Bull. Math. Soc. Sci. Math. Roumanie
Tome 56(104) No. 4, 2013, 497–503

Cohen-Macaulay binomial edge ideals of small deviation

by
Giancarlo Rinaldo

Abstract

We classify all binomial edge ideals that are complete intersection and Cohen-Macaulay almost complete intersection. We also describe an algorithm and provide an implementation to compute the primary decomposition of binomial edge ideals.

Key Words: Binomial ideal, unmixed, Cohen-Macaulay.
2010 Mathematics Subject Classification: Primary 13F55; Secondary 13H10.

Introduction

In 2010, binomial edge ideals were introduced in [4] and appeared independently also in [6]. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables with coefficients in a field K. Let G be a graph on vertex set $[n]$. For each edge $\{i, j\}$ of G with $i < j$, we associate a binomial $f_{ij} = x_i y_j - x_j y_i$. The ideal J_G of S generated by $f_{ij} = x_i y_j - x_j y_i$ such that $i < j$, is called *the binomial edge ideal* of G. Any ideal generated by a set of 2-minors of a $2 \times n$-matrix of indeterminates may be viewed as the binomial edge ideal of a graph.

Algebraic properties of binomial edge ideals in terms of properties of the underlying graph were studied in [4], [2], [5] and [7]. In [5] and [7] the authors considered the Cohen-Macaulay property of these graphs. However, the classification of Cohen-Macaulay binomial edge ideals in terms of the underlying graphs is still widely open and, as in the case of monomial edge ideals introduced in [9], it seems rather hopeless to give a full classification.

In this paper we consider Cohen-Macaulay and unmixed binomial edge ideals J_G with small deviation, namely the difference between the minimum number of the generators and the height of J_G is less than or equal to 2.

Section 1 contains some preliminaries and notions that we use in the paper. In the beginning of Section 2 we give a complete classification of the complete intersection binomial edge ideals (Theorem 1), that is the case of deviation 0. This result is a consequence of Corollary 1.2 of [5]. We also observe that in general the almost complete intersection, namely deviation 1, binomial edge ideals are not unmixed. A nice example is the claw graph (see Example 1). In Theorem 2 we give a complete classification of Cohen-Macaulay binomial edge ideals that are almost complete intersection and we show that this set coincides with the set of unmixed binomial edge ideals that are almost complete intersection.

In Section 3 we describe an algorithm to compute the primary decomposition of J_G and provide an implementation in CoCoA (see [1]) that is freely downloadable (see [8]). Thanks to this we computed Examples 2 and 3 that are unmixed binomial edge ideals of deviation 2 that are not Cohen-Macaulay.

1 Preliminaries

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset C of $V(G)$ is called a *clique* of G if for all i and j belonging to C with $i \neq j$ one has $\{i,j\} \in E(G)$. A vertex of a graph is called a *cutpoint* if the removal of the vertex increases the number of connected components. A vertex v is a cutpoint of a graph G if and only if there exist $u, w \in V(G)$ such that v is in every path connecting u and w (see Theorem 3.1 of [3]). A subgraph H of G *spans* G if $V(H) = V(G)$. In a connected graph G a *chord* of a tree T that spans G is an edge of G not in T. The number of chords of any spanning tree of a connected graph G, namely $m(G)$, is called the cycle rank of G and is $m(G) = |E(G)| - |V(G)| + 1$ (see Corollary 4.5(a) of [3]). If G has c components then $m(G) = |E(G)| - |V(G)| + c$ (see Corollary 4.5(b) of [3]).

Let $v \notin V(G)$. The *cone* of v on G, namely $\mathrm{cone}(v, G)$, is the graph with vertices $V(G) \cup \{v\}$ and edges $E(G) \cup \{\{u,v\} : u \in V(G)\}$.

Let G_1 and G_2 be graphs. We set $G = G_1 \cup G_2$ (resp. $G = G_1 \sqcup G_2$ where $\sqcup$ is disjoint union) where G is the graph with $V(G) = V(G_1) \cup V(G_2)$ (resp. $V(G) = V(G_1) \sqcup V(G_2)$) and $E(G) = E(G_1) \cup E(G_2)$ (resp. $E(G) = E(G_1) \sqcup E(G_2)$).

Set $V = \{x_1, \dots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$;

(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. A vertex i of Δ is called a free vertex of Δ if i belongs to exactly one facet.

If Δ is a simplicial complex with facets $F_1, \dots, F_q$, we call $\{F_1, \dots, F_q\}$ the facet set of Δ and we denote it by $\mathcal{F}(\Delta)$.

