A Note On $S^1$-Equivariant Maps Between 3-Spheres
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Abstract
In this paper, we study the non-existence of equivariant maps of 2-connected compact manifold with effective circle actions. It was given a necessary condition for a map between 3-spheres to be equivariant for an effective circle action.

Key Words: Inverse limit, Hilbert-Smith conjecture, effective group action, equivariant map.
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1 Introduction
This short note is motivated by the following statement known as the generalized Hilbert-Smith conjecture:

Conjecture: If $G$ is a compact group which acts effectively on a connected finite dimensional manifold, then $G$ is a Lie group. A well known fact ([7]) states Hilbert-Smith conjecture is equivalent the following conjecture:

Conjecture: A $p$-adic group cannot act effectively on a connected finite dimensional manifold.

Therefore constructing effective $p$-adic space plays an important role in the study of the Hilbert-Smith conjecture.

We show that one way to obtain a compact space with an effective $p$-adic group action, is to take inverse limit of inverse systems of effective $S^1$-spaces with bonding maps that satisfy certain equivariant properties.

2 Preliminaries and Background
In this section we give necessary definitions and facts that will be used in the note. Suppose that to every $\alpha$ in a set $I$ directed by the relation $\leq$ corresponds a topological space $X_\alpha$, and that for any $\alpha, \beta \in I$ satisfying $\alpha \leq \beta$ a continuous mapping $f_\beta^\alpha : X_\beta \to X_\alpha$ is defined; suppose further that $f_\gamma f_\alpha = f_\beta$ for any $\alpha, \beta, \gamma \in I$ satisfying $\gamma \leq \alpha \leq \beta$ and that $f_\alpha^\alpha : X_\alpha \to X_\alpha$ is the
identity map for all $\alpha \in I$. In this case we say that the family $S = \{X_\alpha, f_\alpha^\beta, I\}$ is an inverse system of the spaces $X_\alpha$; the mappings $f_\alpha^\beta$ are called bonding mappings of the inverse system $S$.

Let $S = \{X_\alpha, f_\alpha^\beta, I\}$ be an inverse system; an element $(x_\alpha)$ of the cartesian product $\prod_{\alpha \in I} X_\alpha$ is called a thread of $S$ if $f_\alpha^\beta(x_\beta) = x_\alpha$ for any $\alpha, \beta \in I$ satisfying $\alpha \leq \beta$, and the subspace of $\prod_{\alpha \in I} X_\alpha$ consisting of all threads of $S$ is called the limit of the inverse system $S = \{X_\alpha, f_\alpha^\beta, I\}$ and is denoted by $\varprojlim S$ or by $\varprojlim X_\alpha$.

We denote by $\pi_\beta : \varprojlim X_\alpha \to X_\beta$ the restriction of the canonical projection $\prod_{\alpha \in I} X_\alpha \to X_\beta$. Clearly, for any $\alpha, \beta \in I$ such that $\alpha \leq \beta$, the projections $\pi_\alpha$ and $\pi_\beta$ satisfy the equality $\pi_\alpha = f_\alpha^\beta \pi_\beta$. The following results are well known and we refer to [10] for more details.

**Theorem 1.** [10, pg. 99] The limit of inverse system of $T_i$-spaces is a $T_i$-space for $i \leq 3\frac{1}{2}$.

**Theorem 2.** [10, pg. 355] The inverse limit of inverse system $S = \{X_\alpha, f_\alpha^\beta, I\}$ of continua is a continuum.

If $S = \{G_\alpha, f_\alpha^\beta, I\}$ is an inverse system of topological groups ($f_\alpha^\beta$ are continuous group homomorphisms), then $\varprojlim G_\alpha$ is also a topological group with the subspace topology from $\prod_{\alpha \in I} G_\alpha$ and with the group operation $(g_\alpha)(h_\alpha) \mapsto (g_\alpha h_\alpha)$. Further, if all groups $G_\alpha$ in the inverse system are compact then inverse limit is compact group.

