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On a conjecture on the number of polynomials with coefficients in $[n]$

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Abstract

In this paper, we prove a counting result for the number of polynomials with integer coefficients bounded by a positive integer n and having all roots integers.

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1 Introduction

For any two positive integers n and s, define $\mathcal{A}_n{}^{(s)}$ to be the set of polynomials of degree s with all the roots integer numbers, and with all the coefficients in the set $[n] := \{1, 2, \cdots, n\}$. Let $A_n{}^{(s)}$ denote the cardinality of $\mathcal{A}_n{}^{(s)}$. In [2], two of the authors have conjectured that for any positive integer s,

$$\lim_{n\to\infty} \frac{A_n{}^{(s)}}{n \log^s n} = \frac{1}{(s!)^2}.$$

The purpose of this note is to prove the above conjecture, and to show that the following result holds true.

Theorem 1.1. *Fix a positive integer s. Then for all large n,*

$$A_n{}^{(s)} = \frac{n \log^s n}{(s!)^2} + \mathrm{O}_s\left(n \log^{s-1} n\right). \tag{1.1}$$

One can extract secondary terms from the right side of (1.1). Due to the complications when dealing with all the necessary subcases, we will do this only for $s = 2$ and $s = 3$. For $s = 2$, we prove the following result, which improves on Theorem 5.1 from [2].

Theorem 1.2. *There exist constants D_1 and D_2 such that for all large n,*

$$A_n{}^{(2)} = \frac{n\log^2 n}{(2!)^2} + D_1 n\log n + D_2 n + \mathrm{O}\left(n^{2/3}\log n\right). \tag{1.2}$$

For $s = 3$, we have the following result.

Theorem 1.3. *There exist constants B_1, B_2 and B_3 such that for all large n,*

$$A_n{}^{(3)} = \frac{n\log^3 n}{(3!)^2} + B_1 n\log^2 n + B_2 n\log n + B_3 n + \mathrm{O}\left(n^{3/4}\log n\right). \tag{1.3}$$

2 Proof of Theorem 1.1

Proof of Theorem 1.1. We recall the definition of $\mathcal{A}_n{}^{(s)}$,

$$A_n{}^{(s)} = \#\left(\left\{\begin{array}{c} P(x) = a_0 + a_1 x + \cdots + a_s x^s : a_0, \ldots, a_s \in \{1, 2, \ldots, n\}, \\ \text{and all roots of } P(x) \text{ are integers.}\end{array}\right\}\right).$$

One observes that if $P(x) \in A_n{}^{(s)}$ then $P(x) = a_s(x+b_1)\ldots(x+b_s)$; $b_1, \ldots, b_s \in \mathbb{N}$; $a_0 = a_s b_1 \ldots b_s \le n, a_1 = a_s(b_1 \ldots b_{s-1} + b_1 \ldots b_{s-2} b_s + \cdots + b_2 \ldots b_s) \le n, \ldots,\ a_{s-1} = a_s(b_1 + b_2 + \cdots + b_s) \le n$. We will first count the number of polynomials in $\mathcal{A}_n{}^{(s)}$ where the constant coefficient $a_0 = a_s b_1 \ldots b_s$ is largest. Denote

$$S_1{}^{(s)} = \{(a_s, b_1, \ldots, b_s) : 1 \le a_s \le n, b_1 \le b_2 \le \cdots \le b_s, a_s b_1 b_2 \ldots b_s \le n\}.$$

For $0 \le j \le s$, we define

$$T_j{}^{(s)} = \#\left(\left\{\begin{array}{l} (a_s, b_1, \ldots, b_s) : a_s b_1 \ldots b_s \le n, b_1 \le \cdots \le b_s, \text{ with} \\ j \text{ number of equalities between } b_1, b_2, \ldots, b_s \\ \text{and the remaining are strict inequalities.}\end{array}\right\}\right).$$

Then,

$$\#(S_1{}^{(s)}) = T_0{}^{(s)} + \sum_{j=1}^{s} T_j{}^{(s)} = \sum_{1 \le a_s \le n} \sum_{\substack{1 \le b_1 < b_2 \cdots < b_s \\ a_s b_1 b_2 \ldots b_s \le n}} 1 + \sum_{j=1}^{s} T_j{}^{(s)}. \tag{2.1}$$

Now,

$$\begin{aligned} T_0{}^{(s)} &= \sum_{1 \le a_s \le n} \sum_{\substack{1 \le b_1 < b_2 \cdots < b_s \\ a_s b_1 b_2 \ldots b_s \le n}} 1 = \frac{1}{s!} \sum_{a_s \le n} \sum_{\substack{b_1, \ldots, b_s \\ a_s b_1 b_2 \ldots b_s \le n}} 1 = \frac{1}{s!} \sum_{a_s \le n} \sum_{m \le \frac{n}{a_s}} d_s(m) \\ &= \frac{1}{s!} \sum_{m \le n} d_{s+1}(m), \end{aligned} \tag{2.2}$$

where $d_l(m)$ is the generalized divisor function which counts the number of ways a positive integer m can be written as a product of l positive numbers. Estimating the error term in the asymptotic formula for the summatory function of $d_l(m)$ is also the famous general (Dirichlet) divisor problem (or the Piltz divisor problem). For more on the Dirichlet divisor problem and the Piltz divisor problem see [3], [4]. The generalized divisor summatory function is given by

$$\sum_{m\le x} d_l(m) = xP_l(\log x) + \mathrm{O}\left(x^{1-1/l}\log x\right), \tag{2.3}$$

where $P_l(\log x)$ is a polynomial in $\log x$ of degree $l-1$. It is given by

$$P_l(\log x) = Res_{z=1}\frac{\zeta^l(z)x^{z-1}}{z} = \sum_{j=1}^{l} a_{l-j}\log^{l-j} x. \tag{2.4}$$

