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# Algebraic dependence and unicity theorem with a truncation level to 1 of meromorphic mappings sharing moving targets

by
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**Abstract**

The purpose of this article is twofold. The first is to show algebraic dependences of meromorphic mappings in several complex variables into the complex projective spaces with a truncation level to 1. The second is to study the unicity problem with a truncation level to 1 of meromorphic mappings in several complex variables into the complex projective spaces from the viewpoint of the propagation of dependences.

**Key Words**: Algebraic dependence, unicity problem, meromorphic mappings, truncated multiplicities.
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## 1 Introduction

The theory on algebraic dependences of meromorphic mappings in several complex variables into the complex projective spaces for fixed targets is studied by Wilhelm Stoll [St2]. Later, Min Ru [R] generalized Stoll's result to holomorphic curves into the complex projective spaces for moving targets and showed some unicity theorems of holomorphic curves into the complex projective spaces for moving targets. Recently, Viet Duc Pham and Duc Thoan Pham [PP] showed some results on algebraic dependences of meromorphic mappings. For instance, they showed the following theorem on algebraic dependences of meromorphic mappings with a truncation level to 1.

**Theorem A** (see [PP], Theorem 3) *Let $f_1, \cdots, f_\lambda : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ be nonconstant meromorphic mappings. Let $g_i : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ $(1 \le i \le q)$ be moving targets located in general position such that $T(r, g_i) = o(\max_{1\le j\le\lambda} T(r, f_j))$ $(1 \le i \le q)$ and $(f_i, g_j) \not\equiv 0$ for $1 \le i \le \lambda,\ 1 \le j \le q$. Assume that the following conditions are satisfied.*

*i)* $\min\{\nu_{(f_1,g_j)}, 1\} = \cdots = \min\{\nu_{(f_\lambda,g_j)}, 1\}$ *for each* $1 \le j \le q$,

*ii)* $\dim\{z|(f_1, g_i)(z) = (f_1, g_j)(z) = 0\} \le n-2$ *for each* $1 \le i < j \le q$,

*iii) There exists an integer number $l$, $2 \le l \le \lambda$, such that for any increasing sequence $1 \le j_1 < \cdots < j_l \le \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ for every point $z \in \cup_{i=1}^{q}(f_1, g_i)^{-1}(0)$.*

*Then*

*i) If* $q > \dfrac{N(2N+1)\lambda}{\lambda - l + 1}$, *then* $f_1, \cdots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$, *i.e.* $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$ *on* $\mathbb{C}$.

*ii) If* $f_i, 1 \le i \le \lambda$ *are linearly nondegenerate over* $\mathcal{R}\{g_j\}_{j=1}^q$ *and*

$$q > \frac{N(N+2)\lambda}{\lambda - l + 1},$$

*then* $f_1, \cdots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

*iii) If* $f_i, 1 \le i \le \lambda$ *are linearly nondegenerate over* $\mathbb{C}$, $g_i$ $(1 \le i \le q)$ *are constant mappings and* $(q - N - 1)(\lambda - l + 1) > N\lambda$, *then* $f_1, \cdots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

The first purpose of this article is to give an improvement of Theorem A. Namely, we will prove the followings

**Theorem 1.** *Let* $f_1, \cdots, f_\lambda : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ *be nonconstant meromorphic mappings. Let* $g_i : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ $(1 \le i \le q)$ *be moving targets located in general position such that* $T(r, g_i) = o(\max_{1 \le j \le \lambda} T(r, f_j))$ $(1 \le i \le q)$ *and* $(f_i, g_j) \not\equiv 0$ *for each* $1 \le i \le \lambda,\ 1 \le j \le q$. *Assume that the following conditions are satisfied.*

*i)* $\min\{\nu_{(f_1,g_j)}, 1\} = \cdots = \min\{\nu_{(f_\lambda,g_j)}, 1\}$ *for each* $1 \le j \le q$,

*ii)* $\dim\{z | (f_1, g_i)(z) = (f_1, g_j)(z) = 0\} \le n - 2$ *for each* $1 \le i < j \le q$,

*iii) There exists an integer number* $l$, $2 \le l \le \lambda$, *such that for any increasing sequence* $1 \le j_1 < \cdots < j_l \le \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ *for every point* $z \in \cup_{i=1}^q (f_1, g_i)^{-1}(0)$.

*Then*

*i) If* $q > \dfrac{N(2N+1)\lambda - (N-1)(\lambda - 1)}{\lambda - l + 1}$, *then* $f_1, \cdots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$, *i.e.* $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$ *on* $\mathbb{C}$.

*ii) If* $f_1, \cdots, f_\lambda$ *are linearly nondegenerate over* $\mathcal{R}(\{g_j\}_{j=1}^q)$ *and*

$$q > \frac{N(N+2)\lambda - (N-1)(\lambda - 1)}{\lambda - l + 1},$$

*then* $f_1, \cdots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

*iii) If* $f_1, \cdots, f_\lambda$ *are linearly nondegenerate over* $\mathbb{C}$, $g_i$ $(1 \le i \le q)$ *are constant mappings and* $(q - N - 1)((\lambda - 1)(N - 1) + q(\lambda - l + 1)) > qN\lambda$, *then* $f_1, \cdots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

The unicity theorems with truncated multiplicities of meromorphic mappings of $\mathbb{C}^n$ into the complex projective space $\mathbf{P}^N(\mathbb{C})$ sharing a finite set of fixed (or moving) hyperplanes in $\mathbf{P}^N(\mathbb{C})$ have received much attention in the last few decades, and they are related to many problems in Nevanlinna theory and hyperbolic complex analysis (see the reference in [A], [Fu], [DT1], [DT2], [TQ1], [TQ2], [HQT], [Q] for the development in related subjects). We now state a recent result in [TQ1] which is the one of the best results available at present.

