The Exponential Diophantine Equation

\[((2^{2m} - 1)n)^x + (2^{m+1}n)^y = ((2^{2m} + 1)n)^z\]

by

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Abstract

Let \(m, n\) be positive integers. Let \((a, b, c)\) be a primitive Pythagorean triplet with \(a^2 + b^2 = c^2\). In 1956, L. Jeśmanowicz conjectured that the equation \((an)^x + (bn)^y = (cn)^z\) has only the positive integer solution \((x, y, z) = (2, 2, 2)\). In this paper, using certain elementary methods, we prove that if \((a, b, c) = (2^{2m} - 1, 2^{m+1}, 2^{2m} + 1)\), then the above equation has only the positive integer solution \((x, y, z) = (2, 2, 2)\). Thus it can be seen that Jeśmanowicz’s conjecture is true for infinitely many primitive Pythagorean triplets.

Key Words: Exponential diophantine equation, primitive Pythagorean triplet, Jeśmanowicz’s conjecture.

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1 Introduction

Let \(\mathbb{Z}, \mathbb{N}\) be the sets of all integers and positive integers respectively. Let \(m, n\) be positive integers. Let \((a, b, c)\) be a primitive Pythagorean triplet such that

\[a^2 + b^2 = c^2, \quad a, b, c \in \mathbb{N}, \quad \gcd(a, b, c) = 1, \quad 2 \nmid b.\] (1.1)

Then we have

\[a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2, \quad u, v \in \mathbb{N},\]
\[u > v, \quad \gcd(u, v) = 1, \quad 2 \nmid uv.\] (1.2)

In 1956, L. Jeśmanowicz\(^2\) conjectured that the equation

\[(an)^x + (bn)^y = (cn)^z, \quad x, y, z \in \mathbb{N}\] (1.3)

has only the solution \((x, y, z) = (2, 2, 2)\) for any \(n\).

This conjecture has been proved to be true in many special cases (see [7] and its references). But, in general, the problem is not solved as yet.
Most of the results concerning the above conjecture deal with the case that \( n = 1 \), and very little is known about (1.3) for \( n > 1 \). In this paper, we discuss the case that
\[
u = 2^m, \quad v = 1.\tag{1.4}
\]
Substituting (1.4) into (1.2), we have
\[
a = 2^{2m} - 1, \quad b = 2^{m+1}, \quad c = 2^{2m} + 1, \tag{1.5}
\]
and by (1.5), the equation (1.3) can be written as
\[
((2^{2m} - 1)n)^x + (2^{m+1}n)^y = ((2^{2m} + 1)n)^z, \quad x, y, z \in \mathbb{N}. \tag{1.6}
\]
In this connection, by an early result of W. -T. Lu\[^5\], (1.6) has only the solution \((x, y, z) = (2, 2, 2)\) for \( n = 1 \). In 1998, M. -J. Deng and G. L. Cohen\[^1\] showed that if \( m = 1 \), then (1.6) has only the solution \((x, y, z) = (2, 2, 2)\) for \( n > 1 \). Recently, Z. -J. Yang and M. Tang\[^10\] proved a similar result for \( m = 2 \). In this paper, using certain elementary methods, we prove a general result as follows.

**Theorem 1.** For any positive integers \( m \) and \( n \), (1.6) has only the solution \((x, y, z) = (2, 2, 2)\).

Thus it can be seen that Jeśmanowicz’s conjecture is true for infinitely many primitive Pythagorean triplets.

### 2 Preliminaries

Let \( k \) be a positive integer, and let \( P(k) \) denote the product of all distinct prime divisors of \( k \). Further let \( P(1) = 1 \).

**Lemma 2.1.**\[^6\] Let \( t \) be a positive integer. If \( 2^t \equiv 1 \mod 2^k - 1 \), then \( k|t \).

**Lemma 2.2.** Every solution \((X, Y, Z)\) of the equation
\[
X^2 + Y^2 = Z^k, \quad X, Y, Z \in \mathbb{N}, \quad \gcd(X, Y) = 1, \quad 2|Y \tag{2.1}
\]
can be expressed as
\[
Z = A^2 + B^2, \quad A, B \in \mathbb{N}, \quad \gcd(A, B) = 1, \quad 2|B \tag{2.2}
\]
and
\[
X + Y\sqrt{-1} = \lambda_1(A + \lambda_2B\sqrt{-1})^k, \quad \lambda_1, \lambda_2 \in \{\pm 1\}. \tag{2.3}
\]
Moreover, if \( 2^r|Y, \ 2^s||k \) and \( 2^r||B \), then \( r > s \) and \( r = s + t \).