Here z is a complex number and $\zeta(z)$ is the Riemann zeta-function. A.F.Lavrik [1] evaluated the coefficients a_{l-j} explicitly in terms of constants $\gamma_k's$ appearing in the coefficients of Laurent series expansion of the Riemann zeta function about its pole $z=1$,

$$a_{l-j} = \frac{(-1)^{j+1}}{(l-j)!}\left(1+\sum_{k=1}^{j-1}(-1)^k\sum_{\Omega(k)}\frac{l(l-1)\dots(l-r+1)}{r_0!r_1!\dots r_\nu!}{\gamma'_0}^{r_0}\dots{\gamma'_\nu}^{r_\nu}\right).$$

The index $\Omega(k)$ denotes summation over all solutions of the equation $k = r_0+2r_1+\dots+(\nu+1)r_\nu$ in non negative integers $\nu, r_0, \dots, r_\nu$, and $r = r_0+\dots+r_\nu$, while $\gamma'_0, \gamma'_1, \dots$ are the coefficients of the series

$$\sum_{\nu=0}^{\infty}\gamma'_\nu(z-1)^\nu = \zeta(z) - \frac{1}{z-1}.$$

The coefficients $\gamma'_n s$ for $n=0,1,\dots$ are given by,

$$(-1)^n n!\gamma'_n = \lim_{N\to\infty}\left(\sum_{m=1}^{N}\frac{\log^n m}{m} - \frac{\log^{n+1} N}{n+1}\right).$$

These are related to the Stieltjes constants $\gamma_k s$ that occur in the Laurent series expansion of the Riemann zeta function by the relation $\gamma'_n = \frac{(-1)^n}{n!}\gamma_n$. Note that the first coefficient $a_{l-1} = \frac{1}{(l-1)!}$ and so we get

$$P_l(\log x) = \frac{1}{(l-1)!}\log^{l-1} x + \sum_{j=2}^{l} a_{l-j}\log^{l-j} x.$$

This along with (2.2) and (2.3) gives

$$T_0{}^{(s)} = \frac{1}{s!}\sum_{m\le n} d_{s+1}(m) = \frac{n}{s!^2}\log^s n + \mathrm{O}\left(n\log^{(s-1)} n\right). \tag{2.5}$$

In $T_j{}^{(s)}$, since two or more $b_i{}'s$ are equal and they satisfy the same inequality $a_s b_1\dots b_s \le n$, we have $T_j{}^{(s)} = \mathrm{O}\left(n\log^{(s-1)} n\right)$. This along with (2.1) and (2.5) gives

$$\#(S_1{}^{(s)}) = \frac{n}{s!^2}\log^s n + \mathrm{O}\left(n\log^{(s-1)} n\right). \tag{2.6}$$

It remains to consider the cases when the constant coefficient is not the largest. We begin with counting polynomials in which a_0 is not the largest but the previous coefficient a_1 is largest. Then

$$b_2b_3\dots b_s + b_1b_3\dots b_s + \dots + b_1b_2\dots b_{s-1} > b_1\dots b_s.$$

Here at least one of the s terms on the left side is larger than $\frac{b_1\dots b_s}{s}$. Say $b_2\dots b_s > \frac{b_1\dots b_s}{s}$. This implies $b_1 < s$. So one of the roots, b_1 in this case is bounded in terms of s. But for each fixed value of b_1, we have at most $\mathrm{O}\left(n\log^{s-1} n\right)$ choices for the s-tuple $(a_s, b_2, \dots, b_s)$. Similarly, if the next coefficient a_2 is largest then,

$$b_3b_4\dots b_s + b_1b_4\dots b_s + \dots + b_1b_2\dots b_{s-2} > b_1\dots b_s,$$

which implies that at least one of the $\binom{s}{2}$ terms on the left side is larger than $\frac{b_1\dots b_s}{\binom{s}{2}}$. Say $b_3b_4\dots b_s > \frac{b_1\dots b_s}{\binom{s}{2}}$. This gives $b_1b_2 < \binom{s}{2}$. Hence both roots b_1, b_2 are bounded in this case and so for each fixed value of b_1 and b_2, we have at most $\mathrm{O}\left(n\log^{s-2} n\right)$ choices for the $(s-1)$-tuple $(a_s, b_3, b_4\dots, b_s)$. Repeating this argument for all the cases when the last coefficient is not the largest, we finally obtain that the number of polynomials in which the maximum coefficient is attained elsewhere than the last one is $\mathrm{O}\left(n\log^{s-1} n\right)$. This along with (2.6) gives

$$A_n{}^{(s)} = \frac{n}{(s!)^2}\log^s n + \mathrm{O}\left(n\log^{s-1} n\right),$$

which completes the proof of the theorem. ∎

3 Proof of Theorem 1.2

In this section we consider the case $s = 2$.

