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On the Eneström-Kayeya theorem

by

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Abstract

In this paper, we relax the hypothesis and generalize some results concerning the Eneström-Kakeya theorem. The results so obtained considerably improve the bounds in some cases.

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1. Introduction and Statement of Results

If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0 > 0,$$

then $P(z)$ does not vanish in $|z| > 1$. This is a famous result in the distribution of zeros of a polynomial and is known as Eneström-Kakeya Theorem [5, 4, 6-9].

If we apply this result to the polynomial $P(tz)$, $t > 0$, then it can be restated as:

Theorem A. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n t^n \geq a_{n-1} t^{n-1} \geq a_{n-2} t^{n-2} \geq ... \geq a_1 t \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq t$.

By using Schwarz's Lemma, Aziz and Mohammad [1] generalized Eneström-Kakeya theorem in a different way and proved the following :

Theorem B. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real positive coefficients. If $t_1 > t_2 \geq 0$ can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \; r = 1, 2, ..., n+1 \qquad (a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ *lie in* $|z| \leq t_1$.

A result of this type was earlier proved by Egervary [3].
Aziz and Zargar [2] relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following result:

Theorem C. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $k \geq 1$,

$$ka_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leq k.$$

While studying Theorem C, a natural question arises that what happens if we relax the hypothesis of Theorem B in a similar way and only assume that

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0,\ r = 2, 3, ..., n.$$

In this paper, we study such a case and prove a more general result from which many known results follow on a fairly uniform procedure. In fact we prove:

Theorem 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that* $a_j = \alpha_j + i\beta_j$, *where* α_j *and* β_j, *j=0,1,2,...,n are real numbers. If* $t_1 > t_2 \geq 0$ *can be found such that for r= 2,3,...,n*

$$\alpha_r t_1 t_2 + \alpha_{r-1}(t_1 - t_2) - \alpha_{r-2} \geq 0,$$
$$\beta_r t_1 t_2 + \beta_{r-1}(t_1 - t_2) - \beta_{r-2} \geq 0,$$

and for r=n+1, there exists some $k = k_1 + ik_2$ *such that*

$$(\alpha_n + k_1)(t_1 - t_2) - \alpha_{n-1} \geq 0,$$
$$(\beta_n + k_2)(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all the zeros of $P(z)$ *lie in* $|z + \frac{k(t_1-t_2)}{a_n}| \leq R$,

where

$$R = \frac{1}{|a_n|}\{[(\alpha_n + k_1) + (\beta_n + k_2)](t_1 - t_2) + (\alpha_n + \beta_n)t_2 - (\alpha_1 + \beta_1)\frac{t_2}{t_1^{n-1}} - (\alpha_0 + \beta_0)\frac{1}{t_1^{n-1}}$$
$$+(|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|\alpha_0| + |\beta_0|)\frac{t_2}{t_1^n}\}.$$

Remark 1. For any real number $\lambda \geq 1$, if we take $k = a_n(\lambda - 1)$ so that $k_1 = \alpha_n(\lambda - 1)$ and $k_2 = \beta_n(\lambda - 1)$, we immediately have the following:

Corollary 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that* $a_j = \alpha_j + i\beta_j$, *where* α_j *and* β_j, *j=0,1,2,...,n are real numbers. If* $t_1 > t_2 \geq 0$ *can be found such that for r= 2,3,...,n*

$$\alpha_r t_1 t_2 + \alpha_{r-1}(t_1 - t_2) - \alpha_{r-2} \geq 0,$$
$$\beta_r t_1 t_2 + \beta_{r-1}(t_1 - t_2) - \beta_{r-2} \geq 0,$$

and for some $\lambda \geq 1$

$$\lambda\alpha_n(t_1 - t_2) - \alpha_{n-1} \geq 0,$$
$$\lambda\beta_n(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all the zeros of $P(z)$ *lie in* $|z + (\lambda - 1)(t_1 - t_2)| \leq R^*$,

where

$$R^* = \frac{1}{|a_n|}\{\lambda(\alpha_n + \beta_n)(t_1 - t_2) + (\alpha_n + \beta_n)t_2 - (\alpha_1 + \beta_1)\frac{t_2}{t_1^{n-1}} - (\alpha_0 + \beta_0)\frac{1}{t_1^{n-1}}$$
$$+(|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|\alpha_0| + |\beta_0|)\frac{t_2}{t_1^n}\}.$$

The following interesting result also follows from Theorem 1, if we assume that k and all a_j, $j = 0, 1, 2, ..., n$ are real.