**Proof.** By [8, Section 15.2], every solution \((X, Y, Z)\) of (2.1) can be expressed as (2.2) and (2.3). Further, by (2.3), we have
\[
Y = \lambda_1\lambda_2B \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} A^{k-2i-1}(-B^2)^i, \tag{2.4}
\]
where \( \lfloor (k-1)/2 \rfloor \) is the integral part of \((k-1)/2\).
By (2.2), we have \( 2 \nmid A \),

\[
2^{s+t} \mid \lambda_1 \lambda_2 \binom{k}{1} A^{k-1} B \tag{2.5}
\]

and

\[
2^{s+3t} \mid (-1)^i \lambda_1 \lambda_2 \binom{k}{2i+1} A^{k-2i-1} B^{2i+1}
\]

\[
= (-1)^i \lambda_1 \lambda_2 k \binom{k-1}{2i+1} A^{k-2i-1} B^{2i+1}. \quad i \geq 1. \tag{2.6}
\]

Therefore, by (2.5) and (2.6), we get

\[
2^{s+t} \mid \lambda_1 \lambda_2 B \sum_{i=0}^{[(k-1)/2]} \binom{k}{2i+1} A^{k-2i-1} (-B^2)^i. \tag{2.7}
\]

Since \( 2 \mid B \), we see from (2.4) and (2.7) that \( r > s \) and \( r = s + t \). The lemma is proved.

**Lemma 2.3.** Every solution \((X, Y, Z)\) of the equation

\[
X^2 + 2Y^2 = Z^k, \quad X, Y, Z \in \mathbb{N}, \quad \gcd(X, Y) = 1 \tag{2.8}
\]

can be expressed as

\[
Z = A^2 + 2B^2, \quad A, B \in \mathbb{N}, \quad \gcd(A, B) = 1, \ 2 \nmid A \tag{2.9}
\]

and

\[
X + Y \sqrt{-2} = \lambda_1 (A + \lambda_2 B \sqrt{-2})^k, \quad \lambda_1, \lambda_2 \in \{\pm 1\}. \tag{2.10}
\]

Moreover, if \( 2^r \mid Y \), \( 2^s \mid k \) and \( 2^t \mid B \), then \( r \geq s \) and \( r = s + t \).

**Proof.** Notice that \( h(-8) = 1 \), where \( h(-8) \) is the class number of primitive binary quadratic forms of discriminant -8. Therefore, by [3, Theorems 1 and 2], every solution \((X, Y, Z)\) of (2.8) can be expressed as (2.9) and (2.10). Further, by (2.10), we have

\[
Y = \lambda_1 \lambda_2 B \sum_{i=0}^{[(k-1)/2]} \binom{k}{2i+1} A^{k-2i-1} (-B^2)^i. \tag{2.11}
\]

Thus, using the same method as in the proof of Lemma 2.2, we can get from (2.11) that \( r \geq s \) and \( r = s + t \). The lemma is proved.

**Lemma 2.4.**[9] If \( k \geq 3 \), then the equation

\[
X^k + Y^k = Z^k, \quad X, Y, Z \in \mathbb{N} \tag{2.12}
\]

has no solution \((X, Y, Z)\).

**Lemma 2.5.**[4] If \( n > 1 \) and \((x, y, z)\) is a solution of (1.3) with \((x, y, z) \neq (2, 2, 2)\), then one of the following conditions is satisfied:

(i) \( \max\{x, y\} > \min\{x, y\} > z, \quad \gcd(n | c \text{ and } P(n) < P(c)) \).

(ii) \( x > z > y \) and \( \gcd(n | b) \).