Proof of Theorem 1.2. For a fixed positive integer n,

$$A_n{}^{(2)} = \#\left(\left\{\begin{array}{l} a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \{1, 2, \dots, n\} \\ \text{and all roots are integers}\end{array}\right\}\right)$$

If $P(x) \in A_n{}^{(2)}$ then $P(x) = a_2(x + b_1)(x + b_2)$; $b_1, b_2 \in \mathbb{N}$; $a_2b_1b_2 \le n$; $a_2(b_1 + b_2) \le n$ and $a_2 \le n$. Thus,

$$\begin{aligned} A_n{}^{(2)} &= \sum_{1\le a_2\le n}\ \sum_{\substack{1\le b_1\le b_2\\ a_2b_1b_2\le n,\\ a_2(b_1+b_2)\le n}} 1 = \sum_{1\le a_2\le n}\ \sum_{\substack{1\le b_1\le b_2\\ a_2b_1b_2\le n}} 1 - \sum_{1\le a_2\le n}\ \sum_{\substack{1\le b_1\le b_2\\ a_2b_1b_2\le n\\ a_2(b_1+b_2)>n}} 1 \\ &=: S_1{}^{(2)} - S_2{}^{(2)}. \end{aligned} \tag{3.1}$$

Rewrite ${S_1}^{(2)}$ as

$$\begin{aligned}
{S_1}^{(2)} &= \frac{1}{2}\sum_{1\le a_2\le n}\ \sum_{\substack{b_1,b_2\\ a_2b_1b_2\le n}} 1 + \frac{1}{2}\sum_{1\le a_2\le n}\ \sum_{\substack{b_1=b_2\\ a_2b_1b_2\le n}} 1\\
&= \frac{1}{2}\sum_{\substack{b_1,b_2,a_2\\ a_2b_1b_2\le n}} 1 + \frac{1}{2}\sum_{1\le a_2\le n}\ \sum_{\substack{b_2\in\mathbb{N}\\ a_2{b_2}^2\le n}} 1 = \frac{1}{2}\sum_{m\le n} d_3(m)\\
&+ \frac{1}{2}\left(\sum_{1\le a_2\le n^{1/3}}\ \sum_{1\le b_2\le\sqrt{\frac{n}{a_2}}} 1 + \sum_{1\le b_2\le n^{1/3}}\ \sum_{1\le a_2\le \frac{n}{{b_2}^2}} 1 - \sum_{1\le a_2\le n^{1/3}}\ \sum_{1\le b_2\le n^{1/3}} 1\right).
\end{aligned}$$

Observe that the sum in the bracket above is given by

$$\begin{aligned}
&\sum_{1\le a_2\le n^{1/3}}\ \sum_{1\le b_2\le\sqrt{\frac{n}{a_2}}} 1 + \sum_{1\le b_2\le n^{1/3}}\ \sum_{1\le a_2\le \frac{n}{{b_2}^2}} 1 - \sum_{1\le a_2\le n^{1/3}}\ \sum_{1\le b_2\le n^{1/3}}\\
&= C_1 n + \mathrm{O}\left(n^{2/3}\right).
\end{aligned}$$

Using this and the expression for the summatory function for $d_3(m)$ from (2.3) we obtain

$$ {S_1}^{(2)} = \frac{1}{4} n\log^2 n + C_2 n\log n + C_3 n + \mathrm{O}\left(n^{2/3}\log n\right), \tag{3.2}$$

where C_2 and C_3 are constants. Next we estimate ${S_2}^{(2)}$, the number of polynomials in ${\mathcal{A}_n}^{(2)}$ in which the last coefficient is not the largest. Now,

$$ {S_2}^{(2)} = S_{1,1} + S_{1,2} \tag{3.3}$$

where $S_{1,1} = \#(\{(a_2, b_1, b_2) \in \mathbb{N}^3 : b_1 < b_2 \text{ and } a_2b_1b_2 \le n < a_2(b_1 + b_2)\})$ and $S_{1,2} = \#(\{(a_2, b_1, b_2) \in \mathbb{N}^3 : b_1 = b_2 \text{ and } a_2b_1b_2 \le n < a_2(b_1 + b_2)\})$. In $S_{1,1}$, since $b_1b_2 < b_1 + b_2 < 2b_2$, we have $b_1 < 2$. This implies $b_1 = 1$. Then, $a_2b_2 \le n < a_2(1 + b_2)$. This implies $\frac{n}{a_2} - 1 < b_2 \le \frac{n}{a_2}$. Therefore,

$$\begin{aligned}
S_{1,1} &= \#\left(\left\{(a_2, b_2) : a_2 > n^{2/3}, b_2 \ge 1, \frac{n}{a_2} - 1 < b_2 \le \frac{n}{a_2}\right\}\right)\\
&\quad + \#\left(\left\{(a_2, b_2) : 1 \le a_2 \le n^{2/3}, b_2 \ge 1, \frac{n}{a_2} - 1 < b_2 \le \frac{n}{a_2}\right\}\right)\\
&= \sum_{1\le b_2\le n^{1/3}}\left(\frac{n}{b_2} - \frac{n}{(1+b_2)}\right) + \mathrm{O}\left(\sum_{1\le a_2\le n^{2/3}} 1\right)\\
&= n\sum_{b_2>1}\frac{1}{b_2(1+b_2)} - n\sum_{b_2>n^{1/3}}\frac{1}{b_2(1+b_2)} + \mathrm{O}\left(n^{2/3}\right)\\
&= C_4 n + \mathrm{O}\left(n^{2/3}\right).
\end{aligned} \tag{3.4}$$