Let $a_1, \ldots, a_q$ $(q \ge N + 1)$ be $q$ meromorphic mappings of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jN})$ $(1 \leqslant j \leqslant q)$. We say that $a_1, \ldots, a_q$ are located in general position if $\det(a_{j_k l}) \not\equiv 0$ for any $1 \leqslant j_0 < j_1 < \ldots < j_N \leqslant q$.

Let $\mathcal{M}_n$ be the field of all meromorphic functions on $\mathbb{C}^n$. Denote by $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big) \subset \mathcal{M}_n$ the smallest subfield which contains $\mathbb{C}$ and all $\dfrac{a_{jk}}{a_{jl}}$ with $a_{jl} \not\equiv 0$.

Let $f$ be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_N)$. We say that $f$ is linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$ if $f_0, \ldots, f_N$ are linearly independent over $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$.

Let $f$, $a$ be two meromorphic mappings of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ with reduced representations $f = (f_0 : \cdots : f_N)$, $a = (a_0 : \cdots : a_N)$ respectively. Put $(f, a) = \sum_{i=0}^{N} a_i f_i$. We say that $a$ is "small" with respect to $f$ if $T(r, a) = o(T(r, f))$ as $r \to \infty$.

Let $f$ and $a$ be nonconstant meromorphic mappings of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$. Denote by $\nu_{(f,a)}$ the map of $\mathbb{C}^n$ into $\mathbb{Z}$ whose value $\nu_{(f,a)}(z)$ $(z \in \mathbb{C}^n)$ is the intersection multiplicity of the images of $f$ and $a$ at $f(z)$.

Let $f : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ be a meromorphic mapping. Let $\varkappa$ be a positive integer. Let $\{a_j\}_{j=1}^q$ be "small" (with respect to $f$) meromorphic mappings of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ in general position such that

$$\dim\{z \in \mathbb{C}^n : (f, a_i)(z) = (f, a_j)(z) = 0\} \leq n - 2 \quad (1 \leq i < j \leq q).$$

Assume that $f$ is linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^q)$.

Consider the set $\mathcal{F}(f, \{a_j\}_{j=1}^q, \varkappa)$ of all linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^q)$ meromorphic maps $g : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ satisfying the conditions:

(i) $\min\,(\nu_{(f,a_j)}, \varkappa) = \min\,(\nu_{(g,a_j)}, \varkappa) \quad (1 \leq j \leq q)$,

(ii) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : (f, a_j)(z) = 0\}$.

**Theorem B.** [TQ1] *a) If $q = 2N^2 + 4N$ and $N \geq 2$, then $\sharp\ \mathcal{F}(f, \{a_j\}_{j=1}^q, 1) = 1$, where denote by $\sharp\ S$ the cardinality of the set $S$.*

*b) If $q = \dfrac{(3N+1)(N+2)}{2}$ and $N \geq 2$, then $\sharp\ \mathcal{F}(f, \{a_j\}_{j=1}^q, 2) \leq 2$.*

In the above-mentioned theorems, there is a strong assumption on the nondegeneracy of meromorphic mappings over $\mathcal{R}(\{a_j\}_{j=1}^q)$. Thus, naturally arises the study of unicity theorems without this assumption. Inspired of the argument of [TQ1], in [CLY] the authors showed successfully a unicity theorem of such kind. We now state their result.

Let $f : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ be a nonconstant meromorphic mapping. Let $\varkappa$ be a positive integer. Let $\{a_j\}_{j=1}^q$ be "small" (with respect to $f$) meromorphic mappings of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ in general position such that $(f, a_j) \not\equiv 0$ $(1 \leq j \leq q)$ and $\dim\{z \in \mathbb{C}^n : (f, a_i)(z) = (f, a_j)(z) = 0\} \leq n - 2 \quad (1 \leq i < j \leq q)$.

Consider the set $\mathcal{G}(f, \{a_j\}_{j=1}^q, \varkappa)$ of all meromorphic maps $g : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ satisfying the conditions:

(i) $\min\,(\nu_{(f,a_j)}, \varkappa) = \min\,(\nu_{(g,a_j)}, \varkappa) \quad (1 \leq j \leq q)$,

(ii) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : (f, a_j)(z) = 0\}$.

**Theorem C.** [CLY] *If $q = 4N^2 + 2N$ and $N \geq 2$, then $\sharp\ \mathcal{G}(f, \{a_j\}_{j=1}^q, 1) = 1$.*

Therefore the following question arises naturally: Are there any unicity theorems with a truncation level to 1 in the case where $q < 4N^2 + 2N$?

It seems to us that some key techniques in their proofs in [TQ1] and [CLY] could not be used when $q < 4N^2 + 2N$. The second purpose of the present paper to give an answer to the above problem. Our approach is based on Theorem 1 on the propagation of dependences. Namely, we will prove the following.