(iii) \( y > z > x \) and \( \gcd(n | a) \).
3 Proof of Theorem

By the results of [1], [5] and [10], it suffices to prove the theorem for \(m \geq 3\) and \(n > 1\). We now assume that \((x, y, z)\) is a solution of (1.6) with \((x, y, z) \neq (2, 2, 2)\). By Lemma 2.5, we only need to examine the following four cases:

**Case I.** \(x > y > z\), \(P(n) \mid 2^{2m} + 1\) and \(P(n) < P(2^{2m} + 1)\).

Under these assumptions, by (1.6), we get
\[
2^{2m} + 1 = c_1c_2, \quad c_1, c_2 \in \mathbb{N}, \quad \gcd(c_1, c_2) = 1, \quad c_2 > 1,
\]
and
\[
2^{n^{y-z}} = c_1^z, \quad c_1 > 1
\]
and
\[
(2^{2m} - 1)^x n^{x-y} + 2^{(m+1)y} = c_2^z.
\]
Since \(c_1 > 1\) and every prime divisor \(p\) of \(2^{2m} + 1\) satisfies \(p \equiv 1 \pmod{4}\), we have \(c_1 \geq 5\) and \(c_2 \leq (2^{2m} + 1)/5\) by (3.1). Therefore, by (3.3), we get
\[
\left(\frac{2^{2m} + 1}{5}\right)^z \geq c_2^z > \left(\frac{2^{2m} - 1}{2}\right)^x > \left(\frac{2^{2m} + 1}{2}\right)^z,
\]
a contradiction.

**Case II.** \(y > x > z\), \(P(n) \mid 2^{2m} + 1\) and \(P(n) < P(2^{2m} + 1)\).

Using the same method as in the proof of Case I, we can exclude this case immediately.

**Case III.** \(x > z > y\) and \(P(n) \mid b\).

Since \(n > 1\), we get from (1.6) that \(P(n) = 2\),
\[
n^{y-z} = 2^{(m+1)y}
\]
and
\[
(2^{2m} - 1)^x n^{x-z} + 1 = (2^{2m} + 1)^z.
\]
By (3.5), we have
\[
n = 2^r, \quad r \in \mathbb{N}
\]
and
\[
r(z - y) = (m + 1)y.
\]
Substituting (3.7) into (3.6), we get
\[
(2^{2m} - 1)^x \cdot 2^{r(x-z)} + 1 = (2^{2m} + 1)^z.
\]
Since \(2^{2m} + 1 \equiv 2 \pmod{2^{2m} - 1}\), we see from (3.9) that
\[
2^z \equiv 1 \pmod{2^{2m} - 1}.
\]
Applying Lemma 2.1 to (3.10), we obtain \(2m \mid z\) and therefore
\[
z = 2mk, \quad k \in \mathbb{N}.
\]
If $2 \mid k$, then $2^{2m+1} \equiv 2m \equiv 2m \equiv 1 \pmod{2m-1}$ and $\gcd((2^{2m+1})^{mk}+1, (2^{2m+1})^{mk}-1) = 1$. Hence, by (3.9) and (3.11), we get

$$\left(2^{2m+1}\right)^{mk} - 1 = 2^r(x-z)^{-1}(2^{2m+1})^x,$$

and

$$\left(2^{2m+1}\right)^{mk} + 1 = 2(2^{m+1})^x$$

whence we obtain

$$\left(2^{2m+1}\right)^{mk} - 1 = 2^r(x-z)^{-1}(2^{m} - 1)^x,$$

and

$$\left(2^{2m+1}\right)^{mk} + 1 = 2(2^{m} + 1)^x.$$ (3.12)

Since $2^m + 1 \equiv 2 \pmod{2^m - 1}$, we see from (3.15) that $2^x \equiv 1 \pmod{2^m - 1}$. It results that $m \mid x$, that is

$$x = ml, \quad l \in \mathbb{N}. \quad (3.16)$$

Therefore, by (3.9), (3.11) and (3.16), we get

$$\left(\left(2^{2m}-1\right)^l \cdot 2^r \prod_{i=1}^{l-1} \frac{x}{i}\right)^m + 1^m = \left((2^{2m} + 1)^{2k}\right)^m. \quad (3.17)$$

But, since $m \geq 3$, by Lemma 2.4, (3.17) is impossible.

**Case IV.** $y > z > x$ and $P(n) \mid 2^{2m} - 1$.

Then we have

$$2^{2m} - 1 = a_1 \cdot a_2, \quad a_1, a_2 \in \mathbb{N}, \quad \gcd(a_1, a_2) = 1,$$ (3.18)

and

$$a_2^x + 2^{(m+1)y} = (2^{2m} + 1)^z.$$ (3.19)

Let

$$x = 2^\alpha x_1, \quad z = 2^\beta z_1, \quad \alpha, \beta \in \mathbb{Z}, \quad \alpha \geq 0, \quad \beta \geq 0,$$

$$x_1, z_1 \in \mathbb{N}, \quad 2 \nmid x_1 \cdot z_1. \quad (3.20)$$

If $a_2 = 1$, then from (3.20) we get

$$2^{(m+1)y} = (2^{2m} + 1)^z - 1 = \sum_{i=1}^{z} \left(\begin{array}{c} z \\ i \end{array}\right) 2^{2mi}. \quad (3.21)$$

