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Lefschetz fibrations, intersection numbers, and representations of the framed braid group

by
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Abstract

We examine the action of the fundamental group Γ of a Riemann surface with m punctures on the middle dimensional homology of a regular fiber in a Lefschetz fibration, and describe to what extent this action can be recovered from the intersection numbers of vanishing cycles. Basis changes for the vanishing cycles result in a nonlinear action of the framed braid group $\widetilde{\mathcal{B}}$ on m strings on a suitable space of $m \times m$ matrices. This action is determined by a family of cohomologous 1-cocycles $\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ parametrized by distinguished configurations c of embedded paths from the regular value to the critical values. In the case of the disc, we compare this family of cocycles with the Magnus cocycles given by Fox calculus and consider some abelian reductions giving rise to linear representations of braid groups. We also prove that, still in the case of the disc, the intersection numbers along straight lines, which conjecturally make sense in infinite dimensional situations, carry all the relevant information.

Key Words: Lefschetz fibration, braid group, mapping class group, monodromy.
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1 Introduction

Picard–Lefschetz theory can be viewed as a complexification of Morse theory with the stable and unstable manifolds replaced by vanishing cycles and the count of connecting orbits in the Morse–Witten complex replaced by the intersection numbers of the vanishing cycles along suitable paths in the base. Relevant topological information that can be recovered from these data includes, in Morse theory, the homology of the underlying manifold and, in Picard–Lefschetz theory, the monodromy action of the fundamental group of the base on the middle dimensional homology of a regular fiber. That the monodromy action on the vanishing cycles can be recovered from the intersection numbers follows from the **Picard–Lefschetz formula**

$$(\psi_L)_* \alpha = \alpha - (-1)^{n(n+1)/2} \langle L, \alpha \rangle L. \tag{1.1}$$

Here $X \to \Sigma$ is a Lefschetz fibration over a Riemann surface Σ with fibers of complex dimension n, meaning that X is a complex manifold and the map $X \to \Sigma$ is holomorphic and has only nondegenerate critical points. We assume moreover that each singular fiber contains exactly one critical point, and denote by M a regular fiber over $z_0 \in \Sigma$. In equation (1.1), $L \subset M$ is an oriented vanishing cycle associated to a curve from z_0 to a singular point $z \in \Sigma$, $\psi_L \in \mathrm{Diff}(M)$ is the Dehn–Arnold–Seidel twist about L obtained from the (counterclockwise) monodromy around the singular fiber along the same curve, $(\psi_L)_*$ is the induced action on $H_n(M)$ and $\langle\cdot,\cdot\rangle$ denotes the intersection form. Equation (1.1) continues to hold when $X \to \Sigma$ is a symplectic Lefschetz fibration as introduced by Donaldson [7, 8]. In either case the vanishing cycles are embedded Lagrangian spheres and so their self-intersection numbers are $2(-1)^{n/2}$ when n is even and 0 when n is odd. See [2, Chapter I] and [3, §2.1] for a detailed account of Picard–Lefschetz theory and an exhaustive reference list.

The object of the present paper is to study an algebraic setting, based on equation (1.1), which allows one to describe the monodromy action of the fundamental group in terms of intersection matrices. This requires the choice of a distinguished basis of vanishing cycles and the ambiguity in this choice leads to an action of the braid group on distinguished bases [2], which in turn determines an action on a suitable space of matrices. In the case of the disc, this action was previously considered by Bondal [5] in the context of mirror symmetry. Our motivation is different and comes from an attempt of two of the authors (A.O. and D.S.) to understand complexified Floer homology in the spirit of Donaldson–Thomas theory [9]. In this theory the complex symplectic action or Chern–Simons functional is an infinite dimensional analogue of a Lefschetz fibration. While there are no vanishing cycles, one can (conjecturally) still make sense of their intersection numbers along straight lines and build an intersection matrix whose orbit under the braid group might then be viewed as an invariant. Another source of inspiration for the present paper is the work of Seidel [20, 21] about vanishing cycles and mutations.

We assume throughout this paper that the base Σ of our Lefschetz fibration is a compact Riemann surface, possibly with boundary, not diffeomorphic to the 2-sphere. (In particular, Σ is oriented.) Let $Z \subset \Sigma \setminus \partial\Sigma$ be the set of critical values and z_0 be a regular value. If the surface Σ is diffeomorphic to the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| \leq 1\}$ we assume that $z_0 \in \partial\Sigma$. To assemble the intersection numbers into algebraic data it is convenient to choose a collection $c = (c_1, \dots, c_m)$ of ordered embedded paths from a regular value $z_0 \in \Sigma$ to the critical values (see Figure 1). Following [2, 20] we call such a collection a **distinguished configuration** and denote by $\mathcal{C}$ the set of homotopy classes of distinguished configurations. Any distinguished configuration determines an ordering $\{z_1, \dots, z_m\}$ of the set Z of critical values by $z_i := c_i(1)$. It also determines m vanishing cycles $L_1, \dots, L_m \subset M$ as well as m special elements of the fundamental group

$$g_1, \dots, g_m \in \Gamma := \pi_1(\Sigma \setminus Z, z_0)$$

(obtained by encircling z_i counterclockwise along c_i). The orientation of L_i is not determined by the path c_i and can be chosen independently. However, when n is even, the monodromy along g_i changes the orientation of L_i. Thus, to choose orientations consistently, we fix nonzero tangent vectors $v_z \in T_z\Sigma$ for $z \in Z$, choose orientations of the vanishing cycles in the directions of these vectors, and consider only distinguished configurations c that are tangent to the vectors $-v_z$ at their endpoints. We call these **marked distinguished configurations** and denote by $\widetilde{\mathcal{C}}$ the set of homotopy classes of marked distinguished configurations. The oriented vanishing

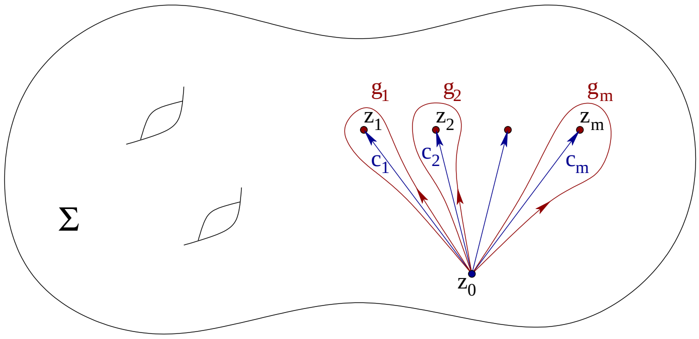

Figure 1: A distinguished configuration.

cycles determine homology classes, still denoted by

$$L_1, \dots, L_m \in H_n(M).$$

These data give rise to a **monodromy character** $\mathcal{N}_c^X : \Gamma \to \mathbb{Z}^{m\times m}$ via

$$\mathcal{N}_c^X(g) := \big(n_{ij}(g)\big)_{i,j=1}^m, \qquad n_{ij}(g) := \langle L_i, \rho(g)L_j\rangle\,. \tag{1.2}$$

Here $\rho : \Gamma \to \mathrm{Aut}(H_n(M))$ denotes the monodromy action of the fundamental group. Any such function $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ satisfies the conditions

$$\begin{aligned} n_{ij}(g^{-1}) &= (-1)^n n_{ji}(g), \\ n_{ii}(1) &= \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}, \end{cases} \\ n_{ij}(gg_kh) &= n_{ij}(gh) - (-1)^{n(n+1)/2} n_{ik}(g) n_{kj}(h) \end{aligned} \tag{1.3}$$

for $g, h \in \Gamma$ and $i, j, k = 1, \dots, m$. The last equation in (1.3) follows from (1.1). Our convention for the composition is that a representative loop of $gh \in \Gamma$ first traverses a representative of h, and then a representative of g.

The part of $H_n(M)$ that is generated by the vanishing cycles under the action of Γ can be recovered as the quotient

$$\mathbb{H}_\mathcal{N} := \Lambda/\ker\,\mathcal{N}, \qquad \Lambda := \mathbb{Z}[\Gamma]^m.$$

Here $\mathbb{Z}[\Gamma]$ is the group ring of Γ, whose elements are thought of as maps $\lambda : \Gamma \to \mathbb{Z}$ with finite support and whose multiplication is the convolution product $(\lambda_1\lambda_2)(h) := \sum_g \lambda_1(hg^{-1})\lambda_2(g)$; the map $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ is regarded as an endomorphism $\mathcal{N} : \Lambda \to \Lambda$ by the convolution product $\lambda \mapsto \big(h \mapsto \sum_g \mathcal{N}(hg^{-1})\lambda(g)\big)$. The $\mathbb{Z}$-module $\mathbb{H}_\mathcal{N}$ is equipped with an intersection pairing, with a Γ-action $\rho_\mathcal{N} : \Gamma \to \mathrm{Aut}(\mathbb{H}_\mathcal{N})$, and with special elements $\mathbb{L}_1, \dots, \mathbb{L}_m \in \mathbb{H}_\mathcal{N}$, defined by

$$\langle \mu, \lambda\rangle := \sum_{g,h\in\Gamma} \mu(h)^T \mathcal{N}(hg^{-1})\lambda(g), \quad \rho_\mathcal{N}(g)[\lambda] := [g_*\lambda], \quad \mathbb{L}_i := [\delta_i]. \tag{1.4}$$

Here Γ acts on Λ by

$$(g_*\lambda)(h) := \lambda(hg),$$

$e_i \in \mathbb{Z}^m$ denotes the standard basis, and $\delta_i : \Gamma \to \mathbb{Z}^m$ is defined by $\delta_i(1) := e_i$ and $\delta_i(g) := 0$ for $g \neq 1$. With these structures $\mathbb{H}_{\mathcal{N}}$ is isomorphic to the submodule of $H_n(M)$ generated by the vanishing cycles modulo the kernel of the intersection form. The isomorphism is induced by the map that assigns to every $\lambda \in \Lambda$ the homology class $\sum_{g,i} \lambda_i(g)\rho(g^{-1})L_i \in H_n(M)$.

The monodromy character $\mathcal{N}_c^X : \Gamma \to \mathbb{Z}^{m\times m}$ depends on the choice of a marked distinguished configuration c and this dependence gives rise to an action of the framed braid group $\widetilde{\mathcal{B}}$ of $\Sigma \setminus \{z_0\}$ on m strings, based at Z, on the space of monodromy characters. More precisely, our distinguished configuration c determines m special elements $g_i =: g_{i,c} \in \Gamma$ obtained by encircling $z_i := c_i(1)$ counterclockwise along c_i. Denote by

$$\mathscr{N}_c := \bigl\{\mathcal{N} = (n_{ij}) : \Gamma \to \mathbb{Z}^{m\times m} \,|\, (1.3)\bigr\}$$

the space of monodromy characters on $(\Gamma, g_{1,c}, \ldots, g_{m,c})$. The framed braid group $\widetilde{\mathcal{B}}$, interpreted as the mapping class group of diffeomorphisms in $\mathrm{Diff}_0(\Sigma, z_0)$ that preserve the set Z and the collection of vectors $\{v_z\}_{z\in Z}$, acts freely and transitively on the space $\widetilde{\mathcal{C}}$ of homotopy classes of marked distinguished configurations (see Sections 3 and 4). Here $\mathrm{Diff}_0(\Sigma, z_0)$ denotes the identity component of the group of diffeomorphisms of Σ that fix z_0. The framed braid group also acts on the fundamental group Γ and, for every $\tau \in \widetilde{\mathcal{B}}$ and every $c \in \widetilde{\mathcal{C}}$, the isomorphism $\tau_* : \Gamma \to \Gamma$ maps $g_{i,c}$ to $g_{i,\tau_* c}$. This action actually determines the (unframed) braid τ (see Section 4).

Our first theorem asserts that there is a canonical family of isomorphisms

$$\mathscr{T}_{\tau,c} : \mathscr{N}_c \to \mathscr{N}_{\tau_* c}$$

which extends the geometric correspondence $\mathcal{N}_c^X \mapsto \mathcal{N}_{\tau_* c}^X$ between monodromy characters associated to different choices of the distinguished configuration. It is an open question if every element $\mathcal{N} \in \mathscr{N}_c$ can be realized by a (symplectic) Lefschetz fibration $X \to \Sigma$ with critical fibers over Z. This question is a refinement of the Hurwitz problem of finding a branched cover with given combinatorial data. The isomorphisms $\mathscr{T}_{\tau,c}$ are determined by a family of cocycles

$$\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$$

with values in the group of invertible matrices over the group ring. To describe them we denote by $\pi_{\sigma,c} \in \mathfrak{S}_m$ the permutation associated to the action of σ on the ordering determined by c. Then the (i,j) entry of the matrix is

$$s_{j,c}(\sigma) := c_i^{-1} \cdot \sigma_* c_j \in \Gamma$$

(first $\sigma_* c_j$, second c_i^{-1}) for $i = \pi_{\sigma,c}(j)$ and it is zero for $i \neq \pi_{\sigma,c}(j)$. We emphasize that $\sigma_* g_{j,c} = s_{j,c}(\sigma)^{-1} g_{i,c} s_{j,c}(\sigma)$ for $i := \pi_{\sigma,c}(j)$. See Figure 2 below for some examples.

In the next theorem we think of an element $\mathcal{M} \in \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ as a function $\mathcal{M} : \Gamma \to \mathbb{Z}^{m\times m}$ with finite support, denote $\mathcal{M}^t(g) := \mathcal{M}(g^{-1})^T$, and multiply matrices using the convolution product (see Section 2).

Theorem A. **(i)** *The maps* $\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ *are injective and satisfy the cocycle and coboundary conditions*

$$\mathcal{S}_c(\sigma\tau) = \mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau), \qquad \mathcal{S}_{\tau_* c}(\sigma) = \mathcal{S}_c(\tau)^{-1}\mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau). \tag{1.5}$$

(ii) *The maps* $\mathcal{S}_c$ *in (i) determine bijections*

$$\mathscr{T}_{\tau,c} : \mathscr{N}_c \to \mathscr{N}_{\tau_* c}, \qquad \mathcal{N} \mapsto \mathcal{S}_c(\tau)^t \mathcal{N} \mathcal{S}_c(\tau).$$

(iii) *Given a Lefschetz fibration* $X \to \Sigma$ *and elements* $c \in \widetilde{\mathcal{C}}$, $\tau \in \widetilde{\mathcal{B}}$, *we have*

$$\mathcal{N}^X_{\tau_* c} = \mathcal{S}_c(\tau)^t \mathcal{N}^X_c \mathcal{S}_c(\tau). \tag{1.6}$$

In terms of Serre's definition of non-abelian cohomology [22, Appendix to Chapter VII] the first equation in (1.5) asserts that $\mathcal{S}_c$ is a cocycle for every $c \in \widetilde{\mathcal{C}}$. The second equation in (1.5) asserts that the cocycles $\mathcal{S}_c$ are all cohomologous and hence define a canonical 1-cohomology class

$$[\mathcal{S}_c] \in H^1(\widetilde{\mathcal{B}}, \mathrm{GL}_m(\mathbb{Z}[\Gamma])).$$

We call it the **Picard–Lefschetz monodromy class**.

In the case $\Sigma = \mathbb{D}$ there is another well known cocycle arising from a topological context, namely the Magnus cocycle

$$\mathcal{M}_c : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma]), \qquad c \in \mathcal{C}.$$

Here $\mathcal{B}$ denotes the usual braid group (with no framing), which we view as a subgroup of $\widetilde{\mathcal{B}}$ using the framing determined by a vector field v on Σ whose only singularity is an attractive point at z_0 and such that $v(z) = v_z$ for all $z \in Z$ (see Section 6). The Magnus cocycle is also related to an intersection pairing [26, 19] and its dependence on the choice of the distinguished configuration $c \in \mathcal{C}$ is similar to the dependence of the Picard–Lefschetz cocycle. The cohomology classes $[\mathcal{M}_c]$ and $[\mathcal{S}_c|_{\mathcal{B}}]$ are distinct and nontrivial in $H^1(\mathcal{B}, \mathrm{GL}_m(\mathbb{Z}[\Gamma]))$. After reduction of Γ to the infinite cyclic group, both cocycles define linear representations of the braid group with coefficients in $\mathbb{Z}[t, t^{-1}]$. In the case of the Magnus cocycle, this is the famous Burau representation [4, 17]. In the case of the Picard–Lefschetz cocycle, this representation was first discovered by Tong–Yang–Ma [25] and is a key ingredient in the classification of m-dimensional representations of the braid group $\mathcal{B}$ on m strings [23]. For pure braids, i.e. braids which do not permute the elements of Z, the Tong–Yang–Ma representation is determined by linking numbers (see Section 6).

We continue our discussion of the planar case $\Sigma = \mathbb{D}$. The group Γ is then isomorphic to the free group Γ_m generated by $g_1, \ldots, g_m$, and it is convenient to switch from the geometric picture in Theorem A to generators and relations. We denote by $\widetilde{\mathcal{B}}_m$ the abstract group generated by $\sigma_2, \ldots, \sigma_m$, $\varepsilon_1, \ldots, \varepsilon_m$ with relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \varepsilon_i\sigma_i = \sigma_i\varepsilon_{i-1}, \qquad \varepsilon_{i-1}\sigma_i = \sigma_i\varepsilon_i. \tag{1.7}$$

All other pairs of generators commute. The choice of an element $c \in \widetilde{\mathcal{C}}$ determines an isomorphism $\widetilde{\mathcal{B}}_m \to \widetilde{\mathcal{B}}$ obtained by identifying the generators σ_i, ε_i with the moves depicted in

Figure 2 ([18], see also Section 4). This gives rise to a contravariant free and transitive action of $\widetilde{\mathcal{B}}_m$ on $\widetilde{\mathcal{C}}$ denoted by $\widetilde{\mathcal{B}}_m \times \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}} : (\sigma, c) \mapsto \sigma^* c$, and to an action of $\widetilde{\mathcal{B}}_m$ on Γ_m via

$$(\sigma_i)_* : \left\{ \begin{array}{ccc} g_{i-1} & \mapsto & g_{i-1} g_i g_{i-1}^{-1}, \\ g_i & \mapsto & g_{i-1}, \end{array} \right. \tag{1.8}$$

$(\sigma_i)_* g_j = g_j$ for $j \neq i-1, i$, and $(\varepsilon_i)_* = \mathrm{Id}$.

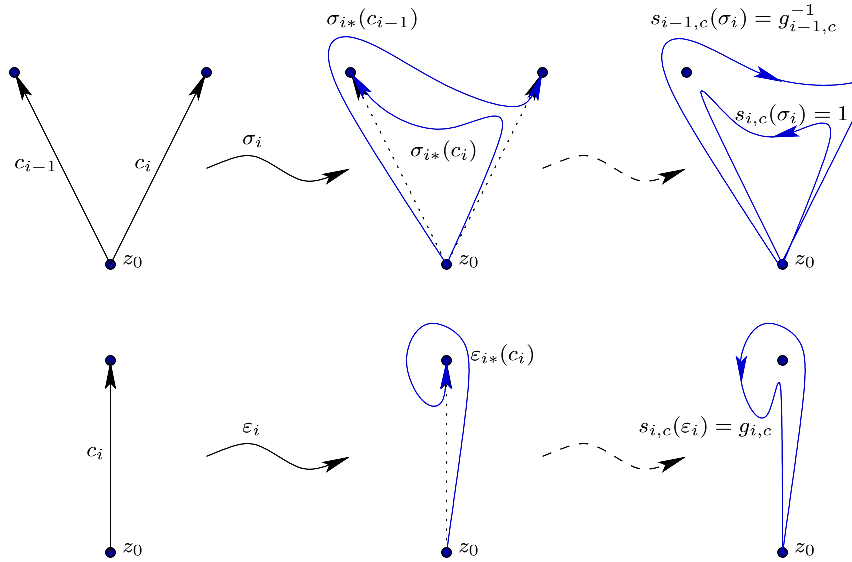

Figure 2: The generators of $\widetilde{\mathcal{B}}_m$.