In $S_{1,2}$, since $b_1 = b_2$, we have $a_2 {b_1}^2 \leq n < 2a_2 b_1$, which implies $b_1 = b_2 = 1$. And so,

$$S_{1,2} = \#(\{a_2 \in \mathbb{N} : a_2 \leq n < 2a_2\}) = \frac{n}{2} + \mathrm{O}(1).$$

Combining (3.3), (3.4) and above, we obtain,

$$S_2{}^{(2)} = C_5 n + \mathrm{O}\left(n^{2/3}\right), \tag{3.5}$$

for a positive constant C_5. Therefore, the above equation along with (3.1) and (3.2) gives us

$$A_n{}^{(2)} = \frac{1}{4} n \log^2 n + C_6 n \log n + C_7 n + \mathrm{O}\left(n^{2/3} \log n\right),$$

where C_6 and C_7 are constants. This completes the proof of the theorem. □

4 Proof of Theorem 1.3

In this section we estimate $A_n{}^{(s)}$ for $s = 3$.

Proof of Theorem 1.3. For a fixed positive integer n,

$$A_n{}^{(3)} = \#\left(\left\{\begin{array}{l} a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in \{1, 2, \ldots, n\} \\ \text{and all roots are integers} \end{array}\right\}\right)$$

If $P(x) \in A_n{}^{(3)}$ then $P(x) = a_3(x + b_1)(x + b_2)(x + b_3)$; $b_1, b_2, b_3 \in \mathbb{N}$; $a_3 b_1 b_2 b_3 \leq n,$; $a_3(b_1 + b_2 + b_3) \leq n,$; $a_3(b_1 b_2 + b_2 b_3 + b_3 b_1) \leq n$ and $a_3 \leq n$.

$$\begin{aligned} A_n{}^{(3)} &= \sum_{1 \leq a_3 \leq n} \sum_{\substack{1 \leq b_1 \leq b_2 \leq b_3 \\ a_3 b_1 b_2 b_3 \leq n, \\ a_3(b_1 b_2 + b_2 b_3 + b_1 b_3) \leq n, \\ a_3(b_1 + b_2 + b_3) \leq n}} 1 \\ &= \sum_{1 \leq a_3 \leq n} \sum_{\substack{1 \leq b_1 \leq b_2 \leq b_3 \\ a_3 b_1 b_2 b_3 \leq n, \\ a_3(b_1 b_2 + b_2 b_3 + b_1 b_3) \leq n}} 1 - \sum_{1 \leq a_3 \leq n} \sum_{\substack{1 \leq b_1 \leq b_2 \leq b_3 \\ a_3 b_1 b_2 b_3 \leq n, \\ a_3(b_1 b_2 + b_2 b_3 + b_1 b_3) \leq n, \\ a_3(b_1 + b_2 + b_3) > n}} 1. \end{aligned} \tag{4.1}$$

Now for any positive integers b_1, b_2, b_3, we have, $b_1 b_2 + b_2 b_3 + b_1 b_3 \geq b_1 + b_2 + b_3$ which implies there exists no positive integers b_1, b_2, b_3, for which the condition, $b_1 b_2 + b_2 b_3 + b_1 b_3 \leq n < b_1 + b_2 + b_3$ holds true. And so the second sum in the above equation equals zero. This gives,

$$\begin{aligned} A_n{}^{(3)} &= \sum_{1 \leq a_3 \leq n} \sum_{\substack{1 \leq b_1 \leq b_2 \leq b_3 \\ a_3 b_1 b_2 b_3 \leq n, \\ a_3(b_1 b_2 + b_2 b_3 + b_1 b_3) \leq n}} 1 \\ &= \sum_{1 \leq a_3 \leq n} \sum_{\substack{1 \leq b_1 \leq b_2 \leq b_3 \\ a_3 b_1 b_2 b_3 \leq n}} 1 - \sum_{1 \leq a_3 \leq n} \sum_{\substack{1 \leq b_1 \leq b_2 \leq b_3 \\ a_3 b_1 b_2 b_3 \leq n, \\ a_3(b_1 b_2 + b_2 b_3 + b_1 b_3) > n}} 1. \end{aligned} \tag{4.2}$$

Define

$$S_1{}^{(3)} := \left\{x = (a_3, b_1, b_2, b_3) \in \mathbb{N}^4 : 1 \le a_3 \le n, a_3b_1b_2b_3 \le n\right\},$$

$$S_2{}^{(3)} := \left\{\begin{array}{c} x = (a_3, b_1, b_2, b_3) \in \mathbb{N}^4 : 1 \le a_3 \le n, 1 \le b_1 \le b_2 \le b_3, \\ a_3b_1b_2b_3 \le n < a_3(b_1b_2 + b_2b_3 + b_1b_3). \end{array}\right\}.$$

Let $T_1{}^{(3)} := \{x \in S_1{}^{(3)} : b_1 < b_2 = b_3\}, T_2{}^{(3)} := \{x \in S_1{}^{(3)} : b_1 = b_2 < b_3\}$, and $T_2{}^{(3)} := \{x \in S_1{}^{(3)} : b_1 = b_2 = b_3\}$. Observe that,

$$\sum_{1 \le a_3 \le n} \sum_{\substack{1 \le b_1 \le b_2 \le b_3, \\ a_3b_1b_2b_3 \le n}} 1 = \frac{1}{3!}\#(S_1{}^{(3)}) + \frac{1}{2}\#(T_1{}^{(3)}) + \frac{1}{2}\#(T_2{}^{(3)}) + \frac{5}{6}\#(T_3{}^{(3)}).$$