The *clique complex* $\Delta(G)$ of G is the simplicial complex whose faces are the cliques of G. Hence a vertex v of a graph G is called *free vertex* if it belongs to only one clique of $\Delta(G)$.

We need notations and results from [4] (section 3) that we recall for the sake of completeness.

Let $T \subseteq [n]$, and let $\overline{T} = [n] \setminus T$. Let $G_1, \dots, G_{c(T)}$ be the connected components of the induced subgraph on $\overline{T}$, namely $G_{\overline{T}}$. For each G_i, denote by $\widetilde{G}_i$ the complete graph on the vertex set $V(G_i)$. We set

$$P_T(G) = (\bigcup_{i \in T} \{x_i, y_i\}, J_{\widetilde{G}_1}, \dots, J_{\widetilde{G}_{c(T)}}), \tag{1.1}$$

$P_T(G)$ is a prime ideal. Then J_G is a radical ideal and

$$J_G = \bigcap_{T \subset [n]} P_T(G)$$

is its primary decomposition (see Corollary 2.2 and Theorem 3.2 of [4]). If there is no possible confusion, we write simply P_T instead of $P_T(G)$. Moreover, $\operatorname{height} P_T = n + |T| - c(T)$ (see [4, Lemma 3.1]). We denote by $\mathcal{M}(G)$ the set of minimal prime ideals of J_G.

If each $i \in T$ is a cutpoint of the graph $G_{\overline{T} \cup \{i\}}$, then we say that T has the *cutpoint property for G*. We denote by $\mathcal{C}(G)$ the set of all $T \subset V(G)$ such that T has the cutpoint property for G.

Lemma 1. *[4] $P_T(G) \in \mathcal{M}(G)$ if and only if $T \in \mathcal{C}(G)$.*

Lemma 2. *[7] Let G be a connected graph. Then J_G is unmixed if and only if for all $T \in \mathcal{C}(G)$ we have $c(T) = |T| + 1$.*

2 Complete intersection and almost complete intersection

Let S be a standard graded polynomial ring over a field K. For a homogeneous ideal $J \subseteq S$, J is called a complete intersection ideal (resp. an almost complete intersection ideal) if J is minimally generated by height J (resp. height $J+1$) elements. A homogeneous ideal has deviation $\mathrm{dev}(J)$ if it is minimally generated by height $I + \mathrm{dev}(J)$ elements. Throughout this section let $S = K[\{x_i, y_i\} : i \in V(G)]$, J_G be the binomial edge ideal of a graph G and $\mu(J_G)$ the minimal number of generators of J_G.

A nice combinatorial interpretation of $\mathrm{dev}(J_G)$ is given by the following

Remark 1. *Suppose that* height $J_G =$ height $P_\emptyset$. *Then* height $J_G = n - c$ *where c are the connected components of G. Therefore*

$$\mathrm{dev}(J_G) = \mu(J_G) - n + c = m(G).$$

Theorem 1. *Let G be a graph. Then J_G is complete intersection if and only if each component of G is a path graph.*

Proof: Let $G = \bigcup_{i=1}^{c} G_i$ where G_i are the connected components of G. Let $n_i = |V(G_i)|$ for $i = 1, \dots, c$. Since G_i is connected it has at least $n_i - 1$ edges, namely the number of edges of a tree. Hence

$$\mu(J_G) \geq \sum_{i=1}^{c} (n_i - 1) = n - c.$$

Since J_G is a complete intersection then it is unmixed, hence height $J_G =$ height $P_\emptyset$. By Remark 1 and since complete intersection implies $\mathrm{dev}(J_G) = 0$ we obtain that $\mu(J_G) = n - c$. Therefore every connected component G_i is a tree. Now the proof is a consequence of Corollary 1.2 of [5]. □

In general almost complete intersection binomial edge ideals are not unmixed as the following example shows.

Example 1. *Let G be the graph on 4 vertices and edges*

$$\{\{1,2\}, \{1,3\}, \{1,4\}\},$$

namely the claw graph. We observe that

$$J_G = P_\emptyset \cap P_{\{1\}}$$

where height $P_\emptyset = 3$ *and* height $P_{\{1\}} = 2$.

Remark 2. *If J_G is an unmixed almost complete intersection binomial edge ideal with c components we have that G has $c-1$ components that are path graphs and 1 that contains only one cycle, namely a unicyclic graph. The proof is similar to the proof of Theorem 1.*

Thanks to Remark 2 we assume from now on that G is a connected unicyclic graph.