The third equation in (1.3) shows that, in the case of the disc, we can switch from matrix valued functions to actual matrices. More precisely, a monodromy character $\mathcal{N} : \Gamma_m \to \mathbb{Z}^{m \times m}$ is uniquely determined by the matrix $N := \mathcal{N}(1)$. The latter satisfies

$$n_{ij} = (-1)^n n_{ji}, \qquad n_{ii} = \left\{ \begin{array}{ll} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}. \end{array} \right. \tag{1.9}$$

We denote by $\mathscr{N}_m$ the space of matrices satisfying (1.9). The map $\mathcal{N}$ is explicitly given by $\mathcal{N}(g) = N \rho_N(g)$, the homomorphism $\rho_N : \Gamma_m \to \mathrm{GL}_m(\mathbb{Z})$ being defined on generators by

$$\rho_N(g_i) = \mathbb{1} - (-1)^{n(n+1)/2} E_i N. \tag{1.10}$$

Here $E_i \in \mathbb{Z}^{m \times m}$ is the matrix with the i-th entry on the diagonal equal to one and zeroes elsewhere. The representation ρ_N induces an action of Γ_m on the quotient module $\mathbb{H}_N := \mathbb{Z}^m / \ker N$ which preserves the intersection form $\langle \lambda, \mu \rangle := \lambda^T N \mu$. Moreover, the triple $(\mathbb{H}_N, \rho_N, \langle \cdot, \cdot \rangle)$ becomes isomorphic to $(\mathbb{H}_{\mathcal{N}}, \rho_{\mathcal{N}}, \langle \cdot, \cdot \rangle)$: see Section 2. The next result rephrases Theorem A for the particular case of the disc, and strengthens it with an additional uniqueness statement. The first part of (ii) has been already proved by Bondal [5, Proposition 2.1] for upper triangular matrices instead of (skew-)symmetric ones. For $k = 2, \dots, m$ we denote by Σ_k the permutation matrix of the transposition $(k-1, k)$ and, for $i = 1, \dots, m$, we denote by D_i the diagonal matrix with i-th entry on the diagonal equal to $(-1)^{n+1}$ and the other diagonal entries equal to 1.

Theorem B. (i) *There is a unique function $S : \widetilde{\mathcal{B}}_m \times \mathscr{N}_m \to \mathrm{GL}_m(\mathbb{Z})$ satisfying the following conditions.*

(Cocycle) *For all $N \in \mathscr{N}_m$ and $\sigma, \tau \in \widetilde{\mathcal{B}}_m$ we have*

$$S(\sigma\tau, N) = S(\sigma, N)S(\tau, S(\sigma, N)^T N S(\sigma, N)). \tag{1.11}$$

(Normalization) *For all $N \in \mathscr{N}_m$ we have $S(1, N) = \mathbb{1}$ and*

$$\begin{aligned} S(\sigma_k, N) &= \Sigma_k - (-1)^{n(n+1)/2} n_{k-1,k} E_{k-1}, \qquad k = 2, \dots, m, \\ S(\varepsilon_i, N) &= D_i, \qquad i = 1, \dots, m. \end{aligned} \tag{1.12}$$

(ii) *The function S in (i) determines a contravariant group action of $\widetilde{\mathcal{B}}_m$ on $\mathscr{N}_m$ via*

$$\sigma^* N := S(\sigma, N)^T N S(\sigma, N)$$

for $N \in \mathscr{N}_m$ and $\sigma \in \widetilde{\mathcal{B}}_m$. This action is compatible with the action of $\widetilde{\mathcal{B}}_m$ on the space of marked distinguished configurations in the sense that, for every Lefschetz fibration $X \to \mathbb{D}$ with singular fibers over Z, every $\sigma \in \widetilde{\mathcal{B}}_m$, and every $c \in \widetilde{\mathcal{C}}$, we have

$$N^X_{\sigma^* c} = \sigma^* N^X_c,$$

where $N^X_c := \mathcal{N}^X_c(1)$.

(iii) *For every $\sigma \in \widetilde{\mathcal{B}}_m$ and every $N \in \mathscr{N}_m$ the matrix $S(\sigma, N)$ induces an isomorphism from $\mathbb{H}_{\sigma^* N}$ to $\mathbb{H}_N$ which preserves the bilinear pairings and satisfies*

$$\rho_{\sigma^* N}(g) = S(\sigma, N)^{-1} \rho_N(\sigma_* g) S(\sigma, N). \tag{1.13}$$

By Theorem B every symplectic Lefschetz fibration $f : X \to \mathbb{D}$ with critical fibers over Z determines a $\widetilde{\mathcal{B}}_m$-equivariant map

$$\widetilde{\mathcal{C}} \to \mathscr{N}_m : c \mapsto N^X_c$$

which can be viewed as an algebraic invariant of X. Our next theorem asserts that this invariant is uniquely determined by the matrix

$$Q^X : Z \times Z \to \mathbb{Z}$$

of intersection numbers of vanishing cycles along straight lines. Here we assume that no straight line connecting two points in Z contains another element of Z; such a set Z is called **admissible**. Denote by $\mathscr{N}_Z$ the space of matrices $Q : Z \times Z \to \mathbb{Z}$ satisfying

$$Q(z,z') = (-1)^n Q(z',z), \qquad Q(z,z) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2(-1)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \tag{1.14}$$

We define a map $\widetilde{\mathcal{C}} \times \mathscr{N}_m \to \mathscr{N}_Z : (c,N) \mapsto Q_{c,N}$ by

$$Q_{c,N}(z_i,z_j) := (N\rho_N(g))_{ij} \tag{1.15}$$

where $z_1,\dots,z_m$ is the ordering of Z given by c, Γ_m is identified with Γ, and

$$g := c_i^{-1} \cdot s_{ij} \ \cdot c_j \in \Gamma.$$

Here the right hand side denotes the based loop obtained by first traversing c_j, then moving clockwise near z_j until reaching the straight line s_{ij} from z_j to z_i, then following s_{ij}, then moving counterclockwise near z_i until reaching c_i, and finally traversing c_i in the opposite direction (see Figure 3). Geometrically, this means that the matrix $Q^X := Q_{c,N_c^X}$ assigns to

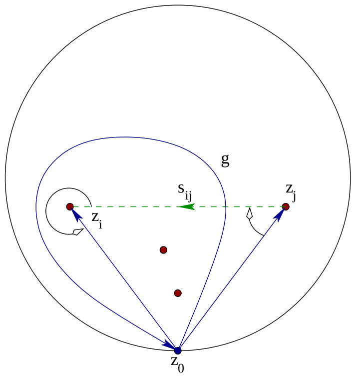

Figure 3: Intersection numbers along straight lines.

a pair $(z,z') \in Z \times Z$ the intersection number of the vanishing cycles along the straight line from z to z', where the orientations at the endpoints are determined by moving the oriented vanishing cycles in the directions v_z and $v_{z'}$ clockwise towards the straight line. (Given a marked distinguished configuration c and the loop g as above, the straight line s_{ji} from z_i to z_j corresponds to the curve $c_j \cdot g^{-1} \cdot c_i^{-1}$.)

Theorem C. *The map $(c,N) \mapsto Q_{c,N}$ defined by (1.15) is invariant under the diagonal action of $\widetilde{\mathcal{B}}_m$ on $\widetilde{\mathcal{C}} \times \mathscr{N}_m$. Moreover, for every $Q \in \mathscr{N}_Z$, there is a unique equivariant map*

$$\widetilde{\mathcal{C}} \to \mathscr{N}_m : c \mapsto N_c$$

such that $Q_{c,N_c} = Q$ for every $c \in \widetilde{\mathcal{C}}$.

Let $f : X \to \mathbb{D}$ be a symplectic Lefschetz fibration with critical fibers over an admissible set $Z \subset \mathrm{int}(\mathbb{D})$. Let $z, z' \in Z$ and $x \in f^{-1}(z)$, $x' \in f^{-1}(z')$ be the associated critical points of f. Then the number $Q^X(z, z')$ is the algebraic count of negative gradient flow lines

$$\dot{u} + \nabla f^\theta(u) = 0, \quad f^\theta := \cos(\theta)\mathrm{Re}\, f + \sin(\theta)\mathrm{Im}\, f, \quad \theta := \arg(z' - z), \tag{1.16}$$

from $x' = \lim_{s\to-\infty} u(s)$ to $x = \lim_{s\to\infty} u(s)$. According to Donaldson–Thomas [9] this count of gradient flow lines is (conjecturally) still meaningful in suitable infinite dimensional settings. It thus gives rise to an intersection matrix Q and hence, by Theorem C, also to an equivariant map $c \mapsto N_c$. A case in point, analogous to symplectic Floer theory, is where f is the complex symplectic action on the loop space of a complex symplectic manifold.

The paper is organized as follows. In Section 2 we explain an algebraic setting for monodromy representations, Section 3 discusses the framed braid group $\widetilde{\mathcal{B}}$, and in Section 4 we prove that $\widetilde{\mathcal{B}}$ acts freely and transitively on the space $\widetilde{\mathcal{C}}$ of distinguished configurations. Theorem A is contained in Theorem 4 from Section 5. We compare in Section 6 the Picard–Lefschetz cocycle with the Magnus cocycle, and discuss some related linear representations of the braid group. Theorem B is proved in Section 7, in Section 8 we introduce monodromy groupoids, in Section 9 we prove Theorem C, and Section 10 illustrates the monodromy representation by an example. We include a brief discussion of some basic properties of Lefschetz fibrations in Section 11, summarizing relevant facts from [2, Chapter I].

2 Monodromy representations

We examine algebraic structures that are relevant in the study of monodromy representations associated to Lefschetz fibrations.

Definition 1. *Fix a positive integer n. Let Γ be a group and $g_1, \dots, g_m$ be pairwise distinct elements of $\Gamma \setminus \{1\}$. A* **monodromy character** *on $(\Gamma, g_1, \dots, g_m)$ is a matrix valued function $\mathcal{N} = (n_{ij}) : \Gamma \to \mathbb{Z}^{m\times m}$ satisfying*

$$n_{ij}(g^{-1}) = (-1)^n n_{ji}(g), \tag{2.1}$$

$$n_{ii}(1) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2(-1)^{n/2}, & \text{if } n \text{ is even,} \end{cases} \tag{2.2}$$

$$n_{ij}(gg_kh) = n_{ij}(gh) - (-1)^{n(n+1)/2} n_{ik}(g)n_{kj}(h) \tag{2.3}$$

for all $g, h \in \Gamma$ and $i, j, k \in \{1, \dots, m\}$. A **monodromy representation** *of $(\Gamma, g_1, \dots, g_m)$ is a tuple $(\mathcal{H}, \rho, \mathcal{L}_1, \dots, \mathcal{L}_m)$ consisting of a $\mathbb{Z}$-module $\mathcal{H}$ together with a nondegenerate bilinear pairing, a representation $\rho : \Gamma \to \mathrm{Aut}(\mathcal{H})$ that preserves the bilinear pairing, and elements $\mathcal{L}_1, \dots, \mathcal{L}_m \in \mathcal{H}$, satisfying*

$$\langle \mathcal{L}, \mathcal{L}' \rangle = (-1)^n \langle \mathcal{L}', \mathcal{L} \rangle, \tag{2.4}$$

$$\langle \mathcal{L}_i, \mathcal{L}_i \rangle = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2(-1)^{n/2}, & \text{if } n \text{ is even,} \end{cases} \tag{2.5}$$

$$\rho(g_i)\mathcal{L} = \mathcal{L} - (-1)^{n(n+1)/2} \langle \mathcal{L}_i, \mathcal{L} \rangle \mathcal{L}_i \tag{2.6}$$

for all $\mathcal{L}, \mathcal{L}' \in \mathcal{H}$ and $i \in \{1, \dots, m\}$. The automorphisms $\rho(g_i)$ are called **Dehn twists** *and the $\mathcal{L}_i$ are called* **vanishing cycles**.

Remark 1. Every monodromy character $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ satisfies

$$n_{ij}(gg_k^{-1}h) = n_{ij}(gh) - (-1)^n(-1)^{n(n+1)/2} n_{ik}(g)n_{kj}(h), \tag{2.7}$$

$$n_{ij}(g_i g) = n_{ij}(gg_j) = (-1)^{n+1} n_{ij}(g). \tag{2.8}$$

Every monodromy representation $(\mathcal{H}, \rho, \mathcal{L}_1, \dots, \mathcal{L}_m)$ satisfies

$$\rho(g_i^{-1})\mathcal{L} = \mathcal{L} - (-1)^n(-1)^{n(n+1)/2} \langle \mathcal{L}_i, \mathcal{L}\rangle \mathcal{L}_i, \tag{2.9}$$

$$\rho(g_i)\mathcal{L}_i = (-1)^{n+1}\mathcal{L}_i. \tag{2.10}$$

Equation (2.7) follows from (2.1) and (2.3) by replacing g, h with h^{-1}, g^{-1} and interchanging i and j. To prove the second equation in (2.8) use equation (2.3) with $k = j$ and $h = 1$; then use (2.2). The proofs of (2.9) and (2.10) are similar.

Every monodromy representation gives rise to a monodromy character and vice versa. If $(\mathcal{H}, \rho, \mathcal{L}_1, \dots, \mathcal{L}_m)$ is a monodromy representation then the associated monodromy character $\mathcal{N}_\rho : \Gamma \to \mathbb{Z}^{m\times m}$ assigns to every $g \in \Gamma$ the intersection matrix

$$n_{ij}(g) := \langle \mathcal{L}_i, \rho(g)\mathcal{L}_j\rangle. \tag{2.11}$$

Conversely, we obtain that every monodromy character $\mathcal{N}$ induces a monodromy representation $(\mathbb{H}_\mathcal{N}, \rho_\mathcal{N}, \mathbb{L}_1, \dots, \mathbb{L}_m)$ as follows.

Denote by $\mathbb{Z}[\Gamma]$ the group ring of Γ. One can think of an element $\lambda \in \mathbb{Z}[\Gamma]$ either as a function $\lambda : \Gamma \to \mathbb{Z}$ with finite support or as a formal linear combination $\lambda = \sum_{g\in\Gamma} \lambda(g)g$. With the first viewpoint, the multiplication in $\mathbb{Z}[\Gamma]$ is the convolution product

$$(\lambda\mu)(h) := \sum_{g\in\Gamma} \lambda(g)\mu(g^{-1}h).$$

The group Γ acts on the group ring by the formula $(g_*\lambda)(h) := \lambda(hg)$ for $\lambda \in \mathbb{Z}[\Gamma]$ and $g, h \in \Gamma$. In the formal sum notation we have

$$g_*\lambda = \sum_h \lambda(hg)h = \sum_h \lambda(h)hg^{-1} = \lambda g^{-1}.$$

For any function $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ we introduce the $\mathbb{Z}$-module

$$\mathbb{H}_\mathcal{N} := \Lambda/\left\{\lambda \in \Lambda \,|\, \mathcal{N}\lambda = 0\right\}, \qquad \Lambda := \mathbb{Z}[\Gamma]^m, \tag{2.12}$$

where $(\mathcal{N}\lambda)(h) := \sum_g \mathcal{N}(hg^{-1})\lambda(g)$ is the convolution product. This abelian group is equipped with a bilinear pairing

$$\langle \mu, \lambda\rangle := \sum_{g,h\in\Gamma} \mu(h)^T \mathcal{N}(hg^{-1})\lambda(g) = \sum_{h\in\Gamma} \mu(h)^T(\mathcal{N}\lambda)(h) \tag{2.13}$$

and a group action $\rho_\mathcal{N} : \Gamma \to \mathrm{Aut}(\mathbb{H}_\mathcal{N})$ defined by

$$\rho_\mathcal{N}(g)[\lambda] := [g_*\lambda], \tag{2.14}$$

which preserves the bilinear pairing. The special elements $\mathbb{L}_1, \dots, \mathbb{L}_m \in \mathbb{H}_{\mathcal{N}}$ are given by

$$\mathbb{L}_i := [\delta_i], \qquad \delta_i(g) := \begin{cases} e_i, & \text{if } g = 1, \\ 0, & \text{if } g \neq 1. \end{cases} \tag{2.15}$$

These structures are well defined for any function $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$. The next lemma asserts that the tuple $(\mathbb{H}_{\mathcal{N}}, \rho_{\mathcal{N}}, \mathbb{L}_1, \dots, \mathbb{L}_m)$ is a monodromy representation whenever $\mathcal{N}$ is a monodromy character.

Lemma 1. (i) *Assume that $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ is a monodromy character. Then the tuple $(\mathbb{H}_{\mathcal{N}}, \rho_{\mathcal{N}}, \mathbb{L}_1, \dots, \mathbb{L}_m)$ defined by (2.12-2.15) is a monodromy representation whose associated character is $\mathcal{N}$.*

(ii) *Let $(\mathcal{H}, \rho, \mathcal{L}_1, \dots, \mathcal{L}_m)$ be a monodromy representation and $\mathcal{N}$ be its character (2.11). Then the map*

$$\Lambda \to \mathcal{H} : \lambda \mapsto \sum_{g\in\Gamma} \sum_{i=1}^{m} \lambda_i(g)\rho(g^{-1})\mathcal{L}_i \tag{2.16}$$

induces an isomorphism of monodromy representations from $\mathbb{H}_{\mathcal{N}}$ to $\mathcal{V}/\mathcal{W}$, where $\mathcal{V} \subset \mathcal{H}$ is the submodule generated by the vanishing cycles $\rho(g)\mathcal{L}_i$ and $\mathcal{W} \subset \mathcal{V}$ is the kernel of the intersection form.

Proof: The $\mathbb{Z}$-module $\mathbb{H}_{\mathcal{N}}$ is isomorphic to the image of the homomorphism defined by $\lambda \mapsto \mathcal{N}\lambda$ and hence is torsion free. That the bilinear pairing in (2.13) is nondegenerate follows directly from the definition. That it satisfies (2.4) follows from (2.1) and that it satisfies (2.5) follows from (2.2) and the identity $\langle \mathcal{L}_i, \mathcal{L}_i \rangle = n_{ii}(1)$. To prove (2.6) fix an index i and an element $\lambda \in \Lambda$. Abbreviate

$$\varepsilon := (-1)^{n(n+1)/2}$$

and define $\lambda' \in \Lambda$ by

$$\lambda' := (g_i)_*\lambda - \lambda + \varepsilon \langle \delta_i, \lambda \rangle \delta_i.$$

Then

$$\lambda'(h) = \lambda(hg_i) - \lambda(h) + \varepsilon \Big(\sum_{g\in\Gamma} e_i^T \mathcal{N}(g^{-1})\lambda(g) \Big) \delta_i(h)$$

and hence

$$\begin{aligned} (\mathcal{N}\lambda')(k) &= \sum_h \mathcal{N}(kh^{-1}) \Big(\lambda(hg_i) - \lambda(h) + \varepsilon \sum_g e_i^T \mathcal{N}(g^{-1})\lambda(g)\delta_i(h) \Big) \\ &= \sum_h \Big(\mathcal{N}(kg_ih^{-1}) - \mathcal{N}(kh^{-1}) + \varepsilon\mathcal{N}(k)E_i\mathcal{N}(h^{-1}) \Big) \lambda(h) \\ &= 0. \end{aligned}$$

The last equation follows from (2.3). Hence the equivalence class of λ' vanishes and so the tuple $(\mathbb{H}_{\mathcal{N}}, \rho_{\mathcal{N}}, \mathbb{L}_1, \dots, \mathbb{L}_m)$ satisfies (2.6). Thus we have proved that $(\mathbb{H}_{\mathcal{N}}, \rho_{\mathcal{N}}, \mathbb{L}_1, \dots, \mathbb{L}_m)$ is a monodromy representation. Its character is

$$\langle \delta_i, g_*\delta_j \rangle = \sum_{h,k} \delta_i(k)^T \mathcal{N}(kh^{-1})\delta_j(hg) = n_{ij}(g).$$

This proves (i).