**Theorem 2.** *Let $f_1, f_2 : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ be nonconstant meromorphic mappings. Let $g_j : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ be moving targets located in general position and*

$$T(r, g_j) = o(\max_{1\le i\le 2}\{T(r, f_i)\})\ (1 \le j \le q)$$

*and $(f_i, g_j) \not\equiv 0$ for each $1 \le i \le 2,\ 1 \le j \le q$. Assume that the following conditions are satisfied.*

*i)* $\min\{\nu_{(f_1,g_j)}, 1\} = \min\{\nu_{(f_2,g_j)}, 1\}$ *for each* $1 \le j \le q$

*ii)* $\dim\{(f_1, g_i)^{-1}(0) \cap (f_1, g_j)^{-1}(0)\} \le n - 2$ *for each* $1 \le i < j \le q$

*iii)* $f_1(z) = f_2(z)$ *for each* $z \in \cup_{j=1}^{q}(f_1, g_j)^{-1}(0)$.

*Then* $f_1 \equiv f_2$ *for each* $q > 4N^2 + 2$.

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## 2 Basic notions and auxiliary results from Nevanlinna theory

**2.1.** We set $||z|| = \big(|z_1|^2 + \cdots + |z_n|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define

$$B(r) := \{z \in \mathbb{C}^n : ||z|| < r\}, \quad S(r) := \{z \in \mathbb{C}^n : ||z|| = r\}\ (0 < r < \infty).$$

Define

$$v_{n-1}(z) := \big(dd^c||z||^2\big)^{n-1} \qquad \text{and}$$

$$\sigma_n(z) := d^c\log||z||^2 \wedge \big(dd^c\log||z||^2\big)^{n-1} \text{on} \quad \mathbb{C}^n \setminus \{0\}.$$

**2.2.** Let $F$ be a nonzero holomorphic function on a domain $\Omega$ in $\mathbb{C}^n$. For a set $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + ... + \alpha_n$ and $\mathcal{D}^\alpha F = \dfrac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1...\partial^{\alpha_n}z_n}$. We define the map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\ \{m : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < m\}\ (z \in \Omega).$$

We mean by a divisor on a domain $\Omega$ in $\mathbb{C}^n$ a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions $F$ and $G$ on a connected neighbourhood $U \subset \Omega$ of $a$ such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\le n - 2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\le n-2$. For a divisor $\nu$ on $\Omega$ we set $|\nu| := \overline{\{z : \nu(z) \ne 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of $\Omega$ or empty.

Take a nonzero meromorphic function $\varphi$ on a domain $\Omega$ in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions $F$ and $G$ on a neighbourhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on $U$ and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define the divisors $\nu_\varphi,\ \nu_\varphi^\infty$ by $\nu_\varphi := \nu_F,\ \nu_\varphi^\infty := \nu_G$, which are independent of choices of $F$ and $G$ and so globally well-defined on $\Omega$.

**2.3.** For a divisor $\nu$ on $\mathbb{C}^n$ and for a positive integer $M$ or $M = \infty$, we define the counting function of $\nu$ by

$$\nu^{(M)}(z) = \min\,\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \int\limits_{|\nu|\,\cap B(t)} \nu(z) v_{n-1} & \text{if } n \geq 2,\\ \sum\limits_{|z|\leq t} \nu(z) & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t)$.

Define

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2n-1}} dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{(M)})$ and denote them by $N^{(M)}(r, \nu)$ respectively.

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi),\ N_\varphi^{(M)}(r) = N^{(M)}(r, \nu_\varphi).$$

For brevity we will omit the character $^{(M)}$ if $M = \infty$.

**2.4.** Let $f : \mathbb{C}^n \longrightarrow \mathbf{P}^N(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbf{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each $f_i$ is a holomorphic function on $\mathbb{C}^n$ and $f(z) = \big(f_0(z) : \cdots : f_N(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension $\geq 2$. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_N|^2\big)^{1/2}$.

The characteristic function of $f$ is defined by

$$T(r, f) = \int\limits_{S(r)} \log\|f\|\sigma_n - \int\limits_{S(1)} \log\|f\|\sigma_n.$$

Let $a$ be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ with reduced representation $a = (a_0 : \cdots : a_N)$. We define

$$m_{f,a}(r) = \int\limits_{S(r)} \log\frac{||f|| \cdot ||a||}{|(f,a)|}\sigma_n - \int\limits_{S(1)} \log\frac{||f|| \cdot ||a||}{|(f,a)|}\sigma_n,$$

where $\|a\| = \big(|a_0|^2 + \cdots + |a_N|^2\big)^{1/2}$.

If $f, a : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ are meromorphic mappings such that $(f, a) \not\equiv 0$, then the first main theorem for moving targets in value distribution theory (see [RS]) states

$$T(r, f) + T(r, a) = m_{f,a}(r) + N_{(f,a)}(r).$$

**2.5.** Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^n$, which are occationally regarded as a meromorphic map into $\mathbf{P}^1(\mathbb{C})$. The proximity function of $\varphi$ is defined by

$$m(r, \varphi) := \int_{S(r)} \log \max\ (|\varphi|, 1)\sigma_n.$$

**2.6.** As usual, by the notation $''\|\ P''$ we mean the assertion $P$ holds for all $r \in [0, \infty)$ excluding a Borel subset $E$ of the interval $[0, \infty)$ with $\int_E dr < \infty$.
**2.7. The First Main Theorem for general position** (see [St2], p. 326). *Let $f_i : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})(\ 1 \leq i \leq \lambda)$ be meromorphic mappings located in general position. Assume that $1 \leq \lambda \leq N + 1$. Then*

$$N(r, \mu_{f_1 \wedge \cdots \wedge f_\lambda}) + m(r, f_1 \wedge \cdots \wedge f_\lambda) \leq \sum_{1 \leq i \leq \lambda} T(r, f_i) + O(1).$$