Using the same method as in the proof of Lemma 2.2, we get

$$2^{2m+\beta} \parallel \sum_{i=1}^{z} \left(\begin{array}{c} z \\ i \end{array}\right) 2^{2mi}. \quad (3.22)$$
Hence, by (3.22) and (3.23), we get
\[
(m + 1)y = 2m + \beta. 
\]  
(3.24)

But, since \( y > z > x \) and \( y \geq 3 \), by (3.21) and (3.24), we get
\[
y > z \geq 2^\beta = 2^{(m+1)y-2m} = 2^{(y-2)m+y} > 2^y,
\]  
(3.25)
a contradiction. So we have \( a_2 > 1 \).

By (3.20) and (3.23), we get
\[
a_2 \equiv 1 \pmod{2^{2m+\beta}}.
\]  
(3.26)

Further, by (3.21) and (3.26), we have
\[
a_2 \equiv \lambda \pmod{2^{2m+\beta-\alpha}},
\]  
(3.27)

where \( \lambda = (-1)^{(a_2-1)/2} \). Since \( a_2 > 1 \), we have \( a_2 + 1 \geq a_2 - \lambda > 0 \). Hence, by (3.27), we get
\[
a_2 \geq 2^{2m+\beta-\alpha} - 1.
\]  
(3.28)

On the other hand, we see from (3.18) and (3.19) that \( a_2 = (2^{2m} - 1)/a_1 < 2^{2m} - 1 \). Therefore, by (3.28), we get
\[
\alpha > \beta. 
\]  
(3.29)

Further, by (3.21) and (3.29), \( x/2^\beta \) is even and \( (z-x)/2^\beta \) is odd. Thus, we find from (3.19) that \( n \) must be a square, namely,
\[
n = l^2, \ l \in \mathbb{N}, \ l > 1, \ 2 \nmid l.
\]  
(3.30)

Substituting (3.30) into (3.20), we get
\[
(a_2^x/2)^2 + 2^{(m+1)y}(l^y-z)^2 = (2^{2m} + 1)^z. 
\]  
(3.31)

If \( 2 \mid (m + 1)y \), then, by (3.31), (2.1) has the solution
\[
(X, Y, Z, k) = (a_2^x/2, 2^{(m+1)y/2}(l^y-z), 2^{2m} + 1, z).
\]  
(3.32)

Applying Lemma 2.2 to (3.32), we have
\[
2^{2m} + 1 = A^2 + B^2, \ A, B \in \mathbb{N}, \ \gcd(A, B) = 1, \ 2 \mid B 
\]  
(3.33)

and
\[
2^{(m+1)y/2-\beta} \mid B. 
\]  
(3.34)

By (3.33) and (3.34), we get
\[
2^m \geq B \geq 2^{(m+1)y/2-\beta},
\]  
(3.35)

whence we obtain
\[
\beta \geq \frac{1}{2}((y - 2)m + y). 
\]  
(3.36)
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Since $m \geq 3$ and $y \geq 3$, we see from (3.21) and (3.36) that $\beta \geq 3$, $y > z \geq 2^\beta \geq 8$ and
\[ y > z \geq 2^\beta \geq 2^{(y-2)(m+y)/2} \geq 2^{2y-3}, \]  
(3.37)
a contradiction.

If $2 \nmid (m+1)y$, then, by (3.31), (2.8) has the solution
\[ (X, Y, Z, k) = \left( \frac{a_x^{2^\beta}}{2}, \frac{2((m+1)y-1)/2y-z}{2}, 2^{2m} + 1, z \right). \]  
(3.38)
Applying Lemma 2.3 to (3.38), we have
\[ 2^{2m} + 1 = A^2 + 2B^2, \quad A, B \in \mathbb{N}, \quad \gcd(A, B) = 1, \quad 2 \nmid A \]  
(3.39)
and
\[ 2^{((m+1)y-1)/2-\beta} \mid B. \]  
(3.40)
Therefore, by (3.39) and (3.40), we get $2^{2m} \geq 2B^2$ and (3.36) holds too. Thus, we can deduce a contradiction as (3.37).

To sum up, the theorem is proved.

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