We prove (ii). The homomorphism (2.16) is obviously surjective. That the preimage of $\mathcal{W}$ under (2.16) is the subspace $\{\lambda \in \Lambda \,|\, \mathcal{N}\lambda = 0\}$ follows from the identity

$$\left\langle \rho(h^{-1})\mathcal{L}_i, \Phi_{\mathcal{N}}(\lambda) \right\rangle = \sum_{g,j} n_{ij}(hg^{-1})\lambda_j(g) = (\mathcal{N}\lambda)_i(h),$$

where

$$\Phi_{\mathcal{N}}(\lambda) := \sum_{g,j} \lambda_j(g)\rho(g^{-1})\mathcal{L}_j.$$

Hence (2.16) induces an isomorphism $\Phi_{\mathcal{N}} : \mathbb{H}_{\mathcal{N}} \to \mathcal{V}/\mathcal{W}$ and it follows directly from the definitions that it satisfies the requirements of the lemma. $\square$

Remark 2. Our geometric motivation for Lemma 1 is the following. Let $\mathcal{N} = \mathcal{N}_c^X : \Gamma \to \mathbb{Z}^{m\times m}$ be the function associated to a Lefschetz fibration $X \to \Sigma$ and a distinguished configuration c via (1.2). Let $V \subset H_n(M)$ be the submodule generated by the vanishing cycles $\rho(g)L_i$ and $W \subset V$ be the kernel of the intersection form on V. Then the homomorphism

$$\Lambda \to V : \lambda \mapsto \sum_{g,i} \lambda_i(g)\rho(g^{-1})L_i$$

descends to a Γ-equivariant isomorphism of $\mathbb{Z}$-modules $\Phi_{\mathcal{N}} : \mathbb{H}_{\mathcal{N}} \to V/W$ that identifies the pairing in (1.4) with the intersection form and maps the element $\mathbb{L}_i$ defined by (1.4) to the equivalence class $[L_i] \in V/W$.

Example 1. Assume that Γ is generated freely by $g_1, \ldots, g_m$. In such a case, the function $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ in Definition 1 is completely determined by $N := \mathcal{N}(1)$. This matrix satisfies

$$N^T = (-1)^n N, \qquad n_{ii} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}. \end{cases}$$

It determines a monodromy representation

$$\mathbb{H}_N := \mathbb{Z}^m / \ker N, \qquad \langle \mu, \lambda \rangle := \mu^T N \lambda,$$

with special elements associated to the standard basis of $\mathbb{Z}^m$ and the action $\rho_N : \Gamma \to \mathrm{Aut}(\mathbb{H}_N)$ uniquely determined by

$$\rho_N(g_i) = \mathbb{1} - (-1)^{n(n+1)/2} E_i N. \tag{2.17}$$

Here $E_i \in \mathbb{Z}^{m\times m}$ denotes the matrix with the i-th entry on the diagonal equal to 1 and zeroes elsewhere. The function $\mathcal{N}$ can be recovered from the matrix N via

$$\mathcal{N}(g) = N\rho_N(g),$$

where $\rho_N : \Gamma \to \mathrm{GL}_m(\mathbb{Z})$ is defined by equation (2.17). Moreover, the monodromy representations $\mathbb{H}_N$ and $\mathbb{H}_{\mathcal{N}}$ are isomorphic. The isomorphism $\mathbb{H}_N \to \mathbb{H}_{\mathcal{N}}$ assigns to each $[v] \in \mathbb{H}_N$ the equivalence class of the function $\lambda_v : \Gamma \to \mathbb{Z}^m$ with value v at 1 and zero elsewhere. The inverse isomorphism is induced by the map $\mathbb{Z}[\Gamma]^m \to \mathbb{Z}^m : \lambda \mapsto \sum_g \rho_N(g^{-1})\lambda(g)$.

The $\mathbb{Z}$-module $M_m(\mathbb{Z}[\Gamma])$ of $m \times m$-matrices with entries in the group ring is naturally an algebra over the group ring. We think of an element $\mathcal{M} \in M_m(\mathbb{Z}[\Gamma])$ also as a function $\mathcal{M} : \Gamma \to \mathbb{Z}^{m\times m}$ with finite support. Hence the product is given by

$$\mathcal{M}\mathcal{N}(g) = \sum_{h\in\Gamma} \mathcal{M}(gh^{-1})\mathcal{N}(h)$$

and this formula continues to be meaningful when only one of the factors has finite support. The group ring $\mathbb{Z}[\Gamma]$ is equipped with an involution $\lambda \mapsto \bar{\lambda}$ given by $\bar{\lambda}(g) := \lambda(g^{-1})$. The **conjugate transpose** of a matrix $\mathcal{M} \in M_m(\mathbb{Z}[\Gamma])$ is defined by

$$\mathcal{M}^t(g) := \overline{\mathcal{M}}(g)^T = \mathcal{M}(g^{-1})^T$$

and it satisfies $(\mathcal{M}\mathcal{N})^t = \mathcal{N}^t\mathcal{M}^t$.

Let $\mathcal{B}$ be a group that acts covariantly on Γ and denote by $\mathcal{B} \times \Gamma \to \Gamma : (\sigma, g) \mapsto \sigma_* g$ the action. In the intended application Γ is the fundamental group of a Riemann surface with m punctures and $\mathcal{B}$ is the braid group on m strings in the same Riemann surface with one puncture. The action of $\mathcal{B}$ on Γ extends linearly to an action on $\mathbb{Z}[\Gamma]$ by algebra automorphisms given by

$$\sigma_*\lambda = \sum_{g\in\Gamma}(\sigma_*\lambda)(g)g := \sum_{g\in\Gamma}\lambda(g)\sigma_* g, \qquad (\sigma_*\lambda)(g) := \lambda((\sigma^{-1})_* g).$$

This action extends to the $\mathbb{Z}$-module $\mathbb{Z}[[\Gamma]]$ of formal sums of elements of Γ with integer coefficients. These correspond to arbitrary integer valued functions on Γ. So $\mathcal{B}$ acts on $M_m(\mathbb{Z}[\Gamma])$ componentwise, or equivalently by

$$\sigma_*\mathcal{M} := \mathcal{M} \circ (\sigma^{-1})_* : \Gamma \to \mathbb{Z}^{m\times m}. \tag{2.18}$$

We then have

$$\sigma_*(\mathcal{M}\mathcal{N}) = (\sigma_*\mathcal{M})(\sigma_*\mathcal{N}), \qquad \sigma_*(\mathcal{M}^t) = (\sigma_*\mathcal{M})^t \tag{2.19}$$

for $\mathcal{M}, \mathcal{N} \in M_m(\mathbb{Z}[\Gamma])$. The action of $\mathcal{B}$ on $M_m(\mathbb{Z}[\Gamma])$ induces an action on the group $\mathrm{GL}_m(\mathbb{Z}[\Gamma])$ of invertible elements of $M_m(\mathbb{Z}[\Gamma])$.

The following notion plays a crucial role in this paper. In our intended application G is the braid group, or the framed braid group, and A is the group $\mathrm{GL}_m(\mathbb{Z}[\Gamma])$ of invertible matrices with entries in the group ring of Γ.

Definition 2 (Serre [22]). *Let* G *and* A *be groups. Suppose that* G *acts covariantly on* A *and denote the action by* $\mathrm{G} \times A \to A : (g, a) \mapsto g_* a$. *A map* $s : \mathrm{G} \to A$ *is called a* **cocycle** *if*

$$s(gh) = s(g)g_* s(h). \tag{2.20}$$

Two cocycles $s_0, s_1 : \mathrm{G} \to A$ *are called* **cohomologous** *if there is an element* $a \in A$ *such that*

$$s_1(g) = a^{-1}s_0(g)g_* a. \tag{2.21}$$

The set of equivalence classes of cocycles is denoted by $H^1(\mathrm{G}, A)$.

Remark 3. The semidirect product $\mathrm{G} \ltimes A$ is equipped with the group operation

$$(g, a) \cdot (h, b) := (gh, ag_*b).$$

A map $s : \mathrm{G} \to A$ is a cocycle if and only if the map $\mathrm{G} \to \mathrm{G} \ltimes A : g \mapsto (g, s(g))$ is a homomorphism (and the homomorphisms associated to two cohomologous cocycles are conjugate by the element $(1, a) \in \mathrm{G} \ltimes A$). Observe that the cocycle condition implies $s(1) = 1$.

Lemma 2. *Every cocycle $\mathcal{S} : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ induces a contravariant action of $\mathcal{B}$ on the space of all functions $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$ via*

$$\sigma^*\mathcal{N} := \big(\mathcal{S}(\sigma)^t \mathcal{N} \mathcal{S}(\sigma)\big) \circ \sigma_*. \tag{2.22}$$

Moreover, two cohomologous cocycles induce conjugate actions.

Proof: For $\sigma, \tau \in \mathcal{B}$ we have

$$\begin{aligned}
(\sigma\tau)^*\mathcal{N} &= \big(\mathcal{S}(\sigma\tau)^t \mathcal{N} \mathcal{S}(\sigma\tau)\big) \circ (\sigma\tau)_* \\
&= \big(\sigma_*(\mathcal{S}(\tau))^t \cdot \mathcal{S}(\sigma)^t \mathcal{N} \mathcal{S}(\sigma) \cdot \sigma_*(\mathcal{S}(\tau))\big) \circ \sigma_* \circ \tau_* \\
&= \big(\mathcal{S}(\tau)^t \cdot \sigma^*\mathcal{N} \cdot \mathcal{S}(\tau)\big) \circ \tau_* = \tau^*\sigma^*\mathcal{N}.
\end{aligned}$$

Here we have used the cocycle condition and (2.19).

Assume two cocycles $\mathcal{S}$ and $\mathcal{S}'$ are cohomologous, i.e. there exists a matrix $\mathcal{A} \in \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ such that $\mathcal{S}'(\sigma) = \mathcal{A}^{-1}\mathcal{S}(\sigma)\sigma_*\mathcal{A}$ for all $\sigma \in \mathcal{B}$. Denote by σ^* and $\sigma^{*'}$ the actions defined by $\mathcal{S}$ and $\mathcal{S}'$ respectively, and denote

$$C_{\mathcal{A}} : \mathrm{Map}(\Gamma, \mathbb{Z}^{m\times m}) \to \mathrm{Map}(\Gamma, \mathbb{Z}^{m\times m}), \qquad \mathcal{N} \mapsto \mathcal{A}^t \mathcal{N} \mathcal{A}. \tag{2.23}$$

A straightforward verification shows that we have $\sigma^{*'} = C_{\mathcal{A}} \circ \sigma^* \circ C_{\mathcal{A}}^{-1}$. □

Proposition 1. *Let $\mathcal{S} : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ be a cocycle. For every $\sigma \in \mathcal{B}$ and every function $\mathcal{N} : \Gamma \to \mathbb{Z}^{m\times m}$, the isomorphism*

$$\Lambda \to \Lambda : \lambda \mapsto \mathcal{S}(\sigma)(\sigma_*\lambda)$$

descends to an isomorphism

$$\mathbb{S}_\sigma : \mathbb{H}_{\sigma^*\mathcal{N}} \to \mathbb{H}_{\mathcal{N}} \tag{2.24}$$

which preserves the bilinear pairings and fits into a commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\rho_{\sigma^*\mathcal{N}}"] \arrow[d, "\sigma_*"'] & \mathrm{Aut}(\mathbb{H}_{\sigma^*\mathcal{N}}) \arrow[d] \\
\Gamma \arrow[r, "\rho_{\mathcal{N}}"'] & \mathrm{Aut}(\mathbb{H}_{\mathcal{N}})
\end{tikzcd} \tag{2.25}$$

with the second vertical arrow given by $\phi \mapsto \mathbb{S}_\sigma \phi \mathbb{S}_\sigma^{-1}$. Moreover, two cohomologous cocycles induce equivalent isomorphisms (2.24).

Proof: For any $\sigma \in \mathcal{B}$ and any function $\mathcal{M} : \Gamma \to \mathbb{Z}^{m\times m}$ the isomorphism $\sigma_* : \Lambda \to \Lambda$ descends to an isomorphism $R_\sigma : \mathbb{H}_{\mathcal{M}\circ\sigma_*} \to \mathbb{H}_{\mathcal{M}}$ because we have $(\mathcal{M}\circ\sigma_*)\lambda = (\sigma^{-1})_*(\mathcal{M}\cdot\sigma_*\lambda)$. That R_σ preserves the bilinear pairings follows from the identity $(\sigma_*\mu)^t\mathcal{M}(\sigma_*\lambda) = \sigma_*(\mu^t(\mathcal{M}\circ\sigma_*)\lambda)$ and the formula $\langle\lambda,\mu\rangle = (\lambda^t\mathcal{M}\mu)(1)$ for the pairing (2.13) on $\mathbb{H}_{\mathcal{M}}$.

For any $\sigma \in \mathcal{B}$ we also have an isomorphism $Q_\sigma : \mathbb{H}_{\mathcal{S}(\sigma)^t\mathcal{N}\mathcal{S}(\sigma)} \to \mathbb{H}_{\mathcal{N}}$ induced by the map $\lambda \mapsto \mathcal{S}(\sigma)\lambda$. This follows from the associativity of the convolution product $(\mathcal{S}^t\mathcal{N}\mathcal{S})\lambda = \mathcal{S}^t(\mathcal{N}\mathcal{S}\lambda)$ and the fact that $\mathcal{S}(\sigma)$ is invertible. That Q_σ preserves the bilinear pairing follows from the identity $(\mathcal{S}\mu)^t\mathcal{N}(\mathcal{S}\lambda) = \mu^t(\mathcal{S}^t\mathcal{N}\mathcal{S})\lambda$.

The isomorphism $\mathbb{S}_\sigma : \mathbb{H}_{\sigma^*\mathcal{N}} \to \mathbb{H}_{\mathcal{N}}$ is the composition $\mathbb{S}_\sigma := Q_\sigma \circ R_\sigma$ where we take $\mathcal{M} := \mathcal{S}(\sigma)^t\mathcal{N}\mathcal{S}(\sigma)$. By what we have already proved, this isomorphism preserves the bilinear pairings. The commutativity of the diagram (2.25) is equivalent to the equation

$$\rho_{\mathcal{N}}(\sigma_* g)\circ\mathbb{S}_\sigma = \mathbb{S}_\sigma\circ\rho_{\sigma^*\mathcal{N}}(g), \qquad g \in \Gamma. \tag{2.26}$$

To prove (2.26) note that

$$(\rho_{\mathcal{N}}(\sigma_* g)\circ\mathbb{S}_\sigma)(\lambda) = \rho_{\mathcal{N}}(\sigma_* g)\big(\mathcal{S}(\sigma)\sigma_*\lambda\big) = \big(\mathcal{S}(\sigma)\sigma_*\lambda\big)(\sigma_* g)^{-1},$$

and

$$(\mathbb{S}_\sigma\circ\rho_{\sigma^*\mathcal{N}}(g))(\lambda) = \mathbb{S}_\sigma(\lambda g^{-1}) = \mathcal{S}(\sigma)\sigma_*(\lambda g^{-1}) = \big(\mathcal{S}(\sigma)\sigma_*\lambda\big)(\sigma_* g)^{-1}.$$

To prove the last assertion, we assume that $\mathcal{S}$ and $\mathcal{S}'$ are two cohomologous cocycles. Thus there is a matrix $\mathcal{A} \in \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ such that $\mathcal{S}'(\sigma) = \mathcal{A}^{-1}\mathcal{S}(\sigma)\sigma_*\mathcal{A}$ for all $\sigma \in \mathcal{B}$. Denote by σ^* and $\sigma^{*'}$ the actions defined by $\mathcal{S}$ and $\mathcal{S}'$ respectively. We proved in Lemma 2 the relation $\sigma^{*'} = C_{\mathcal{A}}\circ\sigma^*\circ C_{\mathcal{A}}^{-1}$, where the map $C_{\mathcal{A}}$ is defined by (2.23). We then have a commutative diagram

$$\begin{tikzcd}
\mathbb{H}_{\sigma^{*\prime}(\mathcal{N})} \arrow[r, "L_{\mathcal{A}}"] \arrow[d, "\mathbb{S}'_\sigma"'] & \mathbb{H}_{\sigma^{*\prime}(\mathcal{N})\mathcal{A}^{-1}} = \mathbb{H}_{\sigma^*((\mathcal{A}^{-1})^t\mathcal{N}\mathcal{A}^{-1})} \arrow[d, "\mathbb{S}_\sigma"] \\
\mathbb{H}_{\mathcal{N}} = \mathbb{H}_{(\mathcal{A}^{-1})^t\mathcal{N}} \arrow[r, "L_{\mathcal{A}}"'] & \mathbb{H}_{(\mathcal{A}^{-1})^t\mathcal{N}\mathcal{A}^{-1}}.
\end{tikzcd}$$

Here the two isomorphisms denoted $L_{\mathcal{A}}$ are induced by the left multiplication $\Lambda \to \Lambda : \lambda \mapsto \mathcal{A}\lambda$. The commutativity of the diagram is checked at the level of Λ using the relation between the cocycles $\mathcal{S}'$ and $\mathcal{S}$. □

3 The framed braid group

In this section we recall the well known correspondence between the braid group and the mapping class group (see [4, 17]) and extend it to the framed braid group introduced by Ko and Smolinsky [18].

Let Σ be a compact oriented 2-manifold, possibly with boundary, let also $Z \subset \Sigma\setminus\partial\Sigma$ be a finite set consisting of m points, and choose a base point $z_0 \in \Sigma\setminus Z$. We assume throughout that Σ is not diffeomorphic to the 2-sphere and that $z_0 \in \partial\Sigma$ whenever Σ is diffeomorphic to the 2-disc. Denote by $\mathrm{Diff}(\Sigma, z_0)$ the group of all diffeomorphisms of Σ that fix z_0 and by $\mathrm{Diff}_0(\Sigma, z_0) \subset \mathrm{Diff}(\Sigma, z_0)$ the identity component. Define

$$\mathcal{G} := \{\phi \in \mathrm{Diff}_0(\Sigma, z_0) \,|\, \phi(Z) = Z\},$$
$$\mathcal{G}_0 := \{\phi \in \mathcal{G} \,|\, \exists \phi_t \in \mathcal{G} \text{ s.t. } \phi_0 = \mathrm{id},\, \phi_1 = \phi\}.$$

Here $[0,1] \to \mathrm{Diff}(\Sigma, z_0) : t \mapsto \phi_t$ is a smooth isotopy of diffeomorphisms fixing the base point. Thus $\mathcal{G}_0$ is the identity component of $\mathcal{G}$. We refer to the quotient

$$\mathcal{G}/\mathcal{G}_0 = \pi_0(\mathcal{G})$$

as the **mapping class group**. It is naturally isomorphic to the braid group.