Let $V$ be a complex vector space of dimension $N \geq 1$. The vectors $\{v_1, \cdots, v_k\}$ are said to be in general position if for each selection of integers $1 \leq i_1 < \cdots < i_p \leq k$ with $p \leq N$, then $v_{i_1} \wedge \cdots \wedge v_{i_p} \neq 0$. The vectors $\{v_1, \cdots, v_k\}$ are said to be in special position if they are not in general position. Take $1 \leq p \leq k$. Then $\{v_1, \cdots, v_k\}$ are said to be in $p$-special position if for each selection of integers $1 \leq i_1 < \cdots < i_p \leq k$, the vectors $v_{i_1}, \cdots, v_{i_p}$ are in special position.
**2.8. The Second Main Theorem for general position** (see [St2], Theorem 2.1, p.320). *Let $M$ be a connected complex manifold of dimension $m$. Let $A$ be a pure $(m-1)$-dimensional analytic subset of $M$. Let $V$ be a complex vector space of dimension $n + 1 > 1$. Let $p$ and $k$ be integers with $1 \leq p \leq k \leq n + 1$. Let $f_j : M \to P(V), 1 \leq j \leq k$, be meromorphic mappings. Assume that $f_1, ..., f_k$ are in general position. Also assume that $f_1, ..., f_k$ are in $p$-special position on $A$. Then we have*

$$\mu_{f_1 \wedge \cdots \wedge f_k} \geq (k - p + 1)\nu_A.$$

We state here the "second main theorem type" for meromorphic mappings intersecting moving targets with truncated counting functions, which is due to Min Ru [R] and Thai-Quang [TQ3].
**2.9. The Second Main Theorem for moving target** (see [TQ3], Corollary1). *Let $f : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ be a meromorphic mapping. Let $\mathcal{A} = \{a_1, ..., a_q\}(q \geq 2N + 1)$ be a set of $q$ meromorphic mappings of $\mathbb{C}^n$ into $\mathbf{P}^N(\mathbb{C})$ located in general position such that $(f, a_i) \not\equiv 0$ for each $1 \leq i \leq q$. Then*

$$\|\quad \frac{q}{2N+1} \cdot T(r, f) \leq \sum_{i=1}^{q} N^{(N)}_{(f,a_i)}(r) + O\big(\max_{1 \leq i \leq q} T(r, a_i)\big) + O\big(\log^+ T(r, f)\big).$$

## 3 Proofs of Main Theorems

# Proof of Theorem 1

It suffices to prove Theorem 1 in the case where $\lambda \leq N+1$. By 3.3.1[PP], we easily get the following claim.

**Claim 3.1.** *Let $h_t : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ $(1 \leq t \leq p \leq N+1)$ be meromorphic mappings with reduced representations $h_t := (h_{t0} : \cdots : h_{tN})$. Let $a_i : \mathbb{C}^n \to \mathbf{P}^N(\mathbb{C})$ $(1 \leq i \leq N+1)$ be moving targets with reduced representations $a_i := (a_{i0} : \cdots : a_{iN})$. Put $\tilde{h}_t := ((h_t, a_1) : \cdots : (h_t, a_{N+1}))$ Assume that $a_1, ..., a_{N+1}$ are in general position and $(h_t, a_i) \not\equiv 0$ $(1 \leq t \leq p,\ 1 \leq i \leq N+1)$. Let $S$ be a pure $(n-1)$-dimensional analytic subset of $\mathbb{C}^n$ such that $S \not\subset (a_1 \wedge \cdots \wedge a_{N+1})^{-1}(0)$. Then $h_1 \wedge \cdots \wedge h_p = 0$ on $S$ if and only if $\tilde{h}_1 \wedge \cdots \wedge \tilde{h}_p = 0$ on $S$.*

Assume that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$.

Consider $\lambda - 1$ arbitrary moving targets $g_{i_1}, \cdots, g_{i_{\lambda-1}}$. Then there exists $g_{i_0}$ with $i_0 \notin \{i_1, ..., i_{\lambda-1}\}$ such that the matrix $A = \begin{pmatrix} (f_1, g_{i_0}) & \cdots & (f_\lambda, g_{i_0}) \\ (f_1, g_{i_1}) & \cdots & (f_\lambda, g_{i_1}) \\ \vdots & \vdots & \vdots \\ (f_1, g_{i_{\lambda-1}}) & \cdots & (f_\lambda, g_{i_{\lambda-1}}) \end{pmatrix}$ is nondegenerate.

Indeed, suppose on contrary. Then the matrix $\begin{pmatrix} (f_1, g_1) & \cdots & (f_\lambda, g_1) \\ (f_1, g_2) & \cdots & (f_\lambda, g_2) \\ \vdots & \vdots & \vdots \\ (f_1, g_q) & \cdots & (f_\lambda, g_q) \end{pmatrix} \not\equiv 0$ is of rank $\leq \lambda - 1$. By Claim 3.1, we have $f_1 \wedge \cdots \wedge f_\lambda = 0$. This is a contradiction.

Put $I = \{i_0, ..., i_{\lambda-1}\}$ and $\bar{I} = \{1, ..., q\} \setminus I$. We now prove the following.