Corollary 2. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \text{ for } r = 2,3,\ldots,n$$

and for some real number $k \geq 0$,

$$(k + a_n)(t_1 - t_2) - a_{n-1} \geq 0,$$

then all the zeros of $P(z)$ *lie in* $|z + \frac{k(t_1 - t_2)}{a_n}| \leq R_1$,

where

$$R_1 = \frac{1}{|a_n|}\{(k + a_n)(t_1 - t_2) + a_n t_2 - a_1\frac{t_2}{t_1^{n-1}} - a_0\frac{1}{t_1^{n-1}} + |a_1 t_1 t_2 + a_0(t_1 - t_2)|\frac{1}{t_1^n} + |a_0|\frac{t_2}{t_1^n}\}.$$

Remark 2. If in particular $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real and positive coefficients satisfying the conditions of Corollary 2, then all the zeros of $P(z)$ lie in

$$|z + \frac{k(t_1 - t_2)}{a_n}| \leq t_1 + \frac{k(t_1 - t_2)}{a_n}.$$

Further, in Remark 2, if we take $k = a_n(\lambda - 1)$ so that $\lambda \geq 1$, we get the following:

Corollary 3. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real and positive coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \; r = 2,3,\ldots,n$$

and for some $\lambda \geq 1$,

$$\lambda a_n(t_1 - t_2) - a_{n-1} \geq 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+(\lambda-1)(t_1-t_2)| \leq \lambda t_1-(\lambda-1)t_2.$$

Theorem B is a special case of Corollary 3 when $\lambda = 1$. This Theorem also follows from Remark 2, when k=0.

In Theorem 1, if we assume that $t_2 = 0$, then we have the following:

Corollary 4. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j, j=0,1,2,...,n are real numbers. If $t > 0$ can be found such that for some $k = k_1 + ik_2$*

$$(k_1+\alpha_n)t^n \geq \alpha_{n-1}t^{n-1} \geq \alpha_{n-2}t^{n-2} \geq ... \geq \alpha_1 t \geq \alpha_0,$$
$$(k_2+\beta_n)t^n \geq \beta_{n-1}t^{n-1} \geq \beta_{n-2}t^{n-2} \geq ... \geq \beta_1 t \geq \beta_0,$$

then all the zeros of $P(z)$ lie in

$$|z+\tfrac{kt}{a_n}| \leq R_2,$$

where

$$R_2 = \tfrac{t}{|a_n|}\{(k_1+\alpha_n)+(k_2+\beta_n)-\tfrac{1}{t^n}[\alpha_0+\beta_0-|\alpha_0|-|\beta_0|]\}.$$

In Corollary 4, if we further assume that all the coefficients of $P(z)$ are real, then $\beta_j = 0$, $j = 0, 1, 2, .., n$ and we get the following:

Corollary 5. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients and for any $t > 0$, there exists some $k \geq 0$, such that*

$$(k+a_n)t^n \geq a_{n-1}t^{n-1} \geq a_{n-2}t^{n-2} \geq ... \geq a_1 t \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z+\tfrac{kt}{a_n}| \leq \tfrac{t}{|a_n|}\{(k+a_n)-\tfrac{1}{t^n}(a_0-|a_0|)\}.$$

Remark 3. Theorem C is a special case of Corollary 5, if we take $k = (\lambda-1)a_n$, $t = 1$ and $a_0 \geq 0$.