From (4.2) and above, we obtain,

$$A_n{}^{(3)} = \frac{1}{3!}\#(S_1{}^{(3)}) + \frac{1}{2}\#(T_1{}^{(3)}) + \frac{1}{2}\#(T_2{}^{(3)}) + \frac{5}{6}\#(T_2{}^{(3)}) - \#(S_2{}^{(3)}). \tag{4.3}$$

Rewriting $S_1{}^{(3)}$ in terms of the generalized divisor function $d_4(n)$ and using (2.3), we have,

$$\begin{aligned}\#(S_1{}^{(3)}) = \sum_{m \le n} d_4(m) &= \frac{1}{3!} n \log^3 n + C_8 n \log^2 n + C_9 n \log n + C_{10} n \\ &+ \mathrm{O}\left(n^{3/4} \log n\right),\end{aligned} \tag{4.4}$$

where C_8, C_9 and C_{10} are constants. To estimate $\#(S_2{}^{(3)})$ in (4.3), we again consider cases distinguished by the relations between the positive integers b_1, b_2 and b_3. Let $A_1 := \{x \in S_2{}^{(3)} : b_1 < b_2 < b_3\}, A_2 := \{x \in S_2{}^{(3)} : b_1 = b_2 < b_3\}, A_3 := \{x \in S_2{}^{(3)} : b_1 < b_2 = b_3\}$, and finally $A_4 := \{x \in S_2{}^{(3)} : b_1 = b_2 = b_3\}$. So,

$$\#(S_2{}^{(3)}) = \#(A_1) + \#(A_2) + \#(A_3) + \#(A_4). \tag{4.5}$$

We first estimate $\#(A_4)$. In A_4 since $b_1 = b_2 = b_3$, we have $a_3{b_1}^3 < 3a_3{b_1}^2$, which implies $b_1 = 1, 2$, and so,

$$\#(A_4) = \sum_{\frac{n}{3} < a_3 \le n} 1 + \sum_{\frac{n}{12} < a_3 \le \frac{n}{8}} 1 = C_{11} n + \mathrm{O}\left(1\right), \tag{4.6}$$

where C_{10} is a positive constant. Next we estimate A_3 where b_1 being the smallest takes the values 1 and 2 only. Then, $\#(A_3) = \#(A_{3,1}) + \#(A_{3,2})$, where $A_{3,1} := \{(a_3, b_1, b_2, b_3) \in A_3 : b_1 = 1\}$, and $A_{3,2} := \{(a_3, b_1, b_2, b_3) \in A_3 : b_1 = 2\}$. In $A_{3,2}$, since, $2a_3{b_2}^2 < a_3(4b_2 + {b_2}^2)$, this implies, $b_2 < 4$ and so $b_2 = b_3 = 3$ since $b_1 = 2 < b_2 = b_3$. Thus,

$$\#(A_{3,2}) = \sum_{\frac{n}{21} < a_3 \le \frac{n}{18}} 1 = C_{12} n + \mathrm{O}\left(1\right), \tag{4.7}$$

for some positive constant C_{11}. To estimate $A_{3,1}$ we write it as $A_{3,1} = A_{3,1,1} \setminus A_{3,1,2}$ where, $A_{3,1,1} := \{(a_3, b_1, b_2, b_3) \in A_{3,1} : a_3 {b_2}^2 \leq n\}$, and $A_{3,1,2} := \{(a_3, b_1, b_2, b_3) \in A_{3,1} : a_3({b_2}^2 + 2b_2) \leq n\}$. Now,

$$\begin{aligned}\#(A_{3,1,1}) = &\sum_{1 \leq a_3 \leq n^{1/3}} \sum_{1 \leq b_2 \leq \sqrt{\frac{n}{a_3}}} 1 + \sum_{1 \leq b_2 \leq n^{1/3}} \sum_{1 \leq a_3 \leq \frac{n}{{b_2}^2}} 1 \\ &- \sum_{1 \leq a_3 \leq n^{1/3}} \sum_{1 \leq b_2 \leq n^{1/3}} 1 = C_{13} n + \mathrm{O}\left(n^{2/3}\right).\end{aligned}$$

And

$$\begin{aligned}\#(A_{3,1,2}) = &\sum_{1 \leq a_3 \leq n^{1/3}} \sum_{1 \leq b_2 \leq \sqrt{\frac{n}{a_3} + 1} - 1} 1 + \sum_{1 \leq b_1 \leq n^{1/3}} \sum_{1 \leq a_3 \leq \frac{n}{{b_2}^2 + 2b_2}} 1 \\ &- \sum_{1 \leq a_3 \leq n^{1/3}} \sum_{1 \leq b_2 \leq n^{1/3}} 1 = C_{14} n + \mathrm{O}\left(n^{2/3}\right).\end{aligned}$$