**Claim 3.2.** *For every $z \in \mathbb{C}^n \setminus (A \cup \bigcup_{t=1}^{\lambda} I(f_t) \cup (g_{i_0} \wedge \cdots \wedge g_{i_{\lambda-1}})^{-1}(0))$, we have*

$$\sum_{i \in I} (\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, g_i)}(z)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z), 1\}) + \sum_{i \in \bar{I}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, g_i)}(z), 1\} \leq \lambda \mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z), \tag{3.3}$$

*where $\tilde{f}_t := ((f_t, g_{i_0}) : \cdots : (f_t, g_{i_{\lambda-1}}))$ and $I(f_t)$ denotes the indeterminacy locus of the meromorphic mapping $f_t$, $(1 \leq t \leq \lambda)$.*

We now prove Claim 3.2

Put $\mathcal{A} := \cup_{i \in I} (f_1, g_i)^{-1}(0), \bar{\mathcal{A}} := \cup_{i \in \bar{I}} (f_1, g_i)^{-1}(0)$ and

$$A = \cup_{1 \leq i < j \leq q} ((f_1, g_i)^{-1}(0) \cap (f_1, g_j)^{-1}(0)).$$

We distinguish two the following cases.

**Case 1.** Let $z_0 \in \mathcal{A} \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i) \cup (g_{i_0} \wedge \cdots \wedge g_{i_{\lambda-1}})^{-1}(0))$ be a regular point of $\mathcal{A}$. Then $z_0$ is a zero of one of the meromorphic functions $\{(f_t, g_i)\}_{i \in I}$. Without loss of generality, we may assume that $z_0$ is a zero of $(f_t, g_{i_0})$. Let $S$ be an irriducible analytic subset of $\mathcal{A}$ containing $z_0$. Suppose that $U$ is an open neighbourhood of $z_0$ in $\mathbb{C}^n$ such that $U \cap (\mathcal{A} \setminus S) = \emptyset$. Choose a holomorphic function $h$ on a neighbourhood $U' \subset U$ of $z_0$ such that $\nu_h = \min_{1\le t\le\lambda}\{\nu_{(f_t, g_{i_0})}\}$ if $z \in S$ and $\nu_h = 0$ if $z \notin S$. Then $(f_t, g_{i_0}) = a_t h$ $(1 \le t \le \lambda)$, where $a_t$ are holomorphic functions. Hence the matrix $\begin{pmatrix} (f_1, g_{i_1}) & \cdots & (f_\lambda, g_{i_1}) \\ \vdots & \vdots & \vdots \\ (f_1, g_{i_{\lambda-1}}) & \cdots & (f_\lambda, g_{i_{\lambda-1}}) \end{pmatrix}$ is of rank $\le \lambda - 1$. Thus, there exist $\lambda$ holomorphic functions $b_1, \cdots, b_\lambda$ such that there is at least $b_t \not\equiv 0$ and

$$\sum_{t=1}^{\lambda} b_t \cdot (f_t, g_{i_j}) = 0 \ (1 \le j \le \lambda - 1).$$

Without loss of generality, we may assume that the set of common zeros of $\{b_t\}_{t=1}^{\lambda}$ is an analytic subset of codimension $\ge 2$. Then, there exists an index $t_1, 1 \le t_1 \le \lambda$ such that $S \not\subset b_{t_1}^{-1}(0)$. We can assume that $t_1 = \lambda$.

Then for each $z \in (U' \cap S) \setminus b_\lambda^{-1}(0)$, we have

$$\begin{aligned}\tilde{f}_1(z) \wedge \cdots \wedge \tilde{f}_\lambda(z) &= \tilde{f}_1(z) \wedge \cdots \wedge \tilde{f}_{\lambda-1}(z) \wedge \left(\tilde{f}_\lambda(z) + \sum_{t=1}^{\lambda-1} \frac{b_t}{b_\lambda} \tilde{f}_t(z)\right) \\ &= \tilde{f}_1(z) \wedge \cdots \wedge \tilde{f}_{\lambda-1}(z) \wedge (V(z)h(z)) \\ &= h(z) \cdot (\tilde{f}_1(z) \wedge \cdots \wedge \tilde{f}_{\lambda-1}(z) \wedge V(z)),\end{aligned}$$

where $V(z) := (a_\lambda + \sum_{t=1}^{\lambda-1} \frac{b_t}{b_\lambda} a_t, 0, \cdots, 0)$.

By the assumption, for any increasing sequence $1 \le j_1 < \cdots < j_l \le \lambda - 1$, we have $f_{j_1} \wedge \cdots \wedge f_{j_l} = 0$ on $S$. It easily follows from Claim 3.1 that $\tilde{f}_{j_1} \wedge \cdots \wedge \tilde{f}_{j_l} = 0$ on $S$. This implies that the family $\{\tilde{f}_1, \cdots \tilde{f}_{\lambda-1}, V\}$ is in $(l+1)$-special position on $S$. By using The Second Main Theorem for general position [St2, Theorem 2.1, p.320], we have

$$\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_{\lambda-1} \wedge V}(z) \ge \lambda - l, \forall z \in S.$$

Hence $\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z) \ge \nu_h(z) + \lambda - l = \min_{1\le t\le\lambda}\{\nu_{(f_t, g_{i_0})}(z)\} + \lambda - l, \forall z \in (U \cup S) \setminus b_{i_1}^{-1}(0)$. In particular, we have

$$\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0) \ge \min_{1\le t\le\lambda}\{\nu_{(f_t, g_i)}(z_0)\} + \lambda - l.$$

This implies that

$$\begin{aligned}&\sum_{i\in I}(\lambda \min_{1\le t\le\lambda}\{\nu_{(f_t, g_i)}(z_0)\} + (\lambda - l)\sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z_0), 1\}) \\ &\qquad + \sum_{i\in \bar{I}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, g_i)}(z_0), 1\} \\ &\qquad = \lambda(\min_{1\le t\le\lambda}\{\nu_{(f_t, g_{i_0})}(z_0)\} + \lambda - l) \le \lambda \mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0).\end{aligned}$$