Finally, assuming the hypothesis of Theorem 1, we can write the disc containing all the zeros of the polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ as

$$|z+\tfrac{(k_1+ik_2)(t_1-t_2)}{a_n}| \leq R$$

If we replace k_1 by $(u-1)\alpha_n$ and k_2 by $(v-1)\beta_n$, we immediately get the following result:

Corollary 6. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j, j=0,1,2,...,n are real numbers. If $t_1 > t_2 \geq 0$ can be found such that for r= 2,3,...,n*

$$\alpha_r t_1 t_2 + \alpha_{r-1}(t_1-t_2) - \alpha_{r-2} \geq 0,$$
$$\beta_r t_1 t_2 + \beta_{r-1}(t_1-t_2) - \beta_{r-2} \geq 0,$$

and for some real numbers u and v, $u \geq 1$, $v \geq 1$ such that

$$u\alpha_n(t_1 - t_2) - \alpha_{n-1} \geq 0,$$
$$v\beta_n(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all the zeros of $P(z)$ lie in $|z + (t_1 - t_2)(\frac{u\alpha_n + iv\beta_n}{a_n} - 1)| \leq R_1^*$,

where

$$R_1^* = \frac{1}{|a_n|}\{(u\alpha_n + v\beta_n)(t_1 - t_2) + (\alpha_n + \beta_n)t_2 - (\alpha_1 + \beta_1)\frac{t_2}{t_1^{n-1}} - (\alpha_0 + \beta_0)\frac{1}{t_1^{n-1}}$$
$$+(|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|\alpha_0| + |\beta_0|)\frac{t_2}{t_1^n}\}.$$

If in Corollary 6, we take $u = \frac{\alpha_{n-1}}{\alpha_n(t_1 - t_2)}$ and $v = \frac{\beta_{n-1}}{\beta_n(t_1 - t_2)}$, so that $u \geq 1, v \geq 1$, we get the following:

Corollary 7. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j, j=0,1,2,...,n are real numbers. If $t_1 > t_2 \geq 0$ can be found such that*

$$\alpha_r t_1 t_2 + \alpha_{r-1}(t_1 - t_2) - \alpha_{r-2} \geq 0, \ r=2,3,\ldots,n$$

$$\leq 0, \ r=n+1$$

$$\beta_r t_1 t_2 + \beta_{r-1}(t_1 - t_2) - \beta_{r-2} \geq 0, \ r=2,3,\ldots,n$$

$$\leq 0, \ r=n+1,$$

then all the zeros of $P(z)$ lie in $|z + \frac{a_{n-1}}{a_n} - (t_1 - t_2)| \leq R_2^*$,

where

$$R_2^* = \frac{1}{|a_n|}\{(\alpha_n + \beta_n)t_2 + (\alpha_{n-1} + \beta_{n-1}) - (\alpha_1 + \beta_1)\frac{t_2}{t_1^{n-1}} - (\alpha_0 + \beta_0)\frac{1}{t_1^{n-1}}$$
$$+|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|\frac{1}{t_1^n} + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|\frac{1}{t_1^n} + (|\alpha_0| + |\beta_0|)\frac{t_2}{t_1^n}\}.$$

In particular, if

$$\alpha_r t_1 t_2 + \alpha_{r-1}(t_1 - t_2) - \alpha_{r-2} \geq 0, \text{ r=1,2,...,n}$$

$$\leq 0, \text{ r=n+1},$$

$$\beta_r t_1 t_2 + \beta_{r-1}(t_1 - t_2) - \beta_{r-2} \geq 0, \text{ r=1,2,...,n}$$

$$\leq 0, \text{ r=n+1},$$

then

$$\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2) \geq 0,$$
$$\beta_1 t_1 t_2 + \beta_0(t_1 - t_2) \geq 0$$

and we get in this case all the zeros of $P(z)$ lie in

$$|z+\frac{a_{n-1}}{a_n}-(t_1-t_2)|\leq\frac{1}{|a_n|}\{(\alpha_n+\beta_n)t_2+(\alpha_{n-1}+\beta_{n-1})\}.$$

Remark 4. A result of Shah and Liman [7, Theorem 1] is a special case of Corollary 7, if we assume that all the coefficients of $P(z)$ are real.
Many other known results and generalizations similarly follows from Theorem 1 with suitable substitutions and we leave to the readers.