Fix an ordering $Z = \{z_1, \dots, z_m\}$ and let $\mathfrak{S}_m$ denote the group of permutations. The **braid group** $\mathcal{B}$ on m strings in $\Sigma \setminus \{z_0\}$ based at Z is defined as the fundamental group of the configuration space of m unordered distinct points in $\Sigma \setminus \{z_0\}$. Think of a braid as an m-tuple of smooth paths $\beta_i : [0,1] \to \Sigma$, $i = 1, \dots, m$, which avoid z_0, are pairwise distinct for each t, and satisfy $\beta_i(0) = z_i$, $\beta_i(1) = z_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_m$. Thus $\mathcal{B}$ is the group of homotopy classes of braids. The composition law is $[\beta] \cdot [\alpha] := [\beta\alpha]$, where $\beta\alpha$ is the braid obtained by first running through α and then through β.

The isomorphism $\Phi : \mathcal{B} \to \mathcal{G}/\mathcal{G}_0$ is defined as follows. Given a braid β choose a smooth isotopy $\{\phi_t\}_{0 \le t \le 1}$ in $\mathrm{Diff}(\Sigma, z_0)$ with $\phi_0 = \mathrm{id}$ satisfying $\phi_t(z_i) = \beta_i(t)$ for $i = 1, \dots, m$ and define $\Phi([\beta]) := [\phi_1]$. To see that this map is well defined choose a smooth family of vector fields $v_t \in \mathrm{Vect}(\Sigma)$ satisfying $v_t(z_0) = 0$ and

$$v_t(\beta_i(t)) = \dot{\beta}_i(t), \qquad i = 1, \dots, m,$$

and let ϕ_t be the isotopy generated by v_t via $\partial_t \phi_t = v_t \circ \phi_t$ and $\phi_0 = \mathrm{id}$. The existence of v_t follows from an easy argument using cutoff functions, and that the isotopy class of ϕ_1 is independent of the choices of β and v_t follows from a parametrized version of the same argument respectively from taking convex combinations of vector fields. We claim that Φ is an isomorphism. (See [4, Theorem 4.3] for a slightly different statement.)

That Φ is a surjective group homomorphism is obvious. Thus we have an exact sequence

$$\pi_1\left(\mathrm{Diff}_0(\Sigma, z_0), \mathrm{id}\right) \longrightarrow \mathcal{B} \xrightarrow{\Phi} \mathcal{G}/\mathcal{G}_0 \longrightarrow 1, \tag{3.1}$$

where the first map sends an isotopy $\{\phi_t\}_{0 \le t \le 1}$ with $\phi_0 = \phi_1 = \mathrm{id}$ to the braid in $\Sigma \setminus \{z_0\}$ defined by $t \mapsto \phi_t(Z)$. Hence the injectivity of Φ follows from the fact that $\mathrm{Diff}_0(\Sigma, z_0)$ is simply connected. (This is why we exclude the 2-sphere and the 2-disc with a base point in the interior: see Remark 4 below.) In fact it is contractible. To see this in the case $\chi(\Sigma) < 0$ we use the fact that the identity component $\mathrm{Diff}_0(\Sigma)$ of $\mathrm{Diff}(\Sigma)$ is contractible [10, 11] and consider the fibration

$$\mathrm{Diff}_0(\Sigma, z_0) \hookrightarrow \mathrm{Diff}_0(\Sigma) \to \widetilde{\Sigma}.$$

The map $\mathrm{Diff}_0(\Sigma) \to \widetilde{\Sigma}$ assigns to every diffeomorphism $\phi \in \mathrm{Diff}_0(\Sigma)$ the homotopy class of the path $[0,1] \to \Sigma : t \mapsto \phi_t(z_0)$ with fixed endpoints, where $[0,1] \to \mathrm{Diff}_0(\Sigma) : t \mapsto \phi_t$ is a smooth isotopy with $\phi_0 = \mathrm{id}$ and $\phi_1 = \phi$. This map is well defined because $\mathrm{Diff}_0(\Sigma)$ is simply connected. If the Euler characteristic is zero Σ is either diffeomorphic to the 2-torus or to the annulus. In both cases $\mathrm{Diff}_0(\Sigma, z_0) = \mathrm{Diff}(\Sigma, z_0) \cap \mathrm{Diff}_0(\Sigma)$ acts freely on $\mathcal{J}(\Sigma)$ and there is a diffeomorphism $\mathrm{Diff}_0(\Sigma, z_0) \times \mathcal{T}(\Sigma) \to \mathcal{J}(\Sigma)$ where $\mathcal{T}(\Sigma) := \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ is diffeomorphic to the upper half space $\mathbb{H}$ in the case of the torus and to an open interval in the case of the annulus. In the case of the disc $\mathbb{D}$ the group $\mathrm{Diff}_0(\mathbb{D})$ of orientation preserving diffeomorphisms acts transitively on $\mathcal{J}(\mathbb{D})$ with isotropy subgroup $\mathrm{PSL}(2, \mathbb{R})$. This action gives rise to a diffeomorphism from $\mathcal{J}(\mathbb{D})$ to the subgroup of all diffeomorphisms that fix a point on the boundary and a point in the interior. Hence the group $\mathrm{Diff}_0(\mathbb{D}, z_0)$ is contractible for every point $z_0 \in \partial\mathbb{D}$.

Remark 4. In the cases $\Sigma = S^2$ and $\Sigma = \mathbb{D}$ with $z_0 \notin \partial\mathbb{D}$ the group $\mathrm{Diff}_0(\Sigma, z_0)$ is not contractible but homotopy equivalent to the circle. It can be deduced from the exact sequence (3.1) that Φ is not injective and that, instead, $\mathcal{B}$ is in these two cases a central extension of the mapping class group $\mathcal{G}/\mathcal{G}_0$ by an infinite cyclic group $\mathcal{Z}$. If $\Sigma = S^2$ then $\mathcal{B}$ can be interpreted as the braid group on m strings in $\mathbb{C}$, and the subgroup $\mathcal{Z}$ is the center of $\mathcal{B}$ for $m \geq 3$ [4, 17]. If $\Sigma = \mathbb{D}$ with $z_0 \notin \partial\mathbb{D}$ then $\mathcal{B}$ can be interpreted as the subgroup of the braid group on $m+1$ strings in $\mathbb{D}$ that fixes the point z_0, and $\mathcal{Z}$ is the center of $\mathcal{B}$ for all $m \geq 1$.

The framed braid group is an extension of the braid group $\mathcal{B}$. Choose nonzero tangent vectors $v_z \in T_z\Sigma$ for $z \in Z$ and define

$$\widetilde{\mathcal{G}} := \left\{\phi \in \mathrm{Diff}_0(\Sigma, z_0) \,|\, \phi(Z) = Z,\, d\phi(z)v_z = v_{\phi(z)}\ \forall z \in Z\right\},$$
$$\widetilde{\mathcal{G}}_0 := \left\{\phi \in \widetilde{\mathcal{G}} \,|\, \exists \phi_t \in \widetilde{\mathcal{G}} \text{ s.t. } \phi_0 = \mathrm{id},\, \phi_1 = \phi\right\}.$$

Thus $\widetilde{\mathcal{G}}_0$ is the identity component of $\widetilde{\mathcal{G}}$. The **marked mapping class group** is the quotient

$$\widetilde{\mathcal{G}}/\widetilde{\mathcal{G}}_0 = \pi_0(\widetilde{\mathcal{G}}).$$

It is naturally isomorphic to the **framed braid group** $\widetilde{\mathcal{B}}$ on m strings in $\Sigma \setminus \{z_0\}$. Fix again an ordering $Z = \{z_1, \ldots, z_m\}$ and denote $v_i := v_{z_i}$ for $i = 1, \ldots, m$. The framed braid group is defined as the fundamental group of the configuration space of m unordered points in the complement of the zero section in the tangent bundle of $\Sigma \setminus \{z_0\}$ whose projections to the base are pairwise distinct. Think of a framed braid as an m-tuple of smooth paths $(\beta_i, \xi_i) : [0, 1] \to T\Sigma$ for $i = 1, \ldots, m$ such that $(\beta_1, \ldots, \beta_m)$ is a braid in $\Sigma \setminus \{z_0\}$ satisfying $\beta_i(0) = z_i$ and $\beta_i(1) = z_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_m$, and each ξ_i is a nowhere vanishing vector field along β_i such that $\xi_i(0) = v_i$ and $\xi_i(1) = v_{\pi(i)}$. Thus $\widetilde{\mathcal{B}}$ is the group of homotopy classes of framed braids.

The isomorphism $\widetilde{\Phi} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{G}}/\widetilde{\mathcal{G}}_0$ is defined as follows. Given a framed braid (β, ξ) choose a smooth isotopy $\{\phi_t\}_{0 \leq t \leq 1}$ in $\mathrm{Diff}(\Sigma, z_0)$ with $\phi_0 = \mathrm{id}$ satisfying

$$\phi_t(z_i) = \beta_i(t), \qquad d\phi_t(z_i)v_i = \xi_i(t), \qquad i = 1, \ldots, m, \tag{3.2}$$

and define $\widetilde{\Phi}([\beta, \xi]) := [\phi_1]$. To see that this map is well defined choose a smooth family of vector fields $v_t \in \mathrm{Vect}(\Sigma)$ satisfying $v_t(z_0) = 0$ and

$$v_t(\beta_i(t)) = \dot{\beta}_i(t), \qquad \nabla_{\xi_i(t)} v_t(\beta_i(t)) = \nabla_t \xi_i(t), \qquad i = 1, \ldots, m. \tag{3.3}$$

(Here ∇ is a torsion free connection on $T\Sigma$ but the second equation in (3.3) is independent of this choice.) Now let ϕ_t be the isotopy generated by v_t via $\partial_t\phi_t = v_t \circ \phi_t$ and $\phi_0 = \mathrm{id}$. Then ϕ_t satisfies (3.2). As above, the existence of v_t follows from an easy argument using cutoff functions, and that the isotopy class of ϕ_1 is independent of the choices of β and v_t follows from a parametrized version of the same argument respectively from taking convex combinations of vector fields. That $\widetilde{\Phi}$ is a surjective group homomorphism is obvious and that it is injective follows again from the definitions and the fact that $\mathrm{Diff}_0(\Sigma, z_0)$ is simply connected.

Remark 5. There is an obvious action of the mapping class group on the braid group induced by the action of $\mathcal{G}$ on $\mathcal{B}$ via

$$\phi_*[\beta_1, \dots, \beta_m] := [\phi \circ \beta_{\pi(1)}, \dots, \phi \circ \beta_{\pi(m)}],$$

for $\phi \in \mathcal{G}$ and a braid $\beta = (\beta_1, \dots, \beta_m)$, where $\pi \in \mathfrak{S}_m$ is defined by $\phi(z_{\pi(i)}) = z_i$. On the other hand we have seen that the mapping class group can be identified with the braid group. The resulting action of the braid group on itself is given by inner automorphisms. In other words

$$\Phi(\alpha)_*\beta = \alpha\beta\alpha^{-1}$$

for $\alpha, \beta \in \mathcal{B}$. The same holds for the framed braid group.

The framed braid group fits into an exact sequence

$$0 \to \mathbb{Z}^m \to \widetilde{\mathcal{B}} \to \mathcal{B} \to 1. \tag{3.4}$$

This extension splits by choosing a nowhere vanishing vector field w on $\Sigma \setminus \{z_0\}$ such that $w_z = v_z$ at each $z \in Z$. The splitting depends on the homotopy class of w relatively to Z, so that it is not unique in general. In the following we shall not distinguish in notation between the mapping class group $\pi_0(\mathcal{G})$ and the braid group $\mathcal{B}$, nor between $\pi_0(\widetilde{\mathcal{G}})$ and $\widetilde{\mathcal{B}}$.

4 Distinguished configurations

Let Σ, Z, z_0 be as in Section 3. An m-tuple $c = (c_1, \dots, c_m)$ of smooth paths $c_i : [0,1] \to \Sigma$ is called a **distinguished configuration** if

(i) each c_i is an embedding with $c_i(0) = z_0$ and, for $i \neq j$, the paths c_i and c_j meet only at z_0;

(ii) $\{c_1(1), \dots, c_m(1)\} = Z$;

(iii) the vectors $\dot{c}_1(0), \dots, \dot{c}_m(0)$ are pairwise linearly independent and are ordered clockwise in $T_{z_0}\Sigma$.

Two distinguished configurations c^0 and c^1 are called **homotopic** if there is a smooth homotopy $\{c^\lambda\}_{0 \le \lambda \le 1}$ of distinguished configurations from c^0 to c^1. We write $c^0 \sim c^1$ if c^0 is homotopic to c^1 and denote the homotopy class of a distinguished configuration c by $[c]$. The set of homotopy classes of distinguished configurations will be denoted by $\mathcal{C}$. Note that each distinguished configuration c determines an ordering $Z = \{z_1, \dots, z_m\}$ via $z_i := c_i(1)$.

Theorem 1. *The braid group $\mathcal{B}$ acts freely and transitively on $\mathcal{C}$ via*

$$\big([\phi],[c_1,\ldots,c_m]\big)\mapsto[\phi\circ c_1,\ldots,\phi\circ c_m]=:[\phi_*c]. \tag{4.1}$$

Proof: We prove that the action is transitive. Given two distinguished configurations c and c' we need to construct an element $\psi\in\mathrm{Diff}_0(\Sigma,z_0)$ such that ψ_*c is homotopic to c'. Up to homotopy we can assume that there is a constant $\varepsilon>0$ such that $c_i(t)=c_i'(t)$ for $0\le t\le\varepsilon$. Now construct an isotopy $[\varepsilon,1]\to\mathrm{Diff}_0(\Sigma,z_0):\lambda\mapsto\phi_\lambda$ satisfying $\phi_1=\mathrm{id}$ and

$$\phi_\lambda(c_i(t))=c_i(\lambda t),\qquad\varepsilon\le\lambda\le1,\quad i=1,\ldots,m,$$

by choosing an appropriate family of vector fields. Choose an analogous isotopy ϕ'_λ for c' and define

$$\psi:=(\phi'_\varepsilon)^{-1}\circ\phi_\varepsilon\in\mathcal{G}.$$

Then $\psi(c_i(t))=c_i'(t)$ as required.

We prove that the action is free when $z_0\notin\partial\Sigma$. The case $z_0\in\partial\Sigma$ is similar. Let c be a distinguished configuration and $\phi\in\mathcal{G}$ such that ϕ_*c is homotopic to c. We prove in five steps that $\phi\in\mathcal{G}_0$.

Step 1. *We may assume that $d\phi(z_0)=\mathbb{1}$.*

It is enough to prove that, for every matrix $A\in\mathrm{GL}^+(n,\mathbb{R})$, there exists a diffeomorphism $\psi:\mathbb{R}^n\to\mathbb{R}^n$, supported in the unit ball and isotopic to the identity through diffeomorphisms with support in the unit ball, that satisfies $\psi(0)=0$ and $d\psi(0)=A$. If A is symmetric and positive definite we may assume that A is a diagonal matrix and choose ψ in the form

$$\psi(x)=(\psi_1(x_1),\ldots,\psi_n(x_n))$$

where each ψ_i is a suitable monotone diffeomorphism of $\mathbb{R}$. If A is orthogonal we choose a smooth path $[0,1]\to\mathrm{SO}(n):r\mapsto A_r$, constant near the ends, with $A_0=A$ and $A_1=\mathbb{1}$ and define

$$\psi(x):=A_{|x|}x.$$

The general case follows by polar decomposition.

Step 2. *We may assume that ϕ agrees with the identity near z_0.*

Let ϕ be a diffeomorphism of $\mathbb{R}^n$ with $\phi(0)=0$ and $d\phi(0)=\mathbb{1}$ and choose a smooth nonincreasing cutoff function $\beta:[0,1]\to[0,1]$ equal to one near zero and vanishing near one. Then, for $\varepsilon>0$ sufficiently small, the formula

$$\phi_\lambda(x):=\lambda\beta(|x|/\varepsilon)x+(1-\lambda\beta(|x|/\varepsilon))\phi(x)$$

defines an isotopy from $\phi_0=\phi$ to a diffeomorphism ϕ_1 equal to the identity near the origin such that, for each λ, ϕ_λ agrees with ϕ outside the ball of radius ε. Now choose a local coordinate chart near z_0 to carry this construction over to Σ.

Step 3. *We may assume that ϕ agrees with the identity near z_0 and $\phi_*c=c$.*

Assume, by Step 2, that ϕ agrees with the identity near z_0. Then the homotopy $\lambda \mapsto c^\lambda$ from $c^0 = c$ to $c^1 = \phi_* c$ can be chosen such that $c^\lambda(t)$ is independent of t for t sufficiently small. Hence there exists a family of vector fields $v^\lambda \in \mathrm{Vect}(\Sigma)$ satisfying

$$v^\lambda(c_i^\lambda(t)) = \partial_\lambda c_i^\lambda(t)$$

for all λ, t, i and $v^\lambda(z) = 0$ for z near z_0. Integrating this family of vector fields yields a diffeomorphism $\psi \in \mathcal{G}_0$ such that $\psi_* \phi_* c = c$.

Step 4. *We may assume that ϕ agrees with the identity near the union of the images of the c_i.*

Assume, by Step 3, that $\phi(c_i(t)) = c_i(t)$ for all t and i and that ϕ agrees with the identity near z_0. If $d\phi(c_i(t)) = 1\!\!1$ for all i and t we can use an interpolation argument as in Step 2 to deform ϕ to a diffeomorphism that satisfies the requirement of Step 4. To achieve the condition $d\phi(c_i(t)) = 1\!\!1$ via a prior deformation we must solve the following problem. Given two smooth functions $a : [0,1] \to \mathbb{R}$ and $b : [0,1] \to (0,\infty)$ find a diffeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ that is equal to the identity outside a small neighborhood of the set $[0,1] \times \{0\}$ and satisfies

$$\psi(x,0) = (x,0), \qquad d\psi(x,0) = \begin{pmatrix} 1 & a(x) \\ 0 & b(x) \end{pmatrix}, \qquad 0 \le x \le 1.$$

It suffices to treat the cases $a(x) \equiv 0$ and $b(x) \equiv 1$. For $b(x) \equiv 1$ one can use an interpolation argument as in the proof of Step 2. For $a(x) \equiv 0$ one can use a parametrized version of the argument for the positive definite case in the proof of Step 1.