**Case 2.** Let $z_0 \in \bar{\mathcal{A}} \setminus (A \cup \bigcup_{t=1}^{\lambda} I(f_t) \cup \{z | g_{i_0} \wedge \cdots \wedge g_{i_{\lambda-1}}(z) = 0\})$ be a regular point of $\bar{\mathcal{A}}$. Then $z_0$ is a zero of one of meromorphic mappings $\{(f_t, g_i)\}_{i \in \bar{I}}$. By the assumption and by Claim 3.1, the family $\{\tilde{f}_1, \cdots, \tilde{f}_\lambda\}$ is in $l$-special position on an irreducible analytic subset of codimension 1 of $\bar{\mathcal{A}}$ which contains $z_0$. By using The Second Main Theorem for general position [St2, Theorem 2.1, p.320], we have

$$\mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0) \geq \lambda - l + 1.$$

Hence

$$\begin{aligned}\sum_{i \in I} (\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, g_i)}(z_0)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z_0), 1\}) \\ + \sum_{i \in \bar{I}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, g_i)}(z_0), 1\} \\ = \lambda(\lambda - l + 1) \leq \lambda \mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z_0).\end{aligned}$$

From the above cases, for each $z \notin A \cup_{t=1}^{\lambda} I(f_t) \cup (g_{i_0} \wedge \cdots \wedge g_{i_{\lambda-1}})^{-1}(0)$, we have

$$\begin{aligned}\sum_{i \in I} (\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, g_i)}(z)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z), 1\}) \\ + \sum_{i \in \bar{I}} \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, g_i)}(z), 1\} \leq \lambda \mu_{\tilde{f}_1 \wedge \cdots \wedge \tilde{f}_\lambda}(z).\end{aligned}$$

Claim 3.2 is proved.

For nonnegative integers $c_1, c_2, ..., c_\lambda$, it is easy to check that

$$\min_{1 \leq t \leq \lambda} c_t \geq \sum_{t=1}^{\lambda} \min\{c_t, N\} - (\lambda - 1)N.$$

Therefore, for every $i \in I \setminus \{i_0\}$, we have

$$\begin{aligned}\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, g_i)}(z)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z), 1\} \geq \lambda \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z), N\} \\ - ((\lambda - 1)N - \lambda + l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_i)}(z), 1\}. \qquad (3.4)\end{aligned}$$

Since $\lambda \min_{1 \leq t \leq \lambda} \{(f_t, g_{i_0})(z)\} \geq \sum_{t=1}^{\lambda} \{\nu_{(f_t, g_{i_0})}(z), 1\}$, it implies that

$$\begin{aligned}\lambda \min_{1 \leq t \leq \lambda} \{\nu_{(f_t, g_{i_0})}(z)\} + (\lambda - l) \sum_{t=1}^{\lambda} \min\{\nu_{(f_t, g_{i_0})}(z), 1\} \\ \geq \sum_{t=1}^{\lambda} (\lambda - l + 1) \min\{\nu_{(f_t, g_{i_0})}(z), 1\}. \qquad (3.5)\end{aligned}$$

Combining (3.3), (3.4) and (3.5), for every $z \in \mathbb{C}^n \setminus (A \cup \bigcup_{t=1}^{\lambda} I(f_t) \cup (g_{i_0} \wedge \cdots \wedge g_{i_{\lambda-1}})^{-1}(0))$, we have

$$\sum_{j=1}^{\lambda-1}\sum_{t=1}^{\lambda}\Big(\lambda \min\{\nu_{(f_t,g_{i_j})}(z), N\} - ((\lambda-1)N-\lambda+l)\min\{\nu_{(f_t,g_{i_j})}(z),1\}\Big)$$
$$+ \sum_{i\in\bar{I}\cup\{i_0\}}\sum_{t=1}^{\lambda}(\lambda-l+1)\min\{\nu_{(f_t,g_i)}(z),1\} \leq \lambda\mu_{\tilde{f}_1\wedge\cdots\wedge\tilde{f}_\lambda}(z)$$

This implies that

$$\sum_{j=1}^{\lambda-1}\sum_{t=1}^{\lambda}\Big(\lambda N^{(N)}_{(f_t,g_{i_j})}(r) - ((\lambda-1)N-\lambda+l)N^{(1)}_{(f_t,g_{i_j})}(r)\Big)$$
$$+ \sum_{i\in\bar{I}\cup\{i_0\}}\sum_{t=1}^{\lambda}(\lambda-l+1)N^{(1)}_{(f_t,g_i)}(r) \leq \lambda N_{\tilde{f}_1\wedge\cdots\wedge\tilde{f}_\lambda}(r) + O\Bigg(\sum_{j=0}^{\lambda} N_{g_{i_0}\wedge\cdots\wedge g_{i_{\lambda-1}}}(r)\Bigg)$$
$$= \lambda N_{\tilde{f}_1\wedge\cdots\wedge\tilde{f}_\lambda}(r) + o\Bigg(\sum_{t=1}^{\lambda} T(r,f_t)\Bigg). \tag{3.6}$$