2. Proof of the Theorem 1

Consider the polynomial
$f(z)=(t_2+z)(t_1-z)P(z)$

$$\begin{aligned}&=-a_nz^{n+2}+(a_n(t_1-t_2)-a_{n-1})z^{n+1}+(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2})z^n+...\\&\quad+(a_2t_1t_2+a_1(t_1-t_2)-a_0)z^2+(a_1t_1t_2+a_0(t_1-t_2))z+a_0t_1t_2\end{aligned}$$

$$\begin{aligned}&=-a_nz^{n+2}-k(t_1-t_2)z^{n+1}+((k+a_n)(t_1-t_2)-a_{n-1})z^{n+1}\\&\quad+(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2})z^n+...\\&\quad+(a_2t_1t_2+a_1(t_1-t_2)-a_0)z^2+(a_1t_1t_2+a_0(t_1-t_2))z+a_0t_1t_2\end{aligned}$$

$$\begin{aligned}&=-a_nz^{n+2}-k(t_1-t_2)z^{n+1}+((k_1+\alpha_n)(t_1-t_2)-\alpha_{n-1})z^{n+1}\\&\quad+(\alpha_nt_1t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2})z^n+...\\&\quad+(\alpha_2t_1t_2+\alpha_1(t_1-t_2)-\alpha_0)z^2+(\alpha_1t_1t_2+\alpha_0(t_1-t_2))z+\alpha_0t_1t_2\\&\quad+i[((k_2+\beta_n)(t_1-t_2)-\beta_{n-1})z^{n+1}+(\beta_nt_1t_2+\beta_{n-1}(t_1-t_2)-\beta_{n-2})z^n+...\\&\quad+(\beta_2t_1t_2+\beta_1(t_1-t_2)-\beta_0)z^2+(\beta_1t_1t_2+\beta_0(t_1-t_2))z+\beta_0t_1t_2].\end{aligned}$$

This gives

$$\begin{aligned}|f(z)|&\geq|a_n||z|^{n+1}|z+\frac{k(t_1-t_2)}{a_n}|-|(k_1+\alpha_n)(t_1-t_2)-\alpha_{n-1}||z|^{n+1}\\&\quad-|\alpha_nt_1t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}||z|^n-...\\&\quad-|\alpha_2t_1t_2+\alpha_1(t_1-t_2)-\alpha_0||z|^2-|\alpha_1t_1t_2+\alpha_0(t_1-t_2)||z|-|\alpha_0t_1t_2|\\&\quad-[|(k_2+\beta_n)(t_1-t_2)-\beta_{n-1}||z|^{n+1}+|\beta_nt_1t_2+\beta_{n-1}(t_1-t_2)-\beta_{n-2}||z|^n+...\\&\quad+|\beta_2t_1t_2+\beta_1(t_1-t_2)-\beta_0||z|^2+|\beta_1t_1t_2+\beta_0(t_1-t_2)||z|+|\beta_0t_1t_2|]\end{aligned}$$

$$\begin{aligned}&=|z|^{n+1}\{|z+\frac{k(t_1-t_2)}{a_n}||a_n|-(|(k_1+\alpha_n)(t_1-t_2)-\alpha_{n-1}|\\&\quad+|(k_2+\beta_n)(t_1-t_2)-\beta_{n-1}|)-(|\alpha_nt_1t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}|\\&\quad+|\beta_nt_1t_2+\beta_{n-1}(t_1-t_2)-\beta_{n-2}|)\frac{1}{|z|}-...\\&\quad-(|\alpha_2t_1t_2+\alpha_1(t_1-t_2)-\alpha_0|+|\beta_2t_1t_2+\beta_1(t_1-t_2)-\beta_0|)\frac{1}{|z|^{n-1}}\\&\quad-(|\alpha_1t_1t_2+\alpha_0(t_1-t_2)|+|\beta_1t_1t_2+\beta_0(t_1-t_2)|)\frac{1}{|z|^n}-(|\alpha_0t_1t_2|+|\beta_0t_1t_2|)\frac{1}{|z|^{n+1}}\}.\end{aligned}$$