Combining the above two estimates and (4.7), we obtain,

$$\#(A_3) = C_{15} n + \mathrm{O}\left(n^{2/3}\right), \tag{4.8}$$

where C_{15} is a constant. Next we estimate A_2 in (4.5). In A_2, since, $b_1 = b_2 < b_3$ and $a_3 b_1 b_2 b_3 < a_3(b_1 b_2 + b_2 b_3 + b_1 b_3)$, we find that $b_1 = b_2 = 1, 2.$ and so we have $\#(A_2) = \#(A_{2,1}) + \#(A_{2,2})$, where $A_{2,1} := \{(a_3, b_1, b_2, b_3) \in A_2 : b_1 = b_2 = 1\}$ and $A_{2,2} := \{(a_3, b_1, b_2, b_3) \in A_2 : b_1 = b_2 = 2\}$. In $A_{2,2}$, since, $4a_3 b_3 \leq n < 4a_3 + 4a_3 b_3$, this implies, $\frac{n}{4a_3} - 1 < b_3 \leq \frac{n}{4a_3}$ which further shows that $b_3 = \mathrm{O}(1)$.

$$\begin{aligned}\#(A_{2,2}) &= \#\left(\left\{(a_3, b_3) : a_3 > n^{3/4}, \frac{n}{4a_3} - 1 < b_3 \leq \frac{n}{4a_3}\right\}\right) \\ &\quad + \#\left(\left\{(a_3, b_3) : 1 \leq a_3 \leq n^{3/4}, \frac{n}{4a_3} - 1 < b_3 \leq \frac{n}{4a_3}\right\}\right) \\ &= \sum_{1 \leq b_3 \leq \frac{n^{1/4}}{4}} \left(\frac{n}{4b_3} - \frac{n}{4(1 + b_3)}\right) + \mathrm{O}\left(\sum_{1 \leq a_3 \leq n^{3/4}} 1\right) \\ &= n \sum_{b_2 > 1} \frac{1}{4b_3(1 + b_3)} - n \sum_{b_3 \geq \frac{n^{1/4}}{4}} \frac{1}{4b_3(1 + b_3)} + \mathrm{O}\left(n^{3/4}\right) \\ &= C_{16} n + \mathrm{O}\left(n^{3/4}\right),\end{aligned} \tag{4.9}$$

where C_{13} is a positive constant. Next as in $A_{3,1}$, we write $A_{2,1}$ as, $A_{2,1} = A_{2,1,1} \setminus A_{2,1,2}$ where, $A_{2,1,1} =: \{(a_3, b_1, b_2, b_3) \in A_{2,1} : a_3 b_3 \leq n\}$, and

$$A_{2,1,2} =: \{(a_3, b_1, b_2, b_3) \in A_{2,1} : a_3(1 + 2b_3) \leq n\}.$$

Using the divisor summatory function ([3]), we obtain,

$$\#(A_{2,1,1}) = \sum_{m\le n} d(m) = n\log n + C_{17}n + \mathrm{O}\left(\sqrt{n}\right). \tag{4.10}$$

We remark that we could use an estimate of the summatory function of divisor function with an error term better than $\mathrm{O}\left(\sqrt{n}\right)$ but this does not improve on our estimation of ${A_n}^{(3)}$.

$$\begin{aligned}\#(A_{2,1,2}) = &\sum_{1\le a_3\le n^{1/2}} \sum_{1\le b_3\le \frac{n}{2a_3}-\frac{1}{2}} 1 + \sum_{1\le b_3\le \frac{\sqrt{n}-1}{2}} \sum_{1\le a_3\le \frac{n}{1+2b_3}} 1 \\ &- \sum_{1\le a_3\le \sqrt{n}} \sum_{1\le b_3\le \frac{\sqrt{n}-1}{2}} 1 \\ = &\, C_{18}n\log n + C_{19}n + \mathrm{O}\left(\sqrt{n}\right).\end{aligned}$$

This along with (4.9) and (4.10) gives,

$$A_2 = C_{20}n\log n + C_{21}n + \mathrm{O}\left(n^{3/4}\right), \tag{4.11}$$

where C_{20} and C_{21} are constants. Next we estimate A_1. Recall that

$$A_1 = \left\{ \begin{matrix} x = (c_3, b_1, b_2, b_3) : 1 \le c_3 \le n, 1 \le b_1 < b_2 < b_3, \\ c_3b_1b_2b_3 \le n < c_3(b_1b_2 + b_2b_3 + b_1b_3) \end{matrix} \right\}.$$

Note that for any $x \in A_1$, we have $b_1 = 1$ or $b_1 = 2$. Indeed,

$$A_{1,1} = \{x \in A_1 : b_1 = 1\}\ ;\ A_{1,2} = \{x \in A_1 : b_1 = 2\},$$

and

$$\#(A_1) = \#(A_{1,1}) + \#(A_{1,2}). \tag{4.12}$$

We have, $A_{1,2} = B_{1,2}$, where

$$B_{1,2} = \left\{ \begin{matrix} (a_3, b_2, b_3) : 1 \le a_3 \le n, 2 < b_2 < b_3, \\ 2a_3b_2b_3 \le n < a_3(2b_2 + 2b_3 + b_2b_3) \end{matrix} \right\}.$$