By The First Main Theorem for general position [St2, p.326] and since $T(r,\tilde{f}_i) \leq T(r,f_i) + o(\max_{1\leq j\leq\lambda} T(r,f_j))$ $(1 \leq i \leq \lambda)$, the equality (3.6) implies that

$$\sum_{j=1}^{\lambda-1}\sum_{t=1}^{\lambda}\Big(\lambda N^{(N)}_{(f_t,g_{i_j})}(r) - ((\lambda-1)N-\lambda+l)N^{(1)}_{(f_t,g_{i_j})}(r)\Big)$$
$$+ (\lambda-l+1)\sum_{i\in\bar{I}\cup\{i_0\}}\sum_{t=1}^{\lambda}N^{(1)}_{(f_t,g_i)}(r) \leq \lambda\sum_{t=1}^{\lambda}T(r,f_t) + o(\max_{1\leq t\leq\lambda}T(r,f_t)).$$

Thus, by summing-up them over all sequences $1 \leq i_1 < \cdots < i_{\lambda-1} \leq q$, we have

$$\sum_{t=1}^{\lambda}\sum_{i=1}^{q}\Bigg(\lambda(\lambda-1)N^{(N)}_{(f_t,g_i)}(r) + \big((q-\lambda+1)(\lambda-l+1)$$
$$- (\lambda-1)((\lambda-1)N-\lambda+l)\big)N^{1}_{(f_t,g_i)}(r)\Bigg) \leq \lambda q\sum_{t=1}^{\lambda}T(r,f_t) + o(\max_{1\leq t\leq\lambda}T(r,f_t)).$$

Since $NN^{(1)}_{(f_t,g_i)}(r) \geq N^{(N)}_{(f_t,g_i)}(r)$, the above inequality implies that

$$\sum_{t=1}^{\lambda}\sum_{i=1}^{q}\Bigg(N\lambda(\lambda-1) + \big((q-\lambda+1)(\lambda-l+1)$$
$$- (\lambda-1)[(\lambda-1)N-\lambda+l]\big)\Bigg)N^{(N)}_{(f_t,g_i)}(r) \leq \lambda Nq\sum_{t=1}^{\lambda}T(r,f_t) + o(\max_{1\leq t\leq\lambda}T(r,f_t)).$$

Thus,

$$\sum_{t=1}^{\lambda}\sum_{i=1}^{q}\Big((q-\lambda+1)(\lambda-l+1)+N(\lambda-1)+(\lambda-1)(\lambda-l)\Big)N^{(N)}_{(f_t,g_i)}(r)$$
$$\leq \lambda Nq\sum_{t=1}^{\lambda}T(r,f_t)+o(\max_{1\leq t\leq\lambda}T(r,f_t)). \tag{3.7}$$

We now prove the assertions of Theorem 1.
i) By applying the Second Main Theorem for moving targets [TQ2] to the left side of (3.7), it implies that

$$\frac{q}{2N+1}\sum_{t=1}^{\lambda}\Big((q-\lambda+1)(\lambda-l+1)+N(\lambda-1)+(\lambda-1)(\lambda-l)\Big)T(r,f_i)$$
$$\leq \lambda qN\sum_{i=1}^{\lambda}T(r,f_i)+o(\max_{1\leq j\leq\lambda}T(r,f_j)).$$

Letting $r\longrightarrow+\infty$, we have

$$q\leq\lambda-1+\frac{(2N+1)N\lambda-(\lambda-1)N-(\lambda-1)(\lambda-l)}{\lambda-l+1}=\frac{(2N+1)N\lambda-(\lambda-1)(N-1)}{\lambda-l+1}.$$

This is a contradiction. Thus, we have $f_1\wedge\cdots\wedge f_\lambda\equiv 0$.
ii) By applying the Second Main Theorem for moving targets [TQ1] to the left side of (3.7), it implies that

$$\frac{q}{N+2}\sum_{t=1}^{\lambda}\Big((q-\lambda+1)(\lambda-l+1)+N(\lambda-1)+(\lambda-1)(\lambda-l)\Big)T(r,f_i)$$
$$\leq \lambda qN\sum_{i=1}^{\lambda}T(r,f_i)+o(\max_{1\leq j\leq\lambda}T(r,f_j)).$$

Letting $r\longrightarrow+\infty$, we have

$$q\leq\lambda-1+\frac{(N+2)N\lambda-(\lambda-1)N-(\lambda-1)(\lambda-l)}{\lambda-l+1}=\frac{(N+2)N\lambda-(\lambda-1)(N-1)}{\lambda-l+1}.$$

This is a contradiction. Thus, we have $f_1\wedge\cdots\wedge f_\lambda\equiv 0$.
iii) By applying the Second Main Theorem for fixed hyperplanes [TQ2] to the left side of (3.7), it implies that

$$(q-N-1)\sum_{i=1}^{\lambda}((q-\lambda+1)(\lambda-l+1)+(\lambda-1)N+(\lambda-1)(\lambda-l))T(r,f_i)$$
$$\leq \lambda qN\sum_{i=1}^{\lambda}T(r,f_i)+o(\max_{1\leq j\leq\lambda}T(r,f_j))$$

Letting $r \longrightarrow +\infty$, we have

$$(q-N-1)((\lambda-1)(N-1)+q(\lambda-l+1)) \le qN\lambda.$$

This is a contradiction. Thus, we have $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$. QED

## Proof of Theorem 2

Suppose that $f_1 \not\equiv f_2$.