For $|z|>t_1$, we have by using hypothesis

$$\begin{aligned}|f(z)|&\geq|z|^{n+1}\{|z+\frac{k(t_1-t_2)}{a_n}||a_n|-(|(k_1+\alpha_n)(t_1-t_2)-\alpha_{n-1}|\\&\quad+|(k_2+\beta_n)(t_1-t_2)-\beta_{n-1}|)-(|\alpha_nt_1t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}|\\&\quad+|\beta_nt_1t_2+\beta_{n-1}(t_1-t_2)-\beta_{n-2}|)\frac{1}{t_1}-...\\&\quad-(|\alpha_2t_1t_2+\alpha_1(t_1-t_2)-\alpha_0|+|\beta_2t_1t_2+\beta_1(t_1-t_2)-\beta_0|)\frac{1}{t_1^{n-1}}\\&\quad-(|\alpha_1t_1t_2+\alpha_0(t_1-t_2)|+|\beta_1t_1t_2+\beta_0(t_1-t_2)|)\frac{1}{t_1^n}-(|\alpha_0t_1t_2|+|\beta_0t_1t_2|)\frac{1}{t_1^{n+1}}\}>0,\end{aligned}$$

if

$|z+\frac{k(t_1-t_2)}{a_n}||a_n| > (k_1+\alpha_n)(t_1-t_2)+(k_2+\beta_n)(t_1-t_2)+\alpha_n t_2+\beta_n t_2$
$-\alpha_1\frac{t_2}{t_1^{n-1}}-\beta_1\frac{t_2}{t_1^{n-1}}-\frac{\alpha_0}{t_1^{n-1}}-\frac{\beta_0}{t_1^{n-1}}+(|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|+|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|)\frac{1}{t_1^n}+|\alpha_0|\frac{t_2}{t_1^n}+|\beta_0|\frac{t_2}{t_1^n}.$

Therefore, for $|z|\geq t_1$, $|f(z)|>0$, if

$|z+\frac{k(t_1-t_2)}{a_n}| > \frac{1}{|a_n|}\{[(\alpha_n+k_1)+(\beta_n+k_2)](t_1-t_2)+(\alpha_n+\beta_n)t_2-(\alpha_1+\beta_1)\frac{t_2}{t_1^{n-1}}$
$-(\alpha_0+\beta_0)\frac{1}{t_1^{n-1}}+(|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|+|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|)\frac{1}{t_1^n}+(|\alpha_0|+|\beta_0|)\frac{t_2}{t_1^n}\}.$

Hence all the zeros of $f(z)$ whose modulus is greater than t_1 lie in the circle

$|z+\frac{k(t_1-t_2)}{a_n}| \leq \frac{1}{|a_n|}\{[(\alpha_n+k_1)+(\beta_n+k_2)](t_1-t_2)+(\alpha_n+\beta_n)t_2-(\alpha_1+\beta_1)\frac{t_2}{t_1^{n-1}}$
$-(\alpha_0+\beta_0)\frac{1}{t_1^{n-1}}+(|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|+|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|)\frac{1}{t_1^n}+(|\alpha_0|+|\beta_0|)\frac{t_2}{t_1^n}\}.$

Since those zeros of $f(z)$ whose modulus is less than t_1 already lie in this circle, we conclude that all the zeros of $f(z)$ and therefore $P(z)$ lie in

$|z+\frac{k(t_1-t_2)}{a_n}| \leq \frac{1}{|a_n|}\{[(\alpha_n+k_1)+(\beta_n+k_2)](t_1-t_2)+(\alpha_n+\beta_n)t_2-(\alpha_1+\beta_1)\frac{t_2}{t_1^{n-1}}$
$-(\alpha_0+\beta_0)\frac{1}{t_1^{n-1}}+(|\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)|+|\beta_1 t_1 t_2+\beta_0(t_1-t_2)|)\frac{1}{t_1^n}+(|\alpha_0|+|\beta_0|)\frac{t_2}{t_1^n}\}.$

This proves Theorem 1 completely.

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