Given a prime number $p$. Set $G_n = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (the usual torus group) for all $n \in \mathbb{N}$ and define $f_n^{n+1} : G_{n+1} \to G_n$, $f_n^{n+1}(z) = z^p$ for all $n \in \mathbb{N}$ and $z \in S^1$. The inverse limit of this inverse system is called the $p$-adic solenoid $T_p$. This inverse limit has the $p$-adic integers as a totally disconnected closed normal subgroup. Solenoids are one of the prototypes of compact abelian groups that are connected but not arc-wise connected.

Let $G$ be a group and $X$ be a topological space. Then $G$ acts on $X$ if there is a continuous function $G \times X \to X$ denoted by $(g, x) \mapsto gx$, such that

- $1x = x$
- $(gh)x = g(hx)$

for all $x \in X$ and $g, h \in G$ (here $1$ is the identity element of $G$). Call $X$ a $G$-space if $G$ acts on $X$. If $X$ is a $G$-space and $x \in X$, then the subspace $G(x) = \{gx \in X : g \in G\}$ is called the orbit of $x$. We denote the set whose elements are the orbits by $X/G$. An action of $G$ on $X$ is said to be transitive if there is precisely one orbit, $X$ itself and it is said to be effective if for any $g \neq 1$ in $G$ there exists an $x$ in $X$ such that $gx \neq x$.

### 3 Main Result

Let $X$ be a $G$-space, $Y$ be an $H$-space and $\varphi : G \to H$ be continuous group homomorphism. A continuous map $f : X \to Y$ is called $\varphi$-equivariant if

$$f(gx) = \varphi(g)f(x)$$

for all $g \in G$ and $x \in X$. 
If \( \{X_\alpha, f_\alpha^\beta, I\} \) is an inverse system of topological spaces and \( \{G_\alpha, \varphi_\alpha^\beta, I\} \) is an inverse system of topological groups, where each \( X_\alpha \) is a \( G_\alpha \)-space and each bonding map \( f_\alpha^\beta \) is \( \varphi_\alpha^\beta \)-equivariant, then we get another inverse system of topological spaces \( \{X_\alpha/G_\alpha, f_\alpha^\beta, I\} \) by passing to orbit spaces. Also, under above conditions \( \varprojlim X_\alpha \) is a \( \varprojlim G_\alpha \)-space with the action given by

\[
(g_\alpha)(x_\alpha) = (g_\alpha x_\alpha)
\]

for \( (g_\alpha) \in \varprojlim G_\alpha \) and \( (x_\alpha) \in \varprojlim X_\alpha \). Singh [9] has proved, for the inverse systems of spaces with bonding maps satisfy certain properties, that \( (\varprojlim X_\alpha)/(\varprojlim G_\alpha) \) is homeomorphic to \( \varprojlim (X_\alpha/G_\alpha) \). As a consequence of this, under the above conditions, if each \( X_\alpha \) is transitive \( G_\alpha \)-space then \( \varprojlim X_\alpha \) is a transitive \( \varprojlim G_\alpha \)-space. Here we consider the following question.

**Problem:** In the above discussion, is it possible to replace ”effective” by ”transitive”.

Before answer the problem, we recall some facts about ordered sets. If \( I \) is a set and \( \leq \) is an order relation on \( X \), and if \( a < b \) (i.e. \( a \leq b \) and \( a \neq b \)), we use the notation \((a, b)\) to denote the set \( \{x : a < x < b\} \); is called an open interval in \( X \). If this set is empty, we call \( a \) the immediate predecessor of \( b \), and we call \( b \) the immediate successor of \( a \).

**Lemma 1.** Let \( \{X_\alpha, f_\alpha^\beta, I\} \) be an inverse system of non-empty Hausdorff topological spaces with a linear order \((\leq)\) on \( I \) and let \( \{G_\alpha, \varphi_\alpha^\beta, I\} \) be an inverse system of topological groups, where each \( X_\alpha \) is an effective \( G_\alpha \)-space and each bonding map \( f_\alpha^\beta \) is \( \varphi_\alpha^\beta \)-equivariant and onto. Further, assume that each element of \( I \) has an immediate predecessor and immediate successor. Then \( X = \varlimsup X_\alpha \) is an effective \( G = \varlimsup G_\alpha \)-space.