Proposition 1. *Let $\mathcal{G}_3$ be the set of graphs such that for all $G \in \mathcal{G}_3$ we have*

$$V(G) = \{u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t\}$$

with $r \geq 1$, $s \geq 1$, $t \geq 1$ and edge set

$$\begin{aligned} E(G) = &\{\{u_i, u_{i+1}\} : i = 1, \dots, r-1\} \cup \{v_i, v_{i+1}\} : i = 1, \dots, s-1\} \cup \\ &\cup \{w_i, w_{i+1}\} : i = 1, \dots, t-1\} \cup \{\{u_1, v_1\}, \{u_1, w_1\}, \{v_1, w_1\}\}. \end{aligned}$$

Then S/J_G is Cohen-Macaulay for all $G \in \mathcal{G}_3$.

Proof: If $r = s = t = 1$ then G is a complete graph, hence it is Cohen-Macaulay. Since u_1, v_1 and w_1 are free vertices in $\Delta(G)$, by Theorem 2.7 of [7] the assertion follows easily. $\square$

Lemma 3. *Let G be the graph with vertex set $V(G) = \{1, \dots, 6\}$ and edge set*

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}.$$

Then S/J_G is Cohen-Macaulay.

Proof: Let H_1 be the graph with one vertex, $V(H_1) = \{1\}$, H_2 the path graph with edges $\{3, 4\}$ and $\{4, 5\}$ and let $G_1 = \text{cone}(2, H_1 \sqcup H_2)$. By Theorem 3.8 of [7] G_1 is Cohen-Macaulay. Now let G_2 be the complete graph on the vertices $V(G_2) = \{5, 6\}$. Then 5 is a free vertex in $\Delta(G_1)$ and $\Delta(G_2)$. By Theorem 2.7 of [7] the assertion follows. $\square$

Proposition 2. *Let $\mathcal{G}_4$ be the set of graphs such that for all $G \in \mathcal{G}_4$ we have*

$$V(G) = \{u_1, u_2, u_3, \dots, u_r, v_1, v_2, v_3, \dots, v_s\}$$

with $r \geq 3$ and $s \geq 3$ and edge set

$$\begin{aligned} E(G) =& \{\{u_i, u_{i+1}\} : i = 1, \dots, r-1\} \cup \{\{v_i, v_{i+1}\} : i = 1, \dots, s-1\} \cup \\ & \cup \{\{u_1, v_1\}, \{u_2, v_2\}\}. \end{aligned}$$

Then S/J_G is Cohen-Macaulay for all $G \in \mathcal{G}_4$.

Proof: Let $s = r = 3$. We observe that

$$\mathcal{C}(G) = \{\emptyset, \{u_2\}, \{v_2\}, \{u_2, v_1\}, \{u_1, v_2\}, \{u_2, v_2\}\}$$

and $J_G \subset S = K[\{x_i, y_i\} : i \in V(G)]$ is unmixed with $\dim S/J_G = 7$. We need to show that $\operatorname{depth} S/J_G \geq 7$. Let

$$J_H = P_\emptyset \cap P_{\{u_2\}} \cap P_{\{v_2\}} \cap P_{\{u_2,v_1\}} \cap P_{\{u_2,v_2\}}$$

and

$$J_{H'} = P_\emptyset \cap P_{\{u_2\}} \cap P_{\{v_2\}} \cap P_{\{u_1,v_2\}} \cap P_{\{u_2,v_2\}}$$

be binomial edge ideals on S. The graphs H and H' are both isomorphic to the graph described in Lemma 3. Hence S/J_H and $S/J_{H'}$ are Cohen-Macaulay with dimension equal to 7. Let

$$J_{H''} = J_H + J_{H'} \subset S.$$

We observe that $H'' = G_1 \cup G_2 \cup G_3$ where G_1 is the complete graph on the vertex set $\{u_1, u_2, v_1, v_2\}$ and G_2 (resp. G_3) is the complete graph on the vertex set $\{u_2, u_3\}$ (resp. $\{v_2, v_3\}$). By Theorem 2.7 of [7], applied twice (or by Theorem 1.1 of [5]) $S/J_{H''}$ is Cohen-Macaulay with dimension equal to 7. Thanks to depth Lemma applied to the following exact sequence

$$0 \longrightarrow S/J_G \longrightarrow S/J_H \oplus S/J_{H'} \longrightarrow S/J_{H''} \longrightarrow 0$$

we obtain that depth S/J_G is greater than or equal to 7. Hence S/J_G is Cohen-Macaulay, too. The assertion follows by Theorem 2.7 of [7] observing that u_3 and v_3 are free vertices in $\Delta(G)$. $\square$

Theorem 2. *Let G be a graph such that J_G is an almost complete intersection binomial edge ideal. The following conditions are equivalent:*

1. *G is in $\mathcal{G}_3 \cup \mathcal{G}_4$ defined as in Propositions 1 and 2;*
2. *S/J_G is CM;*

3. S/J_G *is unmixed.*

Proof: 1) $\Rightarrow$ 2). It follows by Propositions 1 and 2.