Step 5. *We prove that $\phi \in \mathcal{G}_0$.*

Assume, by Step 4, that there exists a coordinate chart $u : \mathbb{D} \to \Sigma$ such that $c_i(t) \in u(\mathbb{D})$ for all i and t and $\phi \circ u = u$. Choose any isotopy

$$[0,1] \to \mathrm{Diff}_0(\Sigma, z_0) : \lambda \mapsto \phi_\lambda$$

from $\phi_0 = \mathrm{id}$ to $\phi_1 = \phi$. By a parametrized version of the argument in Step 1 we may assume that $d\phi_\lambda(z_0) = 1\!\!1$ for every λ. By a parametrized version of the argument in Step 2 we may assume that there is an $\varepsilon > 0$ such that $\phi_\lambda \circ u$ agrees with u on the disc of radius ε for every λ. Choose a diffeomorphism $\psi : \Sigma \to \Sigma$, supported in $u(\mathbb{D})$, such that

$$\psi(u(\{z \in \mathbb{D} \,|\, |z| \le 1 - \varepsilon\})) = u(\{z \in \mathbb{D} \,|\, |z| \le \varepsilon\}).$$

Then $\lambda \mapsto \psi^{-1} \circ \phi_\lambda \circ \psi$ is an isotopy in $\mathcal{G}$ and so $\phi = \psi^{-1} \circ \phi \circ \psi \in \mathcal{G}_0$. This concludes the proof of the theorem. □

Recall from the Introduction that every distinguished configuration c determines elements $g_{1,c}, \ldots, g_{m,c}$ of the fundamental group $\Gamma = \pi_1(\Sigma \setminus Z, z_0)$, where $g_{i,c}$ is the homotopy class of the loop obtained by traversing c_i, encircling z_i counterclockwise, and then traversing c_i in the opposite direction. Clearly, the $g_{i,c}$ depend only on the homotopy class of c. Conversely, we have the following theorem.

Theorem 2. *If $c, c' \in \mathcal{C}$ satisfy $g_{i,c} = g_{i,c'}$ for $i = 1, \ldots, m$ then $c \sim c'$.*

The proof relies on the classical result of Baer, Dehn and Nielsen asserting that, in dimension two, isotopy coincides with homotopy. Specifically, we need the following theorem due to Epstein [12, Theorem 3.1] and Feustel [13] about embedded arcs in 2-manifolds.

The Epstein–Feustel Theorem. *Let S be a compact 2-manifold with boundary and let $\alpha, \beta : [0,1] \to S$ be smooth embeddings such that*

$$\alpha(0) = \beta(0), \qquad \alpha(1) = \beta(1), \qquad \alpha^{-1}(\partial S) = \beta^{-1}(\partial S) = \{\alpha(0), \alpha(1)\}.$$

If α and β are smoothly homotopic with fixed endpoints then there is a smooth ambient isotopy $[0,1] \times S \to S : (\lambda, p) \mapsto \phi_\lambda(p)$ such that

$$\phi_0 = \mathrm{id}, \qquad \phi_1 \circ \alpha = \beta$$

and $\phi_\lambda|_{\partial S} = \mathrm{id}$ for all $\lambda \in [0,1]$.

In the work of Epstein and Feustel the 2-manifold S is triangulated and the isotopy can be chosen piecewise linear whenever the arcs are piecewise linear. In Feustel's theorem the homeomorphism $\phi_\lambda : S \to S$ fixes the endpoints of the arcs. In Epstein's theorem S need not be compact and the ϕ_λ have uniform compact support and are equal to the identity on the boundary of S. To obtain the smooth isotopy in the above formulation, we first approximate the embedded arcs by piecewise linear arcs, then use Epstein's version of the theorem in the piecewise linear setting, then approximate the piecewise linear isotopy by a smooth isotopy, and finally connect two nearby smooth arcs by a smooth isotopy.

Proof of Theorem 2: Let $\zeta_z \subset \Gamma$ be the conjugacy class determined by a small loop encircling the puncture $z \in Z$. Since $\Sigma \neq S^2$ we have $\zeta_z \neq \zeta_{z'}$ whenever $z \neq z'$. Since $g_{i,c} \in \zeta_{c_i(1)}$ and $g_{i,c'} \in \zeta_{c'_i(1)}$ for $i = 1, \ldots, m$, we deduce that c and c' determine the same ordering of Z:

$$Z = \{z_1, \ldots, z_m\}, \qquad z_i := c_i(1) = c'_i(1).$$

Performing an isotopy of the distinguished configuration c', if necessary, we may assume that c'_i agrees with c_i on the interval $[1 - 2\varepsilon, 1]$ for some $\varepsilon > 0$. Next we denote by $B_{2\varepsilon} \subset \mathbb{C}$ the disc of radius 2ε centered at zero and choose embeddings $\psi_i : B_{2\varepsilon} \to \Sigma$ with disjoint images $U_i := \psi_i(B_{2\varepsilon})$ such that $\psi_i(t) = c_i(1-t)$ for $0 \le t < 2\varepsilon$ and $c_i|_{[0,1-2\varepsilon]}$ takes values in the complement of $U_1 \cup \cdots \cup U_m$ for every i. Let $D_i := \psi_i(B_\varepsilon)$ and denote

$$D := D_1 \cup \cdots \cup D_m \subset \Sigma.$$

Then $\Sigma \setminus D$ is a manifold with boundary and the inclusion $\Sigma \setminus D \hookrightarrow \Sigma \setminus Z$ induces an isomorphism of fundamental groups $\pi_1(\Sigma \setminus D, z_0) \cong \Gamma$. We prove in four steps that the distinguished configurations c and c' are homotopic.

Step 1. *For $i = 1, \ldots, m$ let $h_i \in \Gamma$ be the homotopy class of the based loop that traverses $c_i|_{[0,1-\varepsilon]}$ and then $c'_i|_{[0,1-\varepsilon]}$ in the reverse direction. Then, for every i, there is a $k_i \in \mathbb{Z}$ such that $h_i = g_{i,c}^{k_i}$.*

By definition of h_i we have

$$h_i g_{i,c} h_i^{-1} = g_{i,c'} = g_{i,c} \in \Gamma$$

for $i = 1, \dots, m$. Thus h_i commutes with $g_{i,c}$. Moreover, Γ is a free group. If $m > 1$ or $\partial\Sigma \neq \emptyset$ we can choose a basis of Γ such that $g_{i,c}$ is one of the generators and the assertion follows.

If $m = 1$ and $\partial\Sigma = \emptyset$ then Σ has genus $g \geq 1$, by assumption. Hence the group Γ is free of rank $2g$ and we can choose generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ such that $b := g_{1,c} = \prod_{j=1}^{g} [\alpha_j, \beta_j]$. Let $h := h_1$. Since Γ is free and b commutes with h, the subgroup of Γ generated by b and h is free, by the Nielsen–Schreier theorem, and abelian and hence has rank one. Thus there is a $d \in \Gamma$ such that $b = d^r$ and $h = d^k$ for some $r, k \in \mathbb{Z}$. Since $b \neq 1$ we must have $r \neq 0$. We claim that $r = \pm 1$. To see this, let $\Gamma =: \Gamma_1 \supset \Gamma_2 \supset \cdots$ be the lower central series of Γ, with $\Gamma_{\ell+1} := [\Gamma_\ell, \Gamma_1]$ for $\ell \geq 1$. The quotient $H := \Gamma_1/\Gamma_2$ is free abelian of rank $2g$ with generators $[\alpha_1], \dots, [\alpha_g], [\beta_1], \dots, [\beta_g]$. In this quotient the identity $d^r = b$ becomes $r \cdot [d] = [b] = 0$. Since $r \neq 0$ and H has no torsion we obtain $[d] = 0$ in H, i.e. $d \in \Gamma_2$. Since $b \in \Gamma_2$ we can consider the identity $d^r = b$ in the quotient Γ_2/Γ_3. This quotient is canonically isomorphic to the second exterior power of H, by identifying the equivalence class of a commutator $[u, v]$ with $[u] \wedge [v]$, for all $u, v \in \Gamma$. In $\Lambda^2 H$ the equivalence class of b is equal to $\sum_{j=1}^{g} [\alpha_j] \wedge [\beta_j]$. This is a primitive element, and the equation $r \cdot [d] = [b]$ implies $r = \pm 1$. Thus Step 1 is proved.

Step 2. *We may assume without loss of generality that, for each i, the paths $c_i|_{[0,1-\varepsilon]}$ and $c_i'|_{[0,1-\varepsilon]}$ are smoothly homotopic with fixed endpoints in $\Sigma \setminus D$. Moreover, the path $c_i|_{[1-\varepsilon,1]}$ agrees with the path $c_i'|_{[1-\varepsilon,1]}$ for each i.*

Let k_i be as in Step 1 and choose a smooth cutoff function $\rho : [0, 2\varepsilon] \to \mathbb{R}$ such that $\rho(r) = 1$ for $r \leq \varepsilon$ and $\rho(r) = 0$ for $r \geq 3\varepsilon/2$. Define the diffeomorphism $\phi : \Sigma \to \Sigma$ by setting $\phi(\psi_i(z)) := \psi_i(e^{-2\pi \mathrm{i} k_i \rho(|z|)} z)$ for $z \in B_{2\varepsilon}$ and $i = 1, \dots, m$ and by $\phi(p) := p$ for $p \notin U_1 \cup \cdots \cup U_m$. Replacing c' by the equivalent distinguished configuration $\phi_* c'$ and constructing h_i as in Step 1, we obtain $h_i = 1$ and this proves Step 2.

Step 3. *If $z_0 \notin \partial\Sigma$ we may assume without loss of generality that there is a smooth embedding $\psi_0 : B_{2\varepsilon} \to \Sigma$ and real numbers $2\pi > \theta_1 > \cdots > \theta_m \geq 0$ such that the following holds.*

(a) *The closure of $U_0 := \psi_0(B_{2\varepsilon})$ is disjoint from $\overline{U}_i$ for $i = 1, \dots, m$.*

(b) *For each i we have $c_i(t) = c_i'(t) = \psi_0(e^{\mathrm{i}\theta_i} t)$ for $0 \leq t < \varepsilon$, $\left|\psi_0^{-1}(c_i(t))\right| = \left|\psi_0^{-1}(c_i'(t))\right| = t$ for $\varepsilon \leq t < 2\varepsilon$, and $c_i(t), c_i'(t) \notin U_0$ for $2\varepsilon \leq t \leq 1$.*

(c) *The curves $c_i|_{[\varepsilon,1-\varepsilon]}$ and $c_i'|_{[\varepsilon,1-\varepsilon]}$ are smoothly homotopic with fixed endpoints in the submanifold $\Sigma \setminus (D_0 \cup D)$, where $D_0 := \psi_0(B_\varepsilon)$.*

If $z_0 \in \partial\Sigma$ we may assume the same with a smooth embedding $\psi_0 : \{z \in B_{2\varepsilon} \,|\, \mathrm{Im}\, z \geq 0\} \to \Sigma$ of a half disc and with $\pi > \theta_1 > \cdots > \theta_m > 0$.

Assume $z_0 \notin \partial\Sigma$ and choose any embedding $\psi_0 : B_{2\varepsilon} \to \Sigma$ such that $\psi_0(0) = z_0$ and condition (a) holds. Reparametrizing c_i near $t = 0$ we may assume that $\left|\partial_t(\psi_0^{-1} \circ c_i)(0)\right| = 1$. Rotating the embedding, if necessary, we may assume that there are real numbers

$$2\pi > \theta_1 > \theta_2 > \cdots > \theta_m \geq 0$$

such that

$$\partial_t(\psi_0^{-1} \circ c_i)(0) = e^{\mathrm{i}\theta_i}, \qquad i = 1, \dots, m.$$

Shrinking ε, if necessary, we can deform c_i to a curve that satisfies $c_i(t) = \psi_0(e^{\mathrm{i}\theta_i}t)$ for any $0 \le t < 2\varepsilon$. Shrinking ε again we may assume that $c_i(t) \notin U_0$ for $2\varepsilon \le t \le 1$. Applying the same argument to the c_i', using partial Dehn twists supported in U_0, and shrinking ε again, we may assume that the arcs c_i' satisfy the same two properties. Thus condition (b) is fulfilled.

By Step 2 the curves $c_i|_{[0,1-\varepsilon]}$ and $c_i'|_{[0,1-\varepsilon]}$ are homotopic with fixed endpoints in $\Sigma \setminus D$. Now consider the loop γ_i in $\Sigma \setminus (D_0 \cup D)$ obtained by first traversing $c_i|_{[\varepsilon,1-\varepsilon]}$ and then traversing $c_i'|_{[\varepsilon,1-\varepsilon]}$ in the reverse direction. This loop is contractible in $\Sigma \setminus D$ and hence, as a based loop in $\Sigma \setminus (D_0 \cup D)$ with basepoint $c_i(\varepsilon)$, is homotopic to a multiple of $\partial \overline{D}_0$. Using a suitable multiple of a Dehn twist in the annulus $U_0 \setminus D_0$ (as we did in Step 2), we can replace c' by an equivalent distinguished configuration which still satisfies (b) and such that γ_1 is now contractible in $\Sigma \setminus (D_0 \cup D)$. Hence the curves $c_1|_{[\varepsilon,1-\varepsilon]}$ and $c_1'|_{[\varepsilon,1-\varepsilon]}$ are homotopic with fixed endpoints in $\Sigma \setminus (D_0 \cup D)$. For $i \ge 2$ the curves $c_i([\varepsilon, 1-\varepsilon])$ and $c_i'([\varepsilon, 1-\varepsilon])$ in $\Sigma \setminus (D_0 \cup D)$ have endpoints different from $c_1(\varepsilon)$ and $c_1(1-\varepsilon)$. Hence they have well defined intersection numbers with $c_1|_{[\varepsilon,1-\varepsilon]}$. By what we have just proved these agree with the intersection numbers with $c_1'|_{[\varepsilon,1-\varepsilon]}$. Since $c_i([\varepsilon, 1-\varepsilon])$ is disjoint from $c_1([\varepsilon, 1-\varepsilon])$ and $c_i'([\varepsilon, 1-\varepsilon])$ is disjoint from $c_1'([\varepsilon, 1-\varepsilon])$ we deduce that both intersection numbers are zero. Hence the intersection number of $c_1|_{[\varepsilon,1-\varepsilon]}$ with γ_i is zero for $i \ge 2$. Since the loop γ_i is a multiple of $\partial \overline{D}_0$, we deduce that it is contractible in $\Sigma \setminus (D_0 \cup D)$ for $i \ge 2$. This proves Step 3 in the case $z_0 \notin \partial\Sigma$. The proof in the case $z_0 \in \partial\Sigma$ is similar, assertion (c) being simpler to prove.

Step 4. *The distinguished configurations c and c' are homotopic.*

We prove by induction on $\ell \in \{1, \dots, m\}$ that there exists an ambient isotopy

$$[0,1] \times \Sigma \to \Sigma : (\lambda, p) \mapsto \phi_\lambda(p)$$

such that each ϕ_λ is the identity on $D_0 \cup D$ and $\phi_1(c_i(t)) = c_i'(t)$ for $0 \le t \le 1$ and $i = 1, \dots, \ell$. For $\ell = 1$ the existence of the isotopy follows immediately from Step 3, the Epstein–Feustel theorem, and the second assertion of Step 2.

Now suppose by induction that $\ell \in \{2, \dots, m\}$ and that $c_i' = c_i$ for any $i = 1, \dots, \ell - 1$. By Step 3 the curves $c_\ell|_{[\varepsilon,1-\varepsilon]}$ and $c_\ell'|_{[\varepsilon,1-\varepsilon]}$ are homotopic with fixed endpoints in $\Sigma \setminus (D_0 \cup D)$. Choose a smooth open disc $U \subset \Sigma$ (respectively half disc in the case $z_0 \in \partial\Sigma$) such that $\overline{U}$ is an embedded closed disc (respectively half disc) and

$$D_0 \cup D_1 \cup \dots \cup D_{\ell-1} \subset U, \quad \textstyle\bigcup_{i=1}^{\ell-1} c_i([0,1]) \subset U,$$

$$(\overline{D}_\ell \cup \dots \cup \overline{D}_m) \cap \overline{U} = \emptyset, \quad \textstyle\bigcup_{i=\ell}^{m} c_i((\varepsilon,1]) \cap \overline{U} = \emptyset.$$

Then the inclusion of $\Sigma \setminus (U \cup D)$ into $\Sigma \setminus (D_0 \cup D)$ induces an injection of fundamental groups. Hence the curves $c_\ell|_{[\varepsilon,1-\varepsilon]}$ and $c_\ell'|_{[\varepsilon,1-\varepsilon]}$ are homotopic with fixed endpoints in $\Sigma \setminus (U \cup D)$. Hence the existence of an ambient isotopy satisfying the assertion for ℓ follows from the Epstein–Feustel theorem. This proves Step 4 and the theorem. □

Corollary 1. *Let N be the kernel of the homomorphism $\Gamma \to \pi_1(\Sigma, z_0)$ induced by the inclusion $\Sigma \setminus Z \hookrightarrow \Sigma$. Then the homomorphism*

$$\mathcal{B} \to \mathrm{Aut}(N) : \sigma \mapsto \sigma_* \tag{4.2}$$

obtained by composing $\Phi : \mathcal{B} \to \mathcal{G}/\mathcal{G}_0$ with the canonical action of $\mathcal{G}/\mathcal{G}_0$ on the subgroup N, is injective.

Proof: That the canonical action of $\mathcal{G}/\mathcal{G}_0$ on Γ leaves the subgroup N globally invariant follows from the definition of $\mathcal{G}$. To prove the injectivity, we consider a braid $\sigma \in \mathcal{B}$ such that σ_* is the identity of N. For an arbitrary configuration $c \in \mathcal{C}$, we have

$$g_{i,\sigma_* c} = \sigma_* g_{i,c} = g_{i,c}, \qquad i = 1, \ldots, m.$$

Hence it follows from Theorem 2 that c and $\sigma_* c$ are equivalent distinguished configurations. By Theorem 1 this implies that σ is trivial. □

Remark 6. Assume $\Sigma = \mathbb{D}$ with $z_0 \in \partial\mathbb{D}$. We have $N = \Gamma$ in this case and the map (4.2) is known as the **Artin representation**. A classical theorem by Artin [1, 4, 17] asserts that it is injective and that its image consists of all automorphisms $\phi : \Gamma \to \Gamma$ that satisfy the following two conditions:

(i) ϕ permutes the m conjugacy classes in Γ determined by small loops encircling the m punctures;

(ii) ϕ preserves the homotopy class of $\partial\mathbb{D}$.

Thus Corollary 1 is the injectivity part of Artin's theorem.

Let us choose a nonzero tangent vector $v_z \in T_z\Sigma$ at each puncture $z \in Z$. A **marked distinguished configuration** is a distinguished configuration $c = (c_1, \ldots, c_m)$ satisfying

$$\dot{c}_i(1) = -v_{c_i(1)}, \qquad i = 1, \ldots, m.$$

Observe that the configuration c induces an ordering on the set $\{v_z\}_{z \in Z}$ defined by $v_i := v_{c_i(1)}$ for $i = 1, \ldots, m$. The notion of homotopy carries over to marked distinguished configurations and the set of homotopy classes will be denoted by $\widetilde{\mathcal{C}}$. Now the proof of Theorem 1 carries over word by word to the present situation and shows the following.

Theorem 3. *The framed braid group $\widetilde{\mathcal{B}}$ acts freely and transitively on the set $\widetilde{\mathcal{C}}$ via (4.1).*

Remark 7. Given a marked distinguished configuration $c \in \widetilde{\mathcal{C}}$, one can define elements $\sigma_{2,c}, \ldots, \sigma_{m,c}, \varepsilon_{1,c}, \ldots, \varepsilon_{m,c}$ in $\widetilde{\mathcal{B}}$ as follows.