By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f_1,g_1)}{(f_2,g_1)} \equiv \frac{(f_1,g_2)}{(f_2,g_2)} \equiv \cdots \equiv \frac{(f_1,g_{k_1})}{(f_2,g_{k_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f_1,g_{k_1+1})}{(f_2,g_{k_1+1})} \equiv \cdots \equiv \frac{(f_1,g_{k_2})}{(f_2,g_{k_2})}}_{\text{group 2}}$$

$$\not\equiv \underbrace{\frac{(f_1,g_{k_2+1})}{(f_2,g_{k_2+1})} \equiv \cdots \equiv \frac{(f_1,g_{k_3})}{(f_2,g_{k_3})}}_{\text{group 3}} \not\equiv \cdots \not\equiv \underbrace{\frac{(f_1,g_{k_{s-1}+1})}{(f_2,g_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f_1,g_{k_s})}{(f_2,g_{k_s})}}_{\text{group } s},$$

where $k_s = q$.

For each $1 \le i \le q$, we set

$$\sigma(i) = \begin{cases} i+N & \text{if } i+N \le q, \\ i+N-q & \text{if } i+N > q. \end{cases}$$

and

$$P_i = \frac{(f_1,g_i)}{(f_1,g_i)} - \frac{(f_2,g_{\sigma(i)})}{(f_2,g_{\sigma(i)})}.$$

The number of elements of every group is at most $N$ since $f_1 \not\equiv f_2$. Thus $\dfrac{(f_1,g_i)}{(f_2,g_i)}$ and $\dfrac{(f_1,g_{\sigma(i)})}{(f_2,g_{\sigma(i)})}$ belong to distinct groups. This means that $\dfrac{(f_1,g_i)}{(f_2,g_i)} \not\equiv \dfrac{(f_1,g_{\sigma(i)})}{(f_2,g_{\sigma(i)})}$. Hence $P_i \not\equiv 0$ for each $1 \le i \le q$.

By Claim 3.2, for $\lambda = l = 2$ and $i_0 = \sigma(i)$, we have

$$\sum_{j=i,\sigma(i)} 2\min_{1\le t\le 2}\{\nu_{(f_t,g_j)}(z)\} + \sum_{\substack{j=1\\ j\ne i,\sigma(i)}}^{q} \sum_{t=1}^{2} \min\{\nu_{(f_t,g_j)}(z),1\} \le 2\mu_{\tilde f_1 \wedge \tilde f_2}(z), \tag{3.8}$$

for every $z \notin \mathcal{A} = I(f_1) \cup I(f_2) \cup \bigcup_{1\le s<t\le q}((f_1,g_s)^{-1}(0) \cup (f_1,g_t)^{-1}(0)) \cup (g_i \wedge g_{\sigma(i)})^{-1}(0)$, where $\tilde f_t := ((f_t,g_i):(f_t,g_{\sigma(i)}))$ $(1 \le t \le 2)$.

Since $2\min_{1\le t\le 2}\{\nu_{(f_t,g_j)}(z)\} \ge 2\sum_{t=1}^{2}\min\{\nu_{(f_t,g_j)}(z),N\} - N\sum_{t=1}^{2}\min\{\nu_{(f_t,g_j)}(z),1\}$, the inequality (3.8) implies that

$$\sum_{j=i,\sigma(i)}\left(2\sum_{t=1}^{2}\min\{\nu_{(f_t,g_j)}(z),N\}-N\sum_{t=1}^{2}\min\{\nu_{(f_t,g_j)}(z),1\}\right)$$
$$+\sum_{\substack{j=1\\ j\neq i,\sigma(i)}}^{q}\sum_{t=1}^{2}\min\{\nu_{(f_t,g_j)}(z),1\}\leq 2\mu_{\tilde{f}_1\wedge\tilde{f}_2}(z),$$

for every $z\notin\mathcal{A}=I(f_1)\cup I(f_2)\cup\bigcup_{1\leq s<t\leq q}((f_1,g_s)^{-1}(0)\cup(f_1,g_t)^{-1}(0))\cup(g_i\wedge g_{\sigma(i)})^{-1}(0)$. Repeating the argument in the proof of Theorem 1, the above inequality yields that

$$\sum_{j=i,\sigma(i)}\left(2\sum_{t=1}^{2}N^{(N)}_{(f_t,g_i)}(r)-N\sum_{t=1}^{2}N^{(1)}_{(f_t,g_j)}\right)+\sum_{\substack{j=1\\ i\neq i,\sigma(i)}}^{q}\sum_{t=1}^{2}N^{(1)}_{(f_t,g_i)}(r)\leq 2\mu_{\tilde{f}_1\wedge\tilde{f}_2}(r)+o(\sum_{t=1}^{2}T(r,f_t)).$$

Thus, by summing them up over $i$, we have

$$\sum_{t=1}^{2}\sum_{i=1}^{q}\left(4N^{(N)}_{(f_t,g_i)}(r)+(q-2N-2)N^{(1)}_{(f_t,g_i)}(r)\right)\leq 2q\sum_{t=1,2}T(r,f_t)+o(\max_{1\leq t\leq 2}\{T(r,f_t)\}).$$

Hence

$$\sum_{t=1}^{2}\sum_{i=1}^{q}(q+2N-2)N^{(N)}_{(f_t,g_i)}(r)\leq 2qN\sum_{t=1,2}T(r,f_t)+o(\max_{1\leq i\leq 2}\{T(r,f_t)\}).$$

By applying the Second Main Theorem for moving targets [TQ3, Corollary 1] to the left side of the above inequality, we get

$$\sum_{t=1}^{2}\frac{q}{2N+1}(q+2N-2)T(r,f_t)\leq 2qN\sum_{t=1}^{2}T(r,f_t)+o(\max_{1\leq i\leq 2}\{T(r,f_t)\}).$$

Letting $r\to+\infty$, we have

$$q\leq 4N^2+2.$$

This is a contradiction. Thus, we have $f_1\equiv f_2$.

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