2) $\Rightarrow$ 3). Always true.

3) $\Rightarrow$ 1). Suppose that J_G is unmixed. By Lemma 2 if G is unicyclic then

$$G = C_l \cup \left(\bigcup_{i=1}^{r} P_i \right) \tag{2.1}$$

with $0 \leq r \leq l$, where C_l is a cycle of length l and for all $1 \leq i \leq r$, P_i is a path graph, $|V(P_i) \cap V(C_l)| = 1$ and $|V(P_i) \cap V(P_j)| = 0$ for all $j \neq i$. Suppose that G is not in $\mathcal{G}_3 \cup \mathcal{G}_4$. Then G is an element of one of the following sets:

$$\mathcal{G}'_3 = \{G \text{ satisfies (2.1) with } l = 3\} \setminus \mathcal{G}_3;$$

$$\mathcal{G}'_4 = \{G \text{ satisfies (2.1) with } l = 4\} \setminus \mathcal{G}_4;$$

$$\mathcal{G}'_> = \{G \text{ satisfies (2.1) with } l \geq 5\}.$$

G does not belong to $\mathcal{G}'_3$ since $\mathcal{G}'_3$ is empty. Let $G \in \mathcal{G}'_4$. Then there are two vertices v and v' in C_4 that are not adjacent and have the same degree, that is either 2 or 3. We observe that $T = \{v, v'\}$ has the cutpoint property. If the degree is 2 then $c(T) = |T|$, while if the degree is 3 we have $c(T) = |T| + 2$. In both cases J_G is not unmixed by Lemma 2. Contradiction.

Let $G \in \mathcal{G}'_>$ and let $V(C_l) = \{i_1, i_2, \ldots, i_l\}$ such that $\{i_j, i_{j+1}\}$ is an edge of G with $j = 1, \ldots, l-1$ and $\{i_1, i_l\}$ is an edge of G, too. Since G is unicyclic then $\{i_1, i_3\}$ has the cutpoint property. In fact $G_{V \setminus \{i_1, i_3\}}$ has at least two connected components, one containing the vertex i_2 and another one containing the vertices $\{i_4, \ldots, i_l\}$. Since J_G is unmixed there are exactly 3 connected components in $G_{V \setminus \{i_1, i_3\}}$. We may assume without loss of generality that i_1 has degree 3 and i_3 has degree 2.

By the same argument also $\{i_2, i_4\}$ has the cutpoint property and either i_2 or i_4 has degree 3. Suppose i_4 has degree 3. Then $\{i_1, i_4\}$ has the cutpoint property and $G_{V \setminus \{i_1, i_4\}}$ has 4 connected components. Contradiction. Hence i_2 must have degree 3. Also $\{i_3, i_l\}$ has the cutpoint property and since i_3 has degree 2, then i_l has degree 3. Since $\{i_2, i_l\}$ has the cutpoint property and $G_{V \setminus \{i_2, i_l\}}$ has 4 connected components we obtain a contradiction. $\square$

3 An algorithm to compute primary decomposition

In this section we describe the Algorithm 1 that summarizes the results of Lemma 1 and Proposition 2.1 of [7] and provide an implementation in CoCoA (see [1]) that is freely downloadable (see [8]). This tool helps the research of unmixed binomial edge ideals of deviation greater than or equal to 2.

Algorithm 1 (Computation of $\mathcal{C}(G)$)**.**

Input *A simple connected graph* G *with* $V(G) = [n]$.

Output *The set* $\mathcal{C}(G)$.

1. $S := \{1, \ldots, n\} \setminus \{$*free vertices of* $\Delta(G)\}$
2. $\mathcal{C}(G) = \{\emptyset\}$
3. *For each* $T \subset S$ *and* $1 \leq |T| \leq n-2$ *with* $T = \{v_1, \ldots, v_r\}$ *do*

   3.1 If $c(T) > 1$ *then*

$i := 1$

3.1.1 While $c(T \setminus \{v_i\}) < c(T)$ *and* $i \leq r$ *do*

$i := i + 1$

3.1.2 If $i > r$ *then*

$\mathcal{C}(G) := \mathcal{C}(G) \cup \{T\}$

4. Return $\mathcal{C}(G)$

We give a description of the Algorithm 1.