**Proof:** Suppose that the induced \( G \) action on \( X \) is not effective. Then, there exists an element \( g = (g_\alpha) \in G \setminus \{1\} \) such that \( g x = x \) for each \( x = (x_\alpha) \in X \). Since \( g = (g_\alpha) \in G \setminus \{1\} \), there exists at least an element \( \alpha' \in I \) such that \( p_{\alpha'}(g) = g_{\alpha'} \in G_{\alpha'} \setminus \{1\} \) where \( p_{\alpha'} \) denotes the canonical projection \( p_{\alpha'} : \varprojlim G_\alpha \to G_{\alpha'} \).

Let \( y \) be an arbitrary point of \( X_{\alpha'} \). Since each bonding map \( f_\alpha^\beta \) is onto, we have an element \( z = (x_\alpha) \in \pi^{-1}_{\alpha'}(y) \) \((\pi_{\alpha'}) \) denotes the canonical projection \( \pi_{\alpha'} : \varprojlim X_\alpha \to X_{\alpha'} \) such that

\[
(i) \quad f_{\alpha'}^\gamma(\pi_{\gamma}(z)) = \pi_{\lambda}(z)
\]

for \( \alpha' \leq \lambda \) and immediate successor \( \gamma \) of \( \lambda \)

\[
(ii) \quad f_{\alpha'}^\theta(\pi_{\theta}(z)) = \pi_{\beta}(z)
\]

for \( \theta \leq \alpha' \) and immediate predecessor \( \beta \) of \( \theta \)

Since \( g z = z \), we have \( g_{\alpha'} y = y \). This implies that \( X_{\alpha'} \) is not effective \( G_{\alpha'} \)-space which is a contradiction. \( \square \)

**Example 1.** Let \( p \) be a prime number. Set \( G_n = S^1 = \{z \in \mathbb{C} : |z| = 1\} \) and define \( \varphi_{n+1}^n : G_{n+1} \to G_n, \varphi_{n+1}^n(z) = z^p \). Similarly set \( X_n = \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\} \) and define \( f_{n+1}^n : X_{n+1} \to X_n, f_{n+1}^n(w) = w^p \) for all \( n \in \mathbb{N} \). We consider the effective action of \( G_n \) on \( X_n \) given by multiplication. It is clear that each bonding map \( f_{n+1}^n \) is \( \varphi_{n+1}^n \)-equivariant. So inverse limit of the inverse system \( \{X_n, f_{n+1}^n, N\} \) is an effective compact \( \varprojlim G_n = T_p \)-space. Since the \( p \)-adic group, \( Z_p \), is a subgroup of \( T_p \), \( \varprojlim X_n \) is effective \( Z_p \)-space.
Let $X$ be a compact and 2-connected (i.e. $\pi_i(X) = \{0\}$ for $i \leq 2$) manifold and $q : X \to Y$ be a finite sheeted covering map over a compact manifold $Y$. Consider a continuous map $f : X \to X$ such that $qf = q$. Let $p$ be a prime number and $\varphi : S^1 \to S^1, \varphi(z) = z^p$.

**Theorem 3.** If there is an effective $S^1$ action on $X$ which admits $f$ as a $\varphi$-equivariant map, then the $p$–adic integers acts effectively on $X$.

**Proof:** Suppose there exists an effective $S^1$-action on $X$ such that $f$ is $\varphi$-equivariant map (i.e. $f(zx) = z^pf(x)$ for all $z \in S^1$ and $x \in X$). Let $X_f = \varprojlim (X,f) = \{(x_i)_{i \in \mathbb{N}} : f(x_{i+1}) = x_i\}$ and $T_p = \varprojlim (S^1, \varphi) = \{(z_i)_{i \in \mathbb{N}} : \varphi(z_{i+1}) = z_i^p = z_i\}$ (i.e. $p$-adic solenoid). On the other hand Cohen [6] defined $\varprojlim$ functor and proved that there is a short exact sequence for the homotopy classes of maps from a space $Z$ into the inverse limit of spaces $X_\alpha$, namely

$$0 \to \varprojlim \lim^1 [SZ, X_\alpha] \to \lim[Z, \lim X_\alpha] \to \lim[Z, X_\alpha] \to 0$$

where $SZ$ be the suspension of $Z$ and $[Z,W]$ denotes the homotopy classes of maps from the space $Z$ to $W$. It is clear that $[SZ, W] = [Z, \Omega W]$, where $\Omega W$ loop space of $W$. In our case, we consider the homotopy classes of maps from $S^1$ into the inverse limit, $X_f$, of the inverse system \{X, f\}. Since $\pi_1(X) = \{0\}$, we have $\pi_1(X_f) = \varprojlim [SS^1, X]$.

