaviani kindly explained to me, (14) is a special case of a general result of Ottaviani-Rubei (see [19, Theorem 6.11]).

Indeed, considering the coboundary map $\delta'\colon H^0(E) \to H^1(E \otimes \Omega^1_X)$ associated to the last column of diagram (15) as a quiver we infer that $\delta' \neq 0$. Moreover since $H^0(E)$ is the standard la representation (and hence irreducible), this implies that $H^0(E \otimes F) = 0$. Alternatively, the fact that $H^0(E)$ is irreducible was proved directly in [24]. In this way the proof of Theorem 3.1 is complete. □

Remark 3.2 In the special case of Plücker embedding $X = \mathbb{G}(1,3) \hookrightarrow \mathbb{P}^5$, the normal bundle $N_{X|\mathbb{P}^5}$ is isomorphic to $\mathcal{O}_X(1)$, hence the vanishing (12) becomes $H^1(X, E(-1)) = 0$, and this follows directly from [6, Corollary (4.11) and Theorem (4.17)].

Example 3.3 (Submanifods of $\mathbb{P}^N$ of dimension $\frac{N+3}{2}$) For every $m \geq 3$ consider the Plücker embedding $i'_m\colon \mathbb{G}(1,m) \hookrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ of the Grassmann variety of lines in $\mathbb{P}^m$, and set $X'_m := i'_m(\mathbb{G}(1,m))$. As is well-known X'_m is a 4-defective subvariety of $\mathbb{P}^{\binom{m+1}{2}-1}$, meaning that there is a linear projection $\pi_{L_m}\colon \mathbb{P}^{\binom{m+1}{2}-1} \dashrightarrow \mathbb{P}^{4m-7}$ of center a linear subspace L_m of dimension $\binom{m+1}{2} - (4m-7) - 2$ of $\mathbb{P}^{\binom{m+1}{2}-1}$ which does not intersect X'_m such that the restriction $\pi_{L_m}|X'_m\colon X'_m \to \pi_{L_m}(X'_m)$ is a biregular isomorphism (see [13, Exercise 11.27, page 145]). Therefore we may consider the projective embedding $i_m := \pi_{L_m} \circ i'_m\colon \mathbb{G}(1,m) \hookrightarrow \mathbb{P}^{4m-7}$. If we set $X_m := \pi_{L_m}(X'_m)$, $n := \dim X_m = \dim \mathbb{G}(1,m) = 2(m-1)$ and $N := 4m-7$, it follows that X_m is, via the projective embedding i_m, an n-dimensional closed subvariety of $\mathbb{P}^N$, with $n = \frac{N+3}{2}$. If $m = 3$ or if $m = 4$ the projective embeddings i_m and i'_m coincide, i.e. i_m is one of the Plücker embeddings $i'_3\colon \mathbb{G}(1,3) \hookrightarrow \mathbb{P}^5$ or $i'_4\colon \mathbb{G}(1,4) \hookrightarrow \mathbb{P}^9$. Conversely, if i_m and i'_m coincide then $m = 3$ or $m = 4$. In particular, Theorem 2.4 applies to every rank two vector bundle on the submanifold X_m of dimension $2(m-1)$ of $\mathbb{P}^{4m-7}$, with $m \geq 3$.

Corollary 3.4 *Let $X := \mathbb{G}(1,m)$ be the Grassmann variety of lines in $\mathbb{P}^m$, with $m \geq 3$, and let E be the universal rank two quotient vector bundle on X. Let $X \hookrightarrow \mathbb{P}^{4m-7}$ be any projective embedding of X in $\mathbb{P}^{4m-7}$ (see Example 3.3). Then there exists an integer $l_0 > 0$ such that for every $l \geq l_0$ and for every section $s \in H^0(E(l))$ whose zero-locus $Y := Z(s)$ is smooth of codimension 2 in X, the exact sequence of normal bundles*

$$0 \to N_{Y|X} \to N_{Y|\mathbb{P}^N} \to N_{X|\mathbb{P}^N}|_Y \to 0$$

never splits.

Proof: In this case $\dim X = 2(m-1) = \frac{N+3}{2}$, with $N = 4m-7$. Then the corollary follows from Theorem 3.1 and Theorem 2.4 via Remark 2.5. □

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