- Line 1. By Proposition 2.1 of [7] we can avoid all the free vertices of $\Delta(G)$ in the computation of $\mathcal{C}(G)$.
- Line 2. The empty set is always in $\mathcal{C}(G)$ by Lemma 1.
- Line 3. T can be any subset of S by Lemma 1. Nevertheless, since T has the cutpoint property, $c(T) \geq 2$ (see line 3.1). Therefore the maximum cardinality of T is $n-2$ where the 2 connected components are isolated vertices (if such T exists).
- Line 3.1. We observe that if $c(T) = 1$ then $c(T \setminus \{v_i\}) = c(T)$ for all $v_i \in T$. Hence we discard such T.
- Lines 3.1.1-3.1.2. We check if there exists a $v_i \in T$ that does not satisfy the condition $c(T \setminus \{v_i\}) < c(T)$. If such v_i exists interrupt the current computation. Otherwise add the new set to $\mathcal{C}(G)$ (line 3.1.2).

Thanks to Algorithm 1 we found some unmixed binomial edge ideals of deviation 2 that are not Cohen-Macaulay. We provide two examples. The first one is interesting since it is a bipartite graph. The second one since it has induced 5-cycle subgraphs.

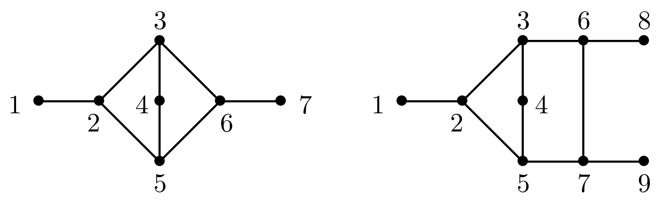

Figure 1:

Example 2. *Let* J_G *the binomial edge ideal associated to the graph with* 7 *vertices in figure 1. Then*

$$\mathcal{C}(G) = \{\emptyset, \{2\}, \{6\}, \{2,6\}, \{3,5\}, \{2,4,6\}\}$$

and is unmixed with $\dim S/J_G = 8$. *Using CoCoA (see [1])* $\operatorname{depth} S/J_G = 7$.

Example 3. *Let* J_G *the binomial edge ideal associated to the graph with* 9 *vertices in figure 1. Then*

$$\begin{aligned}\mathcal{C}(G) = &\{\emptyset, \{2\}, \{6\}, \{7\}, \{2,6\}, \{2,7\}, \{3,5\}, \{3,7\}, \{5,6\}, \{6,7\},\\ &\{2,3,7\}, \{2,4,6\}, \{2,4,7\}, \{2,5,6\}, \{2,6,7\}, \{3,5,6\}, \{3,5,7\},\\ &\{2,4,6,7\}\}\end{aligned}$$

and is unmixed with $\dim S/J_G = 10$. *Using CoCoA (see [1])* $\operatorname{depth} S/J_G = 9$.

References

[1] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra, Available at http://cocoa.dima.unige.it

[2] M. Crupi, G. Rinaldo, Binomial edge ideals with quadratic Gröbner bases, *The Electronic Journal of Combinatorics* **18** (2011), 1–13.

[3] F. Harary, *Graph theory*, Addison-Wesley series in Mathematics, 1972.

[4] J. Herzog, T. Hibi, F. Hreinsdottir, T. Kahle, J. Rauh, Binomial edge ideals and conditional independence statements, *Advances in Applied Mathematics* **45** (2010), 317–333.

[5] V. Ene, J. Herzog, T. Hibi, Cohen-Macaulay binomial edge ideals, *Nagoya Math. J.* **204** (2011), 57–68.

[6] M. Ohtani,Graphs and ideals generated by some 2-minors, *Comm. Algebra* **39** (2011), 905–917.

[7] A. Rauf, G. Rinaldo, Construction of Cohen-Macaulay binomial edge ideals, accepted in *Comm. Algebra* (2013).

[8] G. Rinaldo, An algorithm to compute primary decomposition of binomial edge ideals, available at `http://ww2.unime.it/algebra/rinaldo/binedg`.

[9] R.H. Villarreal, Cohen-Macaulay graphs, *Manuscripta Math.* **66** (1990), 277–293.

Received: 09.11.2012,

Accepted: 09.03.2013.

Department of Mathematics, University of Messina

E-mail: `giancarlo.rinaldo@tiscali.it`