In our case $\varprojlim [SS^1, X] = \text{coker} \psi$ where $\psi : [SS^1, X]^\mathbb{N} \to [SS^1, X]^\mathbb{N}, \psi((x_n)) = (x_n - Sf(x_{n+1}))$ ($Sf : [SS^1, X] \to [SS^1, X]$ induced by $f$). Since $[SS^1, X] = [S^1, \Omega X] = \pi_2(X) = \{0\}$, we have $\pi_1(X_f) = 0$.

Now we identify $Y$ with inverse limit of inverse system \{Y, 1\}. Since $qf = q$, it induces a continuous map $q : X_f \to Y$ given by $q(x_n) = (q(x_n))$. Then, by [11] $g$ will also be a finite sheeted (in fact has the same number of sheet as $q$) covering map over $Y$. However since $X_f$ is simply connected, it is the universal cover of $Y$. Therefore $X_f = X$ by the uniqueness of the universal cover. According to Lemma 1. there is an effective $T_p$ (therefore $Z_p$)-action on $X$.  

We shift now our attention to 3-manifolds. It is well known fact that fundamental group is the most important invariant to distinguish 3-manifolds. In particular, if $X$ is simply connected compact 3-manifold then $X$ has the homology of a 3-sphere by Poincare duality. In fact $X$ is homotopy equivalent to $S^3$ by the Hurewicz theorem. But Poincare conjecture ([2,3,4]) asserts that $S^3$ is the only such manifold. If $X$ is orientable compact manifold of odd dimension then the Euler characteristic is 0 by Poincare duality. If $X$ is not orientable compact manifold then it has a two fold covering by a compact orientable manifold. And it is obvious that the Euler characteristic of this cover is twice the Euler characteristic of $X$. Therefore if $X$ is a non-orientable 3-manifold, then the Euler characteristic of $X$ is 0. This implies that $H_1(X,Z)$ is infinite. Therefore $S^3$ is the only 2-connected compact 3-manifold.

**Remark 1.** A 3-manifold $X$ is called spherical if there exists a finite subgroup $\Gamma$, of $SO(4)$ acting freely by rotations on $S^3$ such that $X = S^3/\Gamma$. On the other hand if $q : S^3 \to Y$ is a finite covering, then the covering transformation group and hence $\pi_1(Y)$ is finite. Thurston’s elliptization conjecture, which was proved in 2003 by G. Perelman ([3,4]), states that a closed
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3-manifold with finite fundamental group is spherical. So \(Y\) must be a spherical 3-manifold. On the other hand \(\Gamma\) is either cyclic, or is a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group by a cyclic group of even order. The most basic example for spherical 3-manifolds are Lens spaces (cyclic case) and links of quotient (alias simple) surface singularities. (see [1] or [8], pp 59-60.)

Recently J. Pardon [5] has proved that there is no effective action of a \(p\)-adic group on any connected 3-manifold. So we have the following theorem.

**Theorem 4.** If \(\Gamma\) is one of the those classes of groups and acts freely by rotations on \(S^3\) and \(\theta\) is an effective \(S^1\) action on \(S^3\) such that it admits a \(\varphi\)-equivariant map \(f : S^3 \to S^3\), then there exists \(x \in S^3\) such that \(f(x) \notin \Gamma(x)\).

**Proof:** Assume \(f(x) \in \Gamma(x)\) for each \(x \in S^3\). Then we have \(qf = q\) where \(q : S^3 \to S^3/\Gamma\) is the canonical, finite sheeted covering map. In accordance with Theorem 3. there is an effective \(p\)-adic group action on \(S^3\) which is a contradiction. \(\square\)

**Corollary 1.** There is no effective \(S^1\)-action on \(S^3\) which admits antipodal map as a \(\varphi\)-equivariant map.

**Proof:** The proof is trivial from Theorem 4. \(\square\)

**References**


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