- For $i = 2, \ldots, m$ we define the framed braid $\sigma_{i,c}$ as follows. We choose an embedded arc $s_i : [0,1] \to \Sigma \setminus \{z_0\}$ from $z_{i-1} = c_{i-1}(1) = s_i(0)$ to $z_i = c_i(1) = s_i(1)$ by catenating $\left(c_{i-1}|_{[\varepsilon,1]}\right)^{-1}$ with a clockwise arc from $c_{i-1}(\varepsilon)$ to $c_i(\varepsilon)$ and with $c_i|_{[\varepsilon,1]}$. Given s_i we choose a braid $\beta = (\beta_1, \ldots, \beta_m)$ such that β_{i-1} runs from z_{i-1} to z_i on the left of s_i, β_i runs from z_i to z_{i-1} on the right of s_i, and $\beta_j \equiv z_j = c_j(1)$ for $j \neq i-1, i$. The framing is determined by a vector field near the union of the curves c_j which is tangent to the curves c_j and has z_0 as an attracting fixed point. The mapping class associated to $\sigma_{i,c}$ is represented by a diffeomorphism supported in an annulus around the geometric image of

β_{i-1} and β_i; it consists of two opposite half Dehn twists, one in each half of this annulus, followed by localized counterclockwise half turns centered at z_{i-1} and z_i. In terms of its action on c, the braid $\sigma_{i,c}$ preserves the curves c_j for $j \neq i-1, i$ and replaces the pair (c_{i-1}, c_i) by $(c_i g_{i-1,c}^{-1}, c_{i-1})$.

- For $i = 1, \ldots, m$ the framed braid $\varepsilon_{i,c}$ is the trivial braid with the framing given by a counterclockwise turn about $z_i = c_i(1)$ and the trivial framing over z_j for $j \neq i$. In terms of its action on c, the braid $\varepsilon_{i,c}$ preserves the curves c_j for $j \neq i$ and replaces c_i by $c_i g_{i,c}$.

In the case $\Sigma = \mathbb{D}$ there is an isomorphism $\phi_c : \widetilde{\mathcal{B}}_m \to \widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}}_m$ is the abstract braid group with generators $\sigma_2, \ldots, \sigma_m$ and $\varepsilon_1, \ldots, \varepsilon_m$ subject to the relations (1.7), as introduced in the Introduction. The isomorphism sends σ_i to $\sigma_{i,c}$ for $i = 2, \ldots, m$ and ε_j to $\varepsilon_{j,c}$ for $j = 1, \ldots, m$. We refer to Figure 2 on page 440 for a pictorial representation of these generators.

5 The Picard–Lefschetz monodromy cocycle

The main result of this section is Theorem 4, which contains Theorem A. We use the notations of the Introduction.

A marked distinguished configuration c and a framed braid $\sigma = [\phi] \in \widetilde{\mathcal{B}}$ with $\phi \in \widetilde{\mathcal{G}}$ determine a permutation $\pi_{\sigma,c} \in \mathfrak{S}_m$ such that $\phi(z_i) = z_{\pi_{\sigma,c}(i)}$ for all $i = 1, \ldots, m$. These permutations satisfy

$$\pi_{\sigma\tau,c} = \pi_{\sigma,c} \circ \pi_{\tau,c}, \qquad \pi_{\sigma,\tau_* c} = \pi_{\tau,c}^{-1} \circ \pi_{\sigma,c} \circ \pi_{\tau,c}. \tag{5.1}$$

For $c \in \widetilde{\mathcal{C}}$ and $j = 1, \ldots, m$ define the function $s_{j,c} : \widetilde{\mathcal{B}} \to \Gamma$ by

$$s_{j,c}(\sigma) := c_i^{-1} \cdot \sigma_* c_j, \qquad i := \pi_{\sigma,c}(j). \tag{5.2}$$

Here the right hand side denotes the catenation of the paths $\sigma_* c_j$ and c_i^{-1} pushed away from z_i in the common tangent direction v_i.

Remark 8. Recall the elements $\sigma_{2,c}, \ldots, \sigma_{m,c} \in \widetilde{\mathcal{B}}$ and $\varepsilon_{1,c}, \ldots, \varepsilon_{m,c} \in \widetilde{\mathcal{B}}$ defined in Remark 7. For $\sigma = \sigma_{k,c}$ the permutation $\pi_{\sigma,c}$ is the transposition of $k-1$ and k, for $\sigma = \varepsilon_{i,c}$ it is the identity. Figure 2 shows that

$$s_{j,c}(\sigma_{k,c}) = \begin{cases} 1, & j \neq k-1, \\ g_{k-1,c}^{-1}, & j = k-1, \end{cases} \qquad s_{j,c}(\varepsilon_{i,c}) = \begin{cases} 1, & j \neq i, \\ g_{i,c}, & j = i. \end{cases}$$

Lemma 3. *The functions $s_{j,c} : \widetilde{\mathcal{B}} \to \Gamma$ defined by* (5.2) *satisfy the conjugation condition*

$$\sigma_* g_{k,c} = s_{k,c}(\sigma)^{-1} g_{i,c} s_{k,c}(\sigma), \qquad i := \pi_{\sigma,c}(k), \tag{5.3}$$

the cocycle condition

$$s_{k,c}(\sigma\tau) = s_{j,c}(\sigma) \sigma_* s_{k,c}(\tau), \qquad j := \pi_{\tau,c}(k), \tag{5.4}$$

and the coboundary condition

$$s_{k,c}(\sigma\tau) = s_{\ell,c}(\tau) s_{k,\tau_* c}(\sigma), \qquad \ell := \pi_{\sigma,\tau_* c}(k) \tag{5.5}$$

for $\sigma, \tau \in \widetilde{\mathcal{B}}$ and $k = 1, \ldots, m$.

Proof: To prove (5.3) we denote $i := \pi_{\sigma,c}(k)$. Then

$$\sigma_* g_{k,c} = g_{k,\sigma_* c} = (\sigma_* c_k)^{-1} \cdot c_i \cdot g_{i,c} \cdot c_i^{-1} \cdot \sigma_* c_k,$$

where the middle term $c_i \cdot g_{i,c} \cdot c_i^{-1}$ represents a counterclockwise turn about z_i. To prove (5.4) we denote $j := \pi_{\tau,c}(k)$ and $i := \pi_{\sigma,c}(j) = \pi_{\sigma\tau,c}(k)$. Then

$$\begin{aligned} s_{j,c}(\sigma)\sigma_* s_{k,c}(\tau) &= c_i^{-1} \cdot \sigma_* c_j \cdot \sigma_*(c_j^{-1} \cdot \tau_* c_k) \\ &= c_i^{-1} \cdot \sigma_* \tau_* c_k \\ &= s_{k,c}(\sigma\tau). \end{aligned}$$

To prove (5.5) we denote $\ell := \pi_{\sigma,\tau_* c}(k)$ and $i := \pi_{\tau,c}(\ell) = \pi_{\sigma\tau,c}(k)$. Then

$$\begin{aligned} s_{\ell,c}(\tau) s_{k,\tau_* c}(\sigma) &= c_i^{-1} \cdot \tau_* c_\ell \cdot (\tau_* c)_\ell^{-1} \cdot \sigma_*(\tau_* c)_k \\ &= c_i^{-1} \cdot \sigma_* \tau_* c_k \\ &= s_{k,c}(\sigma\tau). \end{aligned}$$

This proves the lemma. □

Remark 9. It follows from the definition of the mapping class group $\widetilde{\mathcal{G}}/\widetilde{\mathcal{G}}_0 \cong \widetilde{\mathcal{B}}$ that $\sigma_* g_{k,c}$ is conjugate to $g_{i,c}$ for some i. (When Σ is the disc, this condition appears in Artin's theorem: see Remark 6.) Thus Lemma 3 gives an explicit formula for a conjugating group element, namely the element $s_{k,c}(\sigma)$.

For every marked distinguished configuration c, we define the map $\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ by

$$(\mathcal{S}_c(\sigma))_{ij} := \begin{cases} s_{j,c}(\sigma), & \text{if } i = \pi_{\sigma,c}(j), \\ 0, & \text{if } i \neq \pi_{\sigma,c}(j). \end{cases} \tag{5.6}$$

Remark 10. By Remark 8 we have

$$\mathcal{S}_c(\sigma_{k,c}) := \begin{pmatrix} \mathbb{1}_{k-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & g_{k-1,c}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{m-k} \end{pmatrix}, \quad \mathcal{S}_c(\varepsilon_{i,c}) := \begin{pmatrix} \mathbb{1}_{i-1} & 0 & 0 \\ 0 & g_{i,c} & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}.$$

When Σ is the disc, these conditions uniquely determine the cocycle $\mathcal{S}_c$.

The next theorem contains the statement of Theorem A. The notion of cocycle has been introduced in Definition 2.

Theorem 4. *The maps $\mathcal{S}_c$ with $c \in \widetilde{\mathcal{C}}$ satisfy the following conditions.*

(Homotopy) *If c is homotopic to c' then $\mathcal{S}_c = \mathcal{S}_{c'}$.*

(Injectivity) *Each map $\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ is injective.*

(Cocycle) *For all $c \in \widetilde{\mathcal{C}}$ and $\sigma, \tau \in \widetilde{\mathcal{B}}$ we have*

$$\mathcal{S}_c(\sigma\tau) = \mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau). \tag{5.7}$$

(Coboundary) *For all $c \in \widetilde{\mathcal{C}}$ and $\sigma, \tau \in \widetilde{\mathcal{B}}$ we have*

$$\mathcal{S}_{\tau_* c}(\sigma) = \mathcal{S}_c(\tau)^{-1}\mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau). \tag{5.8}$$

(Monodromy) *For each $c \in \widetilde{\mathcal{C}}$ the formula*

$$\sigma^*\mathcal{N} := \left(\mathcal{S}_c(\sigma)^t \mathcal{N} \mathcal{S}_c(\sigma)\right) \circ \sigma_* \tag{5.9}$$

defines a contravariant group action of $\widetilde{\mathcal{B}}$ on $\mathscr{N}_c$. For $c \in \widetilde{\mathcal{C}}$ and $\tau \in \widetilde{\mathcal{B}}$ the formula

$$\mathscr{T}_{\tau,c}(\mathcal{N}) := \mathcal{S}_c(\tau)^t \mathcal{N} \mathcal{S}_c(\tau) \tag{5.10}$$

defines an equivariant isomorphism from $\mathscr{N}_c$ to $\mathscr{N}_{\tau_* c}$. The isomorphisms satisfy the composition rule $\mathscr{T}_{\sigma,\tau_* c} \circ \mathscr{T}_{\tau,c} = \mathscr{T}_{\sigma\tau,c}$.

(Representation) *For all $c \in \widetilde{\mathcal{C}}$ and $\tau \in \widetilde{\mathcal{B}}$ the matrix $\mathcal{S}_c(\tau) \in \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ induces a collection of isomorphisms*

$$\mathcal{S}_c(\tau) : \mathbb{H}_{\mathscr{T}_{\tau,c}(\mathcal{N})} \to \mathbb{H}_{\mathcal{N}}, \qquad \mathcal{N} \in \mathscr{N}_c,$$

that preserve the structures (2.13-2.15).

(Lefschetz) *If $X \to \Sigma$ is a Lefschetz fibration with singular fibers over Z then*

$$\mathscr{T}_{\tau,c}(\mathcal{N}_c^X) = \mathcal{N}_{\tau_* c}^X$$

for all $c \in \widetilde{\mathcal{C}}$ and $\tau \in \widetilde{\mathcal{B}}$. Moreover, if $\Phi_c^X : \mathbb{H}_{\mathcal{N}_c^X} \to V/W$ denotes the isomorphism of Remark 2, then $\Phi_{\tau_* c}^X = \Phi_c^X \circ \mathcal{S}_c(\tau) : \mathbb{H}_{\mathcal{N}_{\tau_* c}^X} \to V/W$.

(Odd) *If n is odd then the contravariant action of $\widetilde{\mathcal{B}}$ on $\mathscr{N}_c$ descends to $\mathcal{B}$.*

Proof: The *(Homotopy)* condition is obvious. To prove the *(Cocycle)* condition (5.7) we denote $j := \pi_{\tau,c}(k)$ and $i := \pi_{\sigma,c}(j) = \pi_{\sigma\tau,c}(k)$. Then

$$\begin{aligned}
(\mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau))_{ik} &= \sum_{\nu=1}^{m} (\mathcal{S}_c(\sigma))_{i\nu}\sigma_*((\mathcal{S}_c(\tau))_{\nu k}) \\
&= (\mathcal{S}_c(\sigma))_{ij}\sigma_*(\mathcal{S}_c(\tau))_{jk} \\
&= s_{j,c}(\sigma)\sigma_* s_{k,c}(\tau) \\
&= s_{k,c}(\sigma\tau) \\
&= (\mathcal{S}_c(\sigma\tau))_{ik}.
\end{aligned}$$

Here we have used (5.4) and (5.6). For $i \neq \pi_{\sigma\tau,c}(k)$ the (i,k) entry of both matrices $\mathcal{S}_c(\sigma\tau)$ and $\mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau)$ is zero. Thus we have proved (5.7).

To prove the *(Coboundary)* condition (5.8) we set $\ell := \pi_{\sigma,\tau_*c}(k)$ and $i := \pi_{\tau,c}(\ell) = \pi_{\sigma\tau,c}(k)$. Then

$$\begin{aligned}
(\mathcal{S}_c(\tau)\mathcal{S}_{\tau_*c}(\sigma))_{ik} &= \sum_{\nu=1}^{m}(\mathcal{S}_c(\tau))_{i\nu}(\mathcal{S}_{\tau_*c}(\sigma))_{\nu k} \\
&= (\mathcal{S}_c(\tau))_{i\ell}(\mathcal{S}_{\tau_*c}(\sigma))_{\ell k} \\
&= s_{\ell,c}(\tau)s_{k,\tau_*c}(\sigma) \\
&= s_{k,c}(\sigma\tau) \\
&= (\mathcal{S}_c(\sigma\tau))_{j\ell}.
\end{aligned}$$

Here we have used (5.5) and (5.6). For $i \neq \pi_{\sigma\tau,c}(k)$ the (i,k) entry of both matrices $\mathcal{S}_c(\sigma\tau)$ and $\mathcal{S}_c(\tau)\mathcal{S}_{\tau_*c}(\sigma)$ is zero. Thus we have deduced (5.8) from (5.7).

To prove the *(Injectivity)* condition we assume that $c \in \widetilde{\mathcal{C}}$ is a marked distinguished configuration and $\sigma \in \widetilde{\mathcal{B}}$ is a framed braid such that $\mathcal{S}_c(\sigma) = \mathbb{1}$. Then the permutation $\pi_{\sigma,c}$ is the identity and we deduce from (5.3) that

$$g_{i,\sigma_*c} = \sigma_* g_{i,c} = s_{i,c}(\sigma)^{-1} g_{i,c} s_{i,c}(\sigma) = g_{i,c}, \qquad i = 1, \dots, m.$$

Hence it follows from Theorem 2 that c and σ_*c descend to equivalent distinguished configurations in $\mathcal{C}$. By Theorem 1 this implies that σ is a lift of the trivial braid in $\mathcal{B}$. Thus

$$\sigma = \varepsilon_{1,c}^{k_1} \cdots \varepsilon_{m,c}^{k_m}$$

for some integer vector $(k_1, \dots, k_m) \in \mathbb{Z}^m$. Using again the fact that $\mathcal{S}_c(\sigma) = \mathbb{1}$ we obtain that $k_1 = \cdots = k_m = 0$ and hence $\sigma = 1$. Thus we have proved that, for every $c \in \widetilde{\mathcal{C}}$ and every $\sigma \in \widetilde{\mathcal{B}}$, we have

$$\mathcal{S}_c(\sigma) = \mathbb{1} \qquad \Longrightarrow \qquad \sigma = 1.$$

Now let $c \in \widetilde{\mathcal{C}}$ and $\sigma, \tau \in \widetilde{\mathcal{B}}$ be given such that

$$\mathcal{S}_c(\sigma\tau) = \mathcal{S}_c(\tau).$$

Then it follows from the coboundary and cocycle conditions that

$$\mathcal{S}_c(\tau) = \mathcal{S}_c(\sigma\tau) = \mathcal{S}_c(\sigma)\sigma_*\mathcal{S}_c(\tau) = \mathcal{S}_c(\tau)\mathcal{S}_{\tau_*c}(\sigma).$$

Hence $\mathcal{S}_{\tau_*c}(\sigma) = \mathbb{1}$ and so, by what we have already proved, it follows that $\sigma = 1$. This shows that the map $\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ is injective, as claimed.

To prove the *(Monodromy)* condition let $\mathcal{N} = (n_{ij}) \in \mathscr{N}_c$ and $\tau \in \widetilde{\mathcal{B}}$. We must prove that $\mathscr{T}_{\tau,c}(\mathcal{N}) := \mathcal{S}_c(\tau)^t \mathcal{N} \mathcal{S}_c(\tau) \in \mathscr{N}_{\tau_*c}$. To see this denote the entries of $\mathscr{T}_{\tau,c}(\mathcal{N})$ by $\widetilde{n}_{ij}$ and observe that

$$\widetilde{n}_{ij}(g) = n_{i'j'}\big(s_{i,c}(\tau) g s_{j,c}(\tau)^{-1}\big), \qquad i' := \pi_{\tau,c}(i), \qquad j' := \pi_{\tau,c}(j). \tag{5.11}$$

That the functions $\widetilde{n}_{ij}$ satisfy (2.1) and (2.2) is obvious from this formula. To prove (2.3) we abbreviate $\varepsilon := (-1)^{n(n+1)/2}$, $k' := \pi_{\tau,c}(k)$, and compute

$$\begin{aligned}
\widetilde{n}_{ij}(gg_{k,\tau_* c}h) &= n_{i'j'}\big(s_{i,c}(\tau)g(\tau_* g_{k,c})hs_{j,c}(\tau)^{-1}\big)\\
&= n_{i'j'}\big(s_{i,c}(\tau)gs_{k,c}(\tau)^{-1}g_{k',c}s_{k,c}(\tau)hs_{j,c}(\tau)^{-1}\big)\\
&= n_{i'j'}\big(s_{i,c}(\tau)ghs_{j,c}(\tau)^{-1}\big)\\
&\quad -\varepsilon n_{i'k'}\big(s_{i,c}(\tau)gs_{k,c}(\tau)^{-1}\big)n_{k'j'}\big(s_{k,c}(\tau)hs_{j,c}(\tau)^{-1}\big)\\
&= \widetilde{n}_{ij}(gh) - \varepsilon\widetilde{n}_{ik}(g)\widetilde{n}_{kj}(h).
\end{aligned}$$

Here the first and last equations follow from (5.11), the second equation follows from (5.3), and the third follows from (2.3) for $n_{i'j'}$. Thus we have proved that $\mathscr{T}_{\tau,c}(\mathcal{N}) \in \mathscr{N}_{\tau_* c}$, as claimed, and hence also

$$\tau^*\mathcal{N} = \mathscr{T}_{\tau,c}(\mathcal{N}) \circ \tau_* \in \mathscr{N}_c.$$

That equation (5.9) defines a contravariant group action of $\widetilde{\mathcal{B}}$ on $\mathscr{N}_c$ for every $c \in \widetilde{\mathcal{C}}$ is a consequence of Lemma 2. That the map $\mathscr{T}_{\tau,c} : \mathscr{N}_c \to \mathscr{N}_{\tau_* c}$ is equivariant under the action of $\widetilde{\mathcal{B}}$ follows from (5.8). The composition rule follows from the definitions as well as (5.7) and (5.8). This proves the *(Monodromy)* condition.