Note that for any $(a_3, b_2, b_3) \in B_{1,2}$ since $2a_3b_2b_3 < a_3(2b_2 + 2b_3 + b_2b_3)$, we have, $b_2b_3 < 2b_2 + 2b_3 < 4b_3$, and hence $b_2 < 4$ but $b_2 > 2$ and so $b_2 = 3$. Thus, $B_{1,2} = C_{1,2}$, where $C_{1,2} = \{(a_3, b_3) : 1 \le c_3 \le n, b_3 > 3, 6a_3b_3 \le n < a_3(6 + 5b_3)\}$. Again note that for any $(a_3, b_3) \in C_{1,2}$, we have, $6b_3 < 6 + 5b_3$ which together with $b_3 > 3$ implies $b_3 = 4, 5$. Combining all the above facts, we have,

$$\#(A_{1,2}) = \sum_{\frac{n}{26}<c_3\le \frac{n}{24}} 1 + \sum_{\frac{n}{31}<c_3\le \frac{n}{30}} 1 = C_{22}n + \mathrm{O}\left(1\right). \tag{4.13}$$

Finally we estimate $A_{1,1}$. We write $A_{1,1} = D_{1,1} \setminus F_{1,1}$, where $D_{1,1} = \{(a_3, b_2, b_3) : a_3 \leq n, 1 < b_2 < b_3, a_3b_2b_3 \leq n\}$, and $F_{1,1} = \{(a_3, b_2, b_3) : a_3 \leq n, 1 < b_2 < b_3, a_3(b_2 + b_3 + b_2b_3) \leq n\}$. Now,

$$\begin{aligned} D_{1,1} &= \frac{1}{2!} \sum_{m \leq n} d_3(m) - \frac{1}{2} \sum_{a_3 \leq n} \sum_{\substack{b_2 = b_3 \\ a_3b_2b_3 \leq n}} \\ &= C_{23} n \log^2 n + C_{24} n \log n + C_{25} n + \mathrm{O}\left(n^{2/3} \log n\right), \end{aligned} \tag{4.14}$$

where C_{23}, C_{24} and C_{25} are constants. Next we write, $F_{1,1} = F_{1,1,1} \cup F_{1,1,2} \cup F_{1,1,3}$ where, $F_{1,1,1} = \{(a_3, b_2, b_3) \in F_{1,1} : 1 < b_2 < b_3 \leq a_3\}$, $F_{1,1,2} = \{(a_3, b_2, b_3) \in F_{1,1} : 1 < b_2 \leq a_3 < b_3\}$, and $F_{1,1,3} = \{(a_3, b_2, b_3) \in F_{1,1} : 1 \leq a_3 < b_2 < b_3\}$. In order to estimate these we use partial summation. Note that in $F_{1,1,1}$, $b_3 \leq a_3 \leq \dfrac{n}{b_2 + b_3 + b_2b_3}$. Also, since $b_2 < b_3 \leq a_3$, we have that ${b_2b_3}^2 \leq a_3b_2b_3 \leq n$ which implies $b_3 \leq \sqrt{\frac{n}{b_2}}$ and ${b_2}^3 \leq a_3b_2b_3 \leq n$ implies $b_2 \leq n^{1/3}$. Using this and the estimate

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + \mathrm{O}\left(\frac{1}{x}\right),$$

we obtain,

$$\begin{aligned} \#(F_{1,1,1}) &= \sum_{1 < b_2 \leq n^{1/3}} \sum_{b_2 < b_3 \leq \sqrt{\frac{n}{b_2}}} \left\lfloor \frac{n}{b_2 + b_3 + b_2b_3} - b_3 \right\rfloor \\ &= \sum_{1 < b_2 \leq n^{1/3}} \sum_{b_2 < b_3 \leq \sqrt{\frac{n}{b_2}}} \frac{n}{b_2 + b_3 + b_2b_3} \\ &\quad - \sum_{1 < b_2 \leq n^{1/3}} \sum_{b_2 < b_3 \leq \sqrt{\frac{n}{b_2}}} b_3 + \mathrm{O}\left(\sum_{1 < b_2 \leq n^{1/3}} \sum_{b_2 < b_3 \leq \sqrt{\frac{n}{b_2}}} 1 \right) \end{aligned}$$

And so,

$$\begin{aligned}
\#(F_{1,1,1}) = & \sum_{1<b_2\le n^{1/3}} \sum_{b_2<b_3\le\sqrt{\frac{n}{b_2}}} \left(\frac{n}{(b_2+1)(b_3+1)}\right) \\
& + \sum_{1<b_2\le n^{1/3}} \sum_{b_2<b_3\le\sqrt{\frac{n}{b_2}}} \left(\frac{n}{(b_2+1)(b_3+1)(b_2+b_3+b_2b_3)}\right) \\
& - \sum_{1<b_2\le n^{1/3}} \left(\frac{n}{b_2} - {b_2}^2\right) + \mathrm{O}\left(\sum_{1<b_2\le n^{1/3}} \sqrt{\frac{n}{b_2}}\right) \\
= & \, n \sum_{1<b_2\le n^{1/3}} \frac{1}{b_2+1} \sum_{b_2<b_3\le\sqrt{\frac{n}{b_2}}} \frac{1}{b_3+1} \\
& + \sum_{b_2>1} \sum_{b_2<b_3} \frac{n}{(b_2+1)(b_3+1)(b_2+b_3+b_2b_3)} + \mathrm{O}\left(n^{2/3}\right) \\
& - C_{26} n \log n - C_{27} n + \mathrm{O}\left(n^{2/3}\right) \\
= & \, n \sum_{1<b_2\le n^{1/3}} \frac{1}{b_2+1} \log\left(\frac{\sqrt{\frac{n}{b_2}}+1}{b_2+1}\right) + C_{28} n\log n + C_{29} n + \mathrm{O}\left(n^{2/3}\right) \\
= & \, C_{30} n \log^2 n + C_{31} n \log n + C_{32} n + \mathrm{O}\left(n^{2/3}\right), \qquad (4.15)
\end{aligned}$$

where in order to estimate the sum $\sum_{1<b_2\le n^{1/3}} \frac{1}{b_2+1} \log\left(\frac{\sqrt{\frac{n}{b_2}}+1}{b_2+1}\right)$, we use summation by parts formula [5] with $a(n) = \frac{1}{n+1}$, $f(m) = \log\left(\frac{\sqrt{\frac{n}{m}}+1}{m+1}\right)$ and $A(t) = \sum_{n\le t} a(n)$. Next consider the sum $F_{1,1,2}$.

Here we have, $a_3 < b_3 \le \dfrac{\frac{n}{a_3}+1}{1+b_2} - 1, {a_3}^2 b_2 < a_3 b_3 b_2 \le n$ and $b_2 \le n^{1/3}$.

$$\begin{aligned}
\#(F_{1,1,2}) = & \sum_{1<b_2\le n^{1/3}} \sum_{b_2\le a_3<\sqrt{\frac{n}{b_2}}} \left\lfloor \frac{\frac{n}{a_3}+1}{1+b_2} - 1 - a_3 \right\rfloor \\
= & \sum_{1<b_2\le n^{1/3}} \sum_{b_2\le a_3<\sqrt{\frac{n}{b_2}}} \frac{n}{a_3(1+b_2)} - \sum_{1<b_2\le n^{1/3}} \sum_{b_2\le a_3<\sqrt{\frac{n}{b_2}}} \frac{1}{1+b_2} \\
& - \sum_{1<b_2\le n^{1/3}} \sum_{b_2\le a_3<\sqrt{\frac{n}{b_2}}} a_3 + \mathrm{O}\left(\sum_{1<b_2\le n^{1/3}} \sum_{b_2\le a_3<\sqrt{\frac{n}{b_2}}} 1\right).
\end{aligned}$$

Employing summation by parts we get,

$$\begin{aligned}\#(F_{1,1,2}) &= n \sum_{1<b_2\leq n^{1/3}} \frac{1}{1+b_2}\log\left(\frac{\sqrt{\frac{n}{b_2}}}{b_2}\right) - \sum_{1<b_2\leq n^{1/3}} \frac{\sqrt{\frac{n}{b_2}}-b_2}{1+b_2} \\ &\quad - \sum_{1<b_2\leq n^{1/3}} \left(\frac{n}{b_2}-{b_2}^2\right) + \mathrm{O}\left(\sum_{1<b_2\leq n^{1/3}} \sqrt{\frac{n}{b_2}}\right) \\ &= C_{33} n\log^2 n + C_{34} n\log n + C_{35} n + \mathrm{O}\left(n^{2/3}\right). \qquad (4.16)\end{aligned}$$

And finally we estimate $F_{1,1,3}$, where, $b_2 < b_3 \leq \dfrac{\frac{n}{a_3}+1}{1+b_2} - 1, {a_3}^2 b_2 < a_3 b_3 b_2 \leq n$ and $a_3 \leq n^{1/3}$.

$$\#(F_{1,1,3}) = \sum_{1\leq a_3\leq n^{1/3}} \sum_{a_3<b_2\leq\sqrt{\frac{n}{a_3}}} \left\lfloor \frac{\frac{n}{a_3}+1}{1+b_2} - 1 - b_2 \right\rfloor$$

Note that the above sum is similar to the sum in (4.16) and so,

$$\#(F_{1,1,3}) = \#(F_{1,1,2}).$$

This along with (4.14), (4.15) and (4.16) gives,

$$\#(A_{1,1}) = C_{36} n\log^2 n + C_{37} n\log n + C_{38} n + \mathrm{O}\left(n^{2/3}\log n\right).$$

Using this with (4.12) and (4.13), we obtain,

$$\#(A_1) = C_{39} n\log^2 n + C_{40} n\log n + C_{41} n + \mathrm{O}\left(n^{2/3}\log n\right). \qquad (4.17)$$

We are left to estimate the cardinalities of ${T_1}^{(3)}, {T_2}^{(3)}$ and ${T_3}^{(3)}$ in (4.3). Observe that b_1 being the smallest in ${T_1}^{(3)}, {T_2}^{(3)}$ and ${T_3}^{(3)}$, we have, $a_3 {b_1}^3 \leq n$ in all the three sets. And so proceeding as above we have, $\#({T_1}^{(3)}) = C_{43} n\log n + C_{44} n + \mathrm{O}\left(n^{3/4}\right), \#({T_2}^{(3)}) = C_{45} n\log n + C_{46} n + \mathrm{O}\left(n^{3/4}\right)$ and $\#({T_3}^{(3)}) = \mathrm{O}\left(n^{1/2}\right)$.
Finally from (4.3), (4.4), (4.5), (4.6), (4.8), (4.11), (4.17) and above, we obtain

$${A_n}^{(3)} = \frac{n\log^3 n}{(3!)^2} + B_1 n\log^2 n + B_2 n\log n + B_3 n + \mathrm{O}\left(n^{3/4}\log n\right),$$

where B_1, B_2 and B_3 are constants. This completes the proof of the theorem. ∎

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