The *(Representation)* condition follows from the proof of Proposition 1. In order to prove the *(Lefschetz)* condition we fix a symplectic Lefschetz fibration $f : X \to \Sigma$ with critical fibers over Z as well as a marked distinguished configuration $c \in \widetilde{\mathcal{C}}$ and a framed braid $\tau \in \widetilde{\mathcal{B}}$. To emphasize the dependence of the vanishing cycles on the choice of distinguished configuration c, we denote their homology classes by $L_{1,c}, \ldots, L_{m,c}$. Let $\rho : \Gamma \to \mathrm{Aut}(H_n(M))$ be the associated monodromy representation and, for $g \in \Gamma$, let

$$n_{ij}(g) := \langle L_{i,c}, \rho(g)L_{j,c}\rangle$$

denote the entries of the intersection matrix $\mathcal{N}_c^X(g)$. Then

$$L_{i,\tau_* c} = \rho(s_{i,c}(\tau)^{-1})L_{i',c}, \qquad i' := \pi_{\tau,c}(i).$$

Hence the entries of the matrix $\mathcal{N}^X_{\tau_* c}(g) =: (\widetilde{n}_{ij}(g))$ are

$$\begin{aligned}
\widetilde{n}_{ij}(g) &= \langle L_{i,\tau_* c}, \rho(g)L_{j,\tau_* c}\rangle\\
&= \big\langle \rho\big(s_{i,c}(\tau)^{-1}\big)L_{i',c}, \rho\big(gs_{j,c}(\tau)^{-1}\big)L_{j',c}\big\rangle\\
&= \big\langle L_{i',c}, \rho\big(s_{i,c}(\tau)gs_{j,c}(\tau)^{-1}\big)L_{j',c}\big\rangle\\
&= n_{i'j'}\big(s_{i,c}(\tau)gs_{j,c}(\tau)^{-1}\big).
\end{aligned}$$

Hence it follows from (5.11) that $\mathcal{N}^X_{\tau_* c} = \mathscr{T}_{\tau,c}(\mathcal{N}^X_c)$ as claimed. Thus we have proved the *(Lefschetz)* condition.

Now assume that n is odd. Then it follows from (2.8) that

$$n_{ij}(g) = n_{ij}(g_{i,c}g) = n_{ij}(gg_{j,c}) \tag{5.12}$$

for $\mathcal{N} = (n_{ij}) \in \mathscr{N}_c$. Moreover, if $\tau \in \widetilde{\mathcal{B}}$ belongs to the kernel of the homomorphism $\widetilde{\mathcal{B}} \to \mathcal{B}$ then $\pi_{\tau,c} = \mathrm{id} \in \mathfrak{S}_m$ and $s_{i,c}(\tau) = g_{i,c}^{k_i}$ for some $k_i \in \mathbb{Z}$. Hence equations (5.11) and (5.12) show that any such element τ acts trivially on $\mathscr{N}_c$. This proves the theorem. □

6 Comparison with the Magnus cocycle

Let Γ be a free group of finite rank and denote by $\mathrm{Aut}(\Gamma)$ its group of automorphisms. Any choice of a basis $g_1, \ldots, g_m \in \Gamma$ determines a **Magnus cocycle**

$$\mathcal{M} : \mathrm{Aut}(\Gamma) \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$$

defined by

$$\mathcal{M}(\psi) := \left(\overline{\frac{\partial \psi(g_j)}{\partial g_i}} \right)_{i,j=1,\ldots,m}. \tag{6.1}$$

This formula is to be understood in the following way. A **derivation** is a 1-cocycle $d : \Gamma \to \mathbb{Z}[\Gamma]$, i.e. a map that satisfies the equation

$$d(gh) = d(g) + gd(h)$$

for all $g, h \in \Gamma$. In particular we have $d(1) = 0$ and $d(g^{-1}) = -g^{-1}d(g)$ for every $g \in \Gamma$. Examples of derivations are the **Fox derivatives** [15]

$$\frac{\partial}{\partial g_i} : \Gamma \to \mathbb{Z}[\Gamma], \qquad i = 1, \ldots, m,$$

characterized by the condition

$$\frac{\partial g_j}{\partial g_i} = \delta_i^j, \qquad i, j = 1, \ldots, m.$$

Recall that the conjugation $\mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma] : \lambda \mapsto \bar{\lambda}$ is the ring anti-homomorphism defined by $\bar{g} := g^{-1}$ for all $g \in \Gamma \subset \mathbb{Z}[\Gamma]$. This explains the right hand side of (6.1).

The map $\mathcal{M} : \mathrm{Aut}(\Gamma) \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ satisfies the cocycle condition

$$\mathcal{M}(\psi \circ \phi) = \mathcal{M}(\psi) \cdot \psi_* \mathcal{M}(\phi) \tag{6.2}$$

for $\phi, \psi \in \mathrm{Aut}(\Gamma)$. This is proved in Birman [4] as a consequence of the chain rule for Fox calculus. Fox calculus has its origin in the theory of covering spaces. The matrix $\mathcal{M}(\phi)$ represents the action of ϕ on the twisted homology of a bouquet of m circles relative to a base point with coefficients in $\mathbb{Z}[\Gamma]$. The resulting map is a cocycle (instead of a homomorphism) because the lift of a continuous map of the bouquet of circles to its universal cover is not Γ-equivariant.

This construction applies to the braid group of the disc as follows. We return to the geometric setting of Section 3 with $\Sigma = \mathbb{D}$. Thus $Z \subset \mathbb{D}$ is a set of m points in the interior, Γ is the fundamental group of $\mathbb{D} \setminus Z$ based at $z_0 \in \partial\mathbb{D}$, and $\mathcal{B}$ is the braid group on m strings in $\mathbb{D}$ based at Z. The choice of a distinguished configuration $c \in \mathcal{C}$ determines a basis $g_{1,c}, \ldots, g_{m,c}$ of Γ. Since $\mathcal{B}$ acts on Γ we obtain a Magnus cocycle

$$\mathcal{M}_c : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$$

for every distinguished configuration $c \in \mathcal{C}$.

Proposition 2. *The maps $\mathcal{M}_c : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ are injective and satisfy the cocycle and coboundary conditions*

$$\mathcal{M}_c(\sigma\tau) = \mathcal{M}_c(\sigma)\sigma_*\mathcal{M}_c(\tau), \qquad \mathcal{M}_{\tau_* c}(\sigma) = \mathcal{M}_c(\tau)^{-1}\mathcal{M}_c(\sigma)\sigma_*\mathcal{M}_c(\tau) \tag{6.3}$$

for all $c \in \mathcal{C}$ and $\sigma, \tau \in \mathcal{B}$.

Proof: The first equation in (6.3) follows immediately from (6.2) and the second equation can also be derived from the chain rule in Fox calculus. For the sake of completeness we give the details. The chain rule in Fox calculus has the following form. If $g_1, \ldots, g_m$ and $h_1, \ldots, h_m$ are two basis of Γ and $a \in \Gamma$ is an arbitrary element then

$$\frac{\partial a}{\partial g_i} = \sum_{j=1}^{m} \frac{\partial a}{\partial h_j}\frac{\partial h_j}{\partial g_i}, \qquad i = 1, \ldots, m.$$

To prove the first formula in (6.3) we take

$$g_i := g_{i,c}, \qquad h_j := \sigma_* g_{j,c}, \qquad a := \sigma_*\tau_* g_{k,c}.$$

Then the chain rule asserts that

$$\frac{\partial(\sigma\tau)_* g_{k,c}}{\partial g_{i,c}} = \sum_{j=1}^{m} \frac{\partial \sigma_*\tau_* g_{k,c}}{\partial \sigma_* g_{j,c}}\frac{\partial \sigma_* g_{j,c}}{\partial g_{i,c}} = \sum_{j=1}^{m} \left(\sigma_* \frac{\partial \tau_* g_{k,c}}{\partial g_{j,c}}\right)\frac{\partial \sigma_* g_{j,c}}{\partial g_{i,c}}.$$

The first equation in (6.3) follows by conjugation. To prove the second equation in (6.3) we choose

$$g_i := g_{i,c}, \qquad h_j := \tau_* g_{j,c}, \qquad a := \sigma_*\tau_* g_{k,c}.$$

Then the chain rule asserts that

$$\frac{\partial(\sigma\tau)_* g_{k,c}}{\partial g_{i,c}} = \sum_{j=1}^{m} \frac{\partial \sigma_*\tau_* g_{k,c}}{\partial \tau_* g_{j,c}}\frac{\partial \tau_* g_{j,c}}{\partial g_{i,c}} = \sum_{j=1}^{m} \frac{\partial \sigma_* g_{k,\tau_* c}}{\partial g_{j,\tau_* c}}\frac{\partial \tau_* g_{j,c}}{\partial g_{i,c}}.$$

Hence conjugation gives $\mathcal{M}_c(\sigma\tau) = \mathcal{M}_c(\tau)\mathcal{M}_{\tau_* c}(\sigma)$ and so the second equation in (6.3) follows from the first.

Injectivity of $\mathcal{M}_c$ is a consequence of the fundamental formula in Fox calculus [15]. It has the form

$$a - 1 = \sum_{i=1}^{m} \frac{\partial a}{\partial g_{i,c}}(g_{i,c} - 1) \tag{6.4}$$

for all $a \in \Gamma$. Applying this formula to $a = \sigma_* g_{j,c}$ and $a = \tau_* g_{j,c}$ we see that $\mathcal{M}_c(\sigma) = \mathcal{M}_c(\tau)$ if and only if $\sigma_* g = \tau_* g$ for all $g \in \Gamma$, which is equivalent to $\sigma = \tau$ by Artin's theorem (see Remark 6). This proves the proposition. $\square$

The Magnus cocycle $\mathcal{M}_c$ is connected to the Reidemeister intersection pairing (in its relative version)

$$\langle\cdot,\cdot\rangle : H_1(\mathbb{D}\setminus Z, z_0; \mathbb{Z}[\Gamma]) \times H_1(\mathbb{D}\setminus Z, z_0; \mathbb{Z}[\Gamma]) \to \mathbb{Z}[\Gamma]$$

or, equivalently, to the homotopy intersection pairing

$$\omega : \Gamma \times \Gamma \to \mathbb{Z}[\Gamma]$$

introduced by Turaev [26] and Perron [19]. Let Ω_c be the $m\times m$-matrix with coefficients in $\mathbb{Z}[\Gamma]$ which represents ω in the basis $(g_{1,c},\dots,g_{m,c})$. Perron shows that

$$\mathcal{M}_c(\tau)^t \cdot \Omega_c \cdot \mathcal{M}_c(\tau) = \Omega_{\tau_* c} \tag{6.5}$$

for any $c\in\mathcal{C}$ and $\tau\in\mathcal{B}$. This identity can be deduced from the topological interpretation of the Magnus cocycle, according to which $\mathcal{M}_c(\tau)$ is (after conjugation) the matrix representing the homomorphism

$$\tau_* : H_1(\mathbb{D}\setminus Z, z_0; \mathbb{Z}[\Gamma]) \to H_1(\mathbb{D}\setminus Z, z_0; \mathbb{Z}[\Gamma])$$

with respect to a basis given by lifts of $g_{1,c},\dots,g_{m,c}$.

Remark 11. By the definition of Fox derivatives we have

$$\frac{\partial(g_{i-1,c}g_{i,c}g_{i-1,c}^{-1})}{\partial g_{i-1,c}} = 1 - g_{i-1,c}g_{i,c}g_{i-1,c}^{-1}, \qquad \frac{\partial(g_{i-1,c}g_{i,c}g_{i-1,c}^{-1})}{\partial g_{i,c}} = g_{i-1,c}.$$

Hence the Magnus cocycle satisfies

$$\mathcal{M}_c(\sigma_{i,c}) := \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 & 0 \\ 0 & 1 - g_{i-1,c}g_{i,c}^{-1}g_{i-1,c}^{-1} & 1 & 0 \\ 0 & g_{i-1,c}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

for all $i = 2,\dots,m$.

In order to compare the Magnus cocycle with the Picard–Lefschetz cocycle, we need to restrict the latter to an embedded image of $\mathcal{B}$ in $\widetilde{\mathcal{B}}$. For this we fix, as in the previous sections, a non-zero tangent vector v_z at each $z\in Z$. Moreover we fix a contractible neighborhood U of z_0 in $\Sigma=\mathbb{D}$ whose closure does not meet Z, and we fix a vector field v on U whose only singularity is an attractive point at z_0. Relative homotopy classes of nowhere vanishing vector fields on Σ that restrict to v on $Z\cup U$ are parametrized by $H^1(\Sigma, U\cup Z;\mathbb{Z})\cong\mathbb{Z}^m$. We **choose** such a relative homotopy class ξ. The class ξ defines a section

$$\mathcal{B} \overset{\xi}{\hookrightarrow} \widetilde{\mathcal{B}}$$

of the short exact sequence (3.4), which assigns to every braid the framing defined by any representative w of ξ. Furthermore, we can associate to each $\gamma\in\mathcal{C}$ the homotopy class $[c]\in\widetilde{\mathcal{C}}$ where c is a distinguished configuration representing γ which is tangent to w for some representative w of ξ. This defines a section

$$\mathcal{C} \overset{\xi}{\hookrightarrow} \widetilde{\mathcal{C}}$$

of the canonical projection $\widetilde{\mathcal{C}} \to \mathcal{C}$. The free and transitive action of $\widetilde{\mathcal{B}}$ on $\widetilde{\mathcal{C}}$ restricts to a free and transitive action of $\xi(\mathcal{B})$ on $\xi(\mathcal{C})$. In the sequel, we shall identify $\mathcal{B}$ and $\mathcal{C}$ with their respective images by ξ.

For every $c \in \mathcal{C}$, the cocycles $\mathcal{M}_c$ show similarities with the cocycles $\mathcal{S}_c|_{\mathcal{B}}$. Both of them are injective maps $\mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ and, according to the formulas (1.5) and (6.3), they behave in the same way under change of $c \in \mathcal{C}$. Moreover, the formulas (1.6) and (6.5) show that they can both be interpreted as matrices of basis change for certain geometrically defined bilinear forms. However, in contrast to $\mathcal{M}_c$, the entries of $\mathcal{S}_c$ are elements of $\Gamma \subset \mathbb{Z}[\Gamma]$. The framed braids $\sigma_{2,c}, \ldots, \sigma_{m,c}$ belong to the embedded image of $\mathcal{B}$ in $\widetilde{\mathcal{B}}$ defined by the homotopy class of vector fields ξ. We see from the formulas in Remarks 10 and 11 that $\mathcal{M}_c(\sigma_{i,c})$ differs from $\mathcal{S}_c(\sigma_{i,c})$ exactly in one entry.

The Magnus cocycle gives a unified framework for the definition of various linear representations of mapping class groups, including the Burau and Gassner representations [4]. These representations are obtained by reducing Γ to an abelian quotient. Let us apply the same reductions to the Picard–Lefschetz cocycle.

There is a natural homomorphism $\Gamma \to \mathbb{Z}$ which assigns to every loop g in $\mathbb{D} \setminus Z$ the total winding number around the punctures. If we identify $\mathbb{Z}$ with the free group on one generator t^{-1}, then $g_{i,c}$ is mapped to t^{-1}. This induces a ring homomorphism $\mathbb{Z}[\Gamma] \to \mathbb{Z}[t, t^{-1}]$, and hence a group homomorphism $\mathrm{GL}_m(\mathbb{Z}[\Gamma]) \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$. Composition with this homomorphism turns every cocycle into a representation, and turns cohomologous cocycles into conjugate representations. The compositions of $\mathcal{S}_c|_{\mathcal{B}}$ and $\mathcal{M}_c$ with the group homomorphism $\mathrm{GL}_m(\mathbb{Z}[\Gamma]) \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$ will be denoted by

$$\overline{\mathcal{S}}_c : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}]), \qquad \overline{\mathcal{M}}_c : \mathcal{B} \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}]).$$

On the generators $\sigma_{2,c}, \ldots, \sigma_{m,c}$ of $\mathcal{B}$ we have

$$\overline{\mathcal{S}}_c(\sigma_{i,c}) := \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}, \quad \overline{\mathcal{M}}_c(\sigma_{i,c}) := \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 & 0 \\ 0 & 1-t & 1 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}.$$

These representations of the braid group are well known: $\overline{\mathcal{M}}_c$ is the **Burau representation** [6, 4, 17] and $\overline{\mathcal{S}}_c$ is the **Tong–Yang–Ma representation** introduced in [25]. The Burau representation plays an important role in knot theory because of its deep connection with the Alexander polynomial [4, 17]. In contrast to the Burau representation, the Tong–Yang–Ma representation is irreducible. Furthermore, Sysoeva shows in [23] that any irreducible m-dimensional complex representation of the braid group on $m \geq 9$ strings is equivalent to the tensor product of a 1-dimensional representation with a specialization of the latter for some $t \in \mathbb{C} \setminus \{0, 1\}$. (See also [14] for the cases $m \in \{5, 6, 7, 8\}$.)

Proposition 3. *The cocycles $\mathcal{M}_c$ and $\mathcal{S}_c|_{\mathcal{B}}$ define distinct and nontrivial cohomology classes in $H^1(\mathcal{B}, \mathrm{GL}_m(\mathbb{Z}[\Gamma]))$.*

Proof: We have $\mathrm{tr}(\overline{\mathcal{S}}_c(\sigma_{i,c})) = m - 2$ and $\mathrm{tr}(\overline{\mathcal{M}}_c(\sigma_{i,c})) = m - 1 - t$ for all $i = 2, \ldots, m$, while $\mathrm{tr}(\mathrm{Id}) = m$. Thus $\overline{\mathcal{S}}_c$ and $\overline{\mathcal{M}}_c$ are neither conjugate to each other nor conjugate to the trivial representation. □

Remark 12. Since the cocycle $\mathcal{S}_c$ is defined on $\widetilde{\mathcal{B}}$, it gives rise by reduction to an extension

$$\overline{\mathcal{S}}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$$

of the Tong–Yang–Ma representation to the framed braid group. Explicitly, we have

$$\overline{\mathcal{S}}_c(\varepsilon_{i,c}) := \begin{pmatrix} \mathbb{1}_{i-1} & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}, \qquad i = 1, \dots, m.$$

In particular, this formula implies that the Picard–Lefschetz monodromy class in the quotient set $H^1(\widetilde{\mathcal{B}}, \mathrm{GL}_m(\mathbb{Z}[\Gamma]))$ is nontrivial.

Fix an ordering $Z = \{z_1, \dots, z_m\}$ of the punctures. A **pure braid** is a braid $\sigma \in \mathcal{B}$ whose associated permutation of Z is trivial. We denote by $\mathcal{PB} \subset \mathcal{B}$ the subgroup of pure braids. The ordering of Z induces a natural isomorphism between the abelianization of Γ and $\mathbb{Z}^m$ and hence a natural homomorphism from Γ to $\mathbb{Z}^m$. If we identify $\mathbb{Z}^m$ with the free abelian group on m generators $t_1^{-1}, \dots, t_m^{-1}$, this homomorphism sends $g_{i,c}$ to t_i^{-1} for every distinguished configuration c that determines the given ordering of Z. Thus, there is an induced group homomorphism $\mathrm{GL}_m(\mathbb{Z}[\Gamma]) \to \mathrm{GL}_m(\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$. The compositions of $\mathcal{S}_c|_{\mathcal{PB}}$ and $\mathcal{M}_c|_{\mathcal{PB}}$ with this homomorphism will be denoted by

$$\widehat{\mathcal{S}}_c : \mathcal{PB} \to \mathrm{GL}_m(\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]), \qquad \widehat{\mathcal{M}}_c : \mathcal{PB} \to \mathrm{GL}_m(\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]).$$

These are representations of the pure braid group and $\widehat{\mathcal{M}}_c$ is called the **Gassner representation** [16, 4]. Observe that the embedding of the subgroup $\mathcal{PB}$ in $\widetilde{\mathcal{B}}$ is canonical (i.e. it does not depend on the choice of ξ) and, furthermore, the representation $\widehat{\mathcal{S}}_c$ does not depend on c but only on the chosen ordering of Z. Indeed, using equation (5.8) and the fact that $\widehat{\mathcal{S}}_c(\tau)$ is a diagonal matrix for all $\tau \in \mathcal{PB}$, we obtain

$$\widehat{\mathcal{S}}_{\tau_* c}(\sigma) = \widehat{\mathcal{S}}_c(\tau)^{-1}\widehat{\mathcal{S}}_c(\sigma)\widehat{\mathcal{S}}_c(\tau) = \widehat{\mathcal{S}}_c(\sigma)$$

for all $\sigma \in \mathcal{PB}$.

The next proposition gives an explicit formula for $\widehat{\mathcal{S}}_c$ and shows that this representation is completely determined by the linking numbers.

Proposition 4. *For every $\sigma \in \mathcal{PB}$ we have*

$$\widehat{\mathcal{S}}_c(\sigma) = \mathrm{Diag}\left(\prod_{j \neq 1} t_j^{-\ell\mathrm{k}(1,j)}, \dots, \prod_{j \neq m} t_j^{-\ell\mathrm{k}(m,j)}\right) \tag{6.6}$$

where $\ell\mathrm{k}(i,j)$ denotes the linking number of the i-th and j-th components of the closed braid.

Here the **closure** of the braid σ is defined in the usual way [4, 17] by connecting the top and the bottom of $\mathbb{D} \times [0, 1]$ without twisting (see Figure 4 for an illustration).

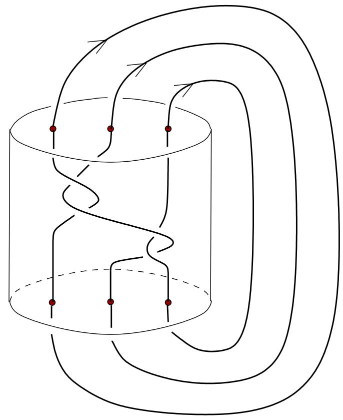

Figure 4: Closure of a pure braid.

Proof of Proposition 4: By definition $\mathcal{S}_c(\sigma)$ is a diagonal matrix with diagonal entries

$$s_{j,c}(\sigma) = c_j^{-1} \cdot \sigma_* c_j \in \Gamma$$

(see equation (5.6)). To understand the corresponding diagonal entry $\widehat{s}_{j,c}(\sigma)$ of $\widehat{\mathcal{S}}_c(\sigma)$ we must express $s_{j,c}(\sigma)$ as a word in the generators $g_{i,c}$ and their inverses. The exponent of t_i is then the total occurence of the factor $g_{i,c}^{-1}$ in this word and we claim that this number is $-\ell\mathrm{k}(i,j)$ for $i \neq j$ and is zero for $i = j$. Equivalently, if we denote by $\widehat{g} \in H_1(\mathbb{D} \setminus Z; \mathbb{Z})$ the homology class of an element $g \in \Gamma$, we must prove that

$$\widehat{s}_{j,c}(\sigma) = \sum_{i \neq j} \ell\mathrm{k}(i,j)\widehat{g}_{i,c}. \tag{6.7}$$

To see this, let $N \subset \mathbb{D} \times [0,1]$ be the complement of the braid σ, viewed as a collection of strings running from $\mathbb{D} \times \{0\}$ to $\mathbb{D} \times \{1\}$. There is a deformation retract $r : N \to \mathbb{D} \setminus Z$ such that, for every $z \in \mathbb{D} \setminus Z$, we have $r(z,0) = z$ and $r(z,1) = \phi^{-1}(z)$ with $\phi \in \mathcal{G}$ a diffeomorphism representing the element in the mapping class group corresponding to the braid σ. This map induces an isomorphism

$$r_* : H_1(N; \mathbb{Z}) \to H_1(\mathbb{D} \setminus Z; \mathbb{Z}).$$

The oriented meridians of the strings of σ form a basis $\mu_1, \ldots, \mu_m$ of $H_1(N; \mathbb{Z})$ and their images under r_* are represented by small loops encircling the elements of Z counterclockwise; thus we have

$$r_*(\mu_i) = \widehat{g}_{i,c}, \qquad i = 1, \ldots, m.$$

Using the distinguished configuration c, we can view the closure of the braid inside N. We denote its components by $K_1, \ldots, K_m$, and the corresponding homology classes in $H_1(N; \mathbb{Z})$ by $\widehat{K}_1, \ldots, \widehat{K}_m$. By the homological definition of the linking numbers, we have

$$\widehat{K}_j = \sum_{i \neq j} \ell\mathrm{k}(i,j)\mu_i.$$

The image by r of the knot $\widehat{K_j}$ is the loop that first traverses $c_j|_{[0,1-\varepsilon]}$ and then $\phi^{-1}\circ c_j|_{[0,1-\varepsilon]}$ in the reverse direction (for some small $\varepsilon > 0$). Hence we obtain that $r_*(\widehat{K}_j) = \widehat{s}_{j,c}(\sigma)$ for all $j \in \{1,\dots,m\}$ and equation (6.7) follows.

Alternatively, one can prove (6.7) as follows. This identity is equivalent to the formula

$$\mathrm{w}(s_{j,c}(\sigma), z_i) = \begin{cases} \ell\mathrm{k}(i,j) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \tag{6.8}$$

where $\mathrm{w}(\gamma, z)$ denotes the winding number of a loop γ about z. We choose a representative of the braid σ in which z_i is constant. Then the j-th component of the braid has the same winding number about z_i as $s_{j,c}(\sigma)$, and this winding number agrees with $\ell\mathrm{k}(i,j)$ by definition of the linking number as an intersection number. □

Remark 13. Proposition 4 implies, by specializing to $t_1 = \cdots = t_m = t$, that the Tong–Yang–Ma representation $\overline{\mathcal{S}}_c$ is trivial on the commutator subgroup $\mathcal{PB}' := [\mathcal{PB}, \mathcal{PB}]$. Thus $\overline{\mathcal{S}}_c$ factors through the quotient $\mathcal{B}/\mathcal{PB}'$, which is an instance of the **extended Coxeter group** in the sense of Tits [24]. In this case the Coxeter group is the symmetric group $\mathfrak{S}_m$, and $\mathcal{B}/\mathcal{PB}'$ is an extension of the latter by the abelian group $\mathcal{PB}/\mathcal{PB}' \cong \mathbb{Z}^{m(m-1)/2}$.

7 Proof of Theorem B

We still specialize to the case where $\Sigma = \mathbb{D}$ is the closed unit disc. In this case Γ is isomorphic to the free group Γ_m generated by $g_1,\dots,g_m$ and $\widetilde{\mathcal{B}}$ is isomorphic to the abstract framed braid group $\widetilde{\mathcal{B}}_m$ with generators $\sigma_2,\dots,\sigma_m,\varepsilon_1,\dots,\varepsilon_m$ and relations (1.7). The isomorphisms depend on the choice of a marked distinguished configuration $c \in \widetilde{\mathcal{C}}$ and will be denoted by

$$\iota_c : \Gamma_m \to \Gamma, \qquad \phi_c : \widetilde{\mathcal{B}}_m \to \widetilde{\mathcal{B}}.$$

The isomorphism ι_c assigns to g_i the special element $g_{i,c}$ obtained by encircling z_i counterclockwise along c_i. The isomorphism ϕ_c assigns to σ_i and ε_i the generators $\sigma_{i,c}$ and $\varepsilon_{i,c}$ of $\widetilde{\mathcal{B}}$ associated to c, as defined in Remark 7. Recall the action of $\widetilde{\mathcal{B}}_m$ on Γ_m by (1.8).

Lemma 4. (i) *The isomorphisms ι_c and ϕ_c satisfy*

$$\iota_{\psi_* c}(g) = \psi_* \iota_c(g), \qquad \phi_{\psi_* c}(\sigma) = \psi \phi_c(\sigma) \psi^{-1},$$

for $g \in \Gamma_m$, $\sigma \in \widetilde{\mathcal{B}}_m$, $c \in \widetilde{\mathcal{C}}$, and $\psi \in \widetilde{\mathcal{B}}$.

(ii) *For every $c \in \widetilde{\mathcal{C}}$ and every $\sigma \in \widetilde{\mathcal{B}}_m$ there is a commutative diagram*

$$\begin{tikzcd} \Gamma_m \arrow[r, "\sigma_*"] \arrow[d, "\iota_c"'] & \Gamma_m \arrow[d, "\iota_c"] \\ \Gamma \arrow[r, "\phi_c(\sigma)_*"'] & \Gamma \end{tikzcd} .$$

(iii) *The formula*

$$\sigma^* c := \phi_c(\sigma)_* c$$

defines a free and transitive contravariant action of $\widetilde{\mathcal{B}}_m$ *on* $\widetilde{\mathcal{C}}$ *and*

$$\iota_{\sigma^* c} = \iota_c \circ \sigma_*$$

for every $\sigma \in \widetilde{\mathcal{B}}_m$ *and every* $c \in \widetilde{\mathcal{C}}$.

Proof: Assertions (i) and (ii) follow immediately from the definitions by checking them on the generators. To prove (iii) we use (i) with $\psi := \phi_c(\sigma)$ to obtain

$$\phi_c(\sigma\tau) = \phi_c(\sigma)\phi_c(\tau) = \phi_{\phi_c(\sigma)_* c}(\tau)\phi_c(\sigma)$$

for $\sigma, \tau \in \widetilde{\mathcal{B}}_m$ and $c \in \widetilde{\mathcal{C}}$. Hence

$$(\sigma\tau)^* c = \phi_c(\sigma\tau)_* c = \phi_{\phi_c(\sigma)_* c}(\tau)_* \phi_c(\sigma)_* c = \phi_{\sigma^* c}(\tau)_*(\sigma^* c) = \tau^* \sigma^* c.$$

That the action is free and transitive follows from Theorem 3. To prove the last equation in (iii) let $g \in \Gamma_m$. Then, by (i) and (ii), we have

$$\iota_{\sigma^* c}(g) = \iota_{\phi_c(\sigma)_* c}(g) = \phi_c(\sigma)_* \iota_c(g) = \iota_c(\sigma_* g).$$

This proves the lemma. □

Proof of Theorem B: Uniqueness is clear. To prove existence, fix a marked distinguished configuration $c \in \widetilde{\mathcal{C}}$, let

$$\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$$

be the cocycle of Theorem A, and define $S : \widetilde{\mathcal{B}}_m \times \mathscr{N}_m \to \mathrm{GL}_m(\mathbb{Z})$ by

$$S(\sigma, N) := \big(\rho_N \cdot (\mathcal{S}_c(\phi_c(\sigma)) \circ \iota_c)\big)(1) \tag{7.1}$$

for $\sigma \in \widetilde{\mathcal{B}}_m$ and $N \in \mathscr{N}_m$. Here we have $\mathcal{S}_c(\phi_c(\sigma)) \circ \iota_c \in \mathrm{GL}_m(\mathbb{Z}[\Gamma_m])$, the representation $\rho_N : \Gamma_m \to \mathrm{GL}_m(\mathbb{Z})$ is given by (1.10), and the term $\rho_N \cdot (\mathcal{S}_c(\phi_c(\sigma)) \circ \iota_c)$ is understood as the convolution product.

We prove that S satisfies (1.12). For $2 \le k \le m$ we have, by Remark 10,

$$\begin{aligned}
S(\sigma_k, N) &= \big(\rho_N(\mathcal{S}_c(\sigma_{k,c}) \circ \iota_c)\big)(1) \\
&= \rho_N(1)\mathcal{S}_c(\sigma_{k,c})(1) + \rho_N(g_{k-1})\mathcal{S}_c(\sigma_{k,c})(g_{k-1,c}^{-1}) \\
&= (\Sigma_k - E_{k,k-1}) + (\mathbb{1} - \varepsilon E_{k-1} N) E_{k,k-1} \\
&= \Sigma_k - \varepsilon n_{k-1,k} E_{k-1}.
\end{aligned}$$

Here $\varepsilon := (-1)^{n(n+1)/2}$ and $E_{k,k-1}$ is the matrix whose entries are all zero except for the entry $(k, k-1)$ which is equal to 1. For $1 \le i \le m$ we have

$$\begin{aligned} S(\varepsilon_i, N) &= \big(\rho_N(\mathcal{S}_c(\varepsilon_{i,c}) \circ \iota_c)\big)(1) \\ &= \rho_N(1)\mathcal{S}_c(\varepsilon_{i,c})(1) + \rho_N(g_i^{-1})\mathcal{S}_c(\varepsilon_{i,c})(g_{i,c}) \\ &= (\mathbb{1} - E_i) + (\mathbb{1} - (-1)^n \varepsilon E_i N)E_i \\ &= \mathbb{1} - (-1)^n \varepsilon n_{ii} E_i \\ &= D_i. \end{aligned}$$

Here the second equation uses the identity

$$\rho_N(g_i^{-1}) = \mathbb{1} - (-1)^n \varepsilon E_i N$$

and the fourth equation follows from (1.9). We also have $(\rho_N \cdot (\mathcal{S}_c(1) \circ \iota_c))(1) = \rho_N(1) = \mathbb{1}$, and the *(Normalization)* condition (1.12) is proved.

We prove that S satisfies (1.11) and (1.13) for all $\sigma, \tau \in \widetilde{\mathcal{B}}_m$ and $N \in \mathcal{N}_m$. The proof is by induction on the word length of σ. The induction step relies on the following two observations.

Claim 1. *If* (1.13) *holds for* σ *and* N*, then* (1.11) *holds for every* τ.

Claim 2. *If* (1.11) *holds for* σ*,* τ*, and* N *and* (1.13) *holds for the pairs* (σ, N) *and* $(\tau, \sigma^* N)$*, then* (1.13) *also holds for the pair* $(\sigma\tau, N)$.

To carry out the induction argument we first observe that (1.13) holds for the generators σ_k and ε_i by direct verification. Hence, by Claim 1, (1.11) also holds whenever σ is a generator. Assume, by induction, that (1.11) and (1.13) hold whenever σ is a word of length at most k. Let σ be a word of length $k+1$. That (1.13) holds for every N follows from Claim 2 by decomposing σ as a product of two words of length at most k. Hence, by Claim 1, (1.11) holds for every τ. Thus it remains to prove Claims 1 and 2.

To prove Claim 1 it is convenient to abbreviate

$$\mathcal{M}_\tau := \mathcal{S}_c(\phi_c(\tau)) \circ \iota_c \in \mathrm{GL}_m(\mathbb{Z}[\Gamma_m])$$

for $\tau \in \widetilde{\mathcal{B}}_m$. Then it follows from Lemma 4 (ii) that the *(Cocycle)* condition (5.7) for $\mathcal{S}_c$ takes the form

$$\mathcal{M}_{\sigma\tau} = \mathcal{M}_\sigma(\sigma_* \mathcal{M}_\tau)$$

for $\sigma, \tau \in \widetilde{\mathcal{B}}_m$. Moreover, equation (1.13) can be written in the form

$$\sigma_*(\rho_{\sigma^* N}) = S(\sigma, N)^{-1} \rho_N S(\sigma, N).$$

This implies

$$\begin{aligned} S(\tau, \sigma^* N) &= \big(\rho_{\sigma^* N} \mathcal{M}_\tau\big)(1) \\ &= \big((\sigma_* \rho_{\sigma^* N})(\sigma_* \mathcal{M}_\tau)\big)(1) \\ &= S(\sigma, N)^{-1} \big(\rho_N S(\sigma, N)(\sigma_* \mathcal{M}_\tau)\big)(1). \end{aligned}$$

On the other hand we have

$$\begin{aligned} S(\sigma\tau, N) &= \big(\rho_N \mathcal{M}_{\sigma\tau}\big)(1) \\ &= \big(\rho_N \mathcal{M}_\sigma(\sigma_* \mathcal{M}_\tau)\big)(1) \\ &= \big(\rho_N S(\sigma, N)(\sigma_* \mathcal{M}_\tau)\big)(1). \end{aligned}$$

Here the third equation follows from the definition of the convolution product and the fact that ρ_N is a group homomorphism. This proves Claim 1.

To prove Claim 2, we first observe that equation (1.11) for the triple (σ, τ, N) implies that $(\sigma\tau)^* N = \tau^* \sigma^* N$. With this understood Claim 2 follows immediately from the definitions. Thus we have proved assertions (i) and (iii) of Theorem B.

To prove assertion (ii), we first recall from Example 1 that any monodromy character $\mathcal{N} : \Gamma_m \to \mathbb{Z}^{m \times m}$ is uniquely determined by the matrix $N := \mathcal{N}(1)$ via

$$\mathcal{N} = N\rho_N$$

where $\rho_N : \Gamma_m \to \mathrm{GL}_m(\mathbb{Z})$ is given by (1.10). This implies that, for every Lefschetz fibration $f : X \to \mathbb{D}$ with critical fibers over Z, we have

$$\mathcal{N}_c^X = N_c^X (\rho_{N_c^X} \circ \iota_c^{-1})$$

Hence assertion (ii) follows from Theorem 4 and the identity

$$\sigma^* N = \big(\mathscr{T}_{\phi_c(\sigma), c}(\mathcal{N})\big)(1), \qquad \mathcal{N} := N(\rho_N \circ \iota_c^{-1}), \tag{7.2}$$

for $c \in \widetilde{\mathcal{C}}$ and $\sigma \in \widetilde{\mathcal{B}}_m$. To prove (7.2) we observe that

$$\rho_N(g)^T N \rho_N(g) = N$$

for $g \in \Gamma_m$ (as can be checked on the generators $g_1, \ldots, g_m$), and that

$$S(\sigma, N) = \sum_{g \in \Gamma_m} \rho_N(g^{-1}) \mathcal{S}_c(\phi_c(\sigma))(\iota_c(g))$$

for $\sigma \in \widetilde{\mathcal{B}}_m$. Hence

$$\begin{aligned} \sigma^* N &= S(\sigma, N)^T N S(\sigma, N) \\ &= \sum_{g,h \in \Gamma_m} \mathcal{S}_c(\phi_c(\sigma))(\iota_c(g))^T \rho_N(g^{-1})^T N \rho_N(h^{-1}) \mathcal{S}_c(\phi_c(\sigma))(\iota_c(h)) \\ &= \sum_{g,h \in \Gamma_m} \mathcal{S}_c(\phi_c(\sigma))(\iota_c(g))^T N \rho_N(gh^{-1}) \mathcal{S}_c(\phi_c(\sigma))(\iota_c(h)) \\ &= \sum_{g,h \in \Gamma} \mathcal{S}_c(\phi_c(\sigma))(g)^T \mathcal{N}(gh^{-1}) \mathcal{S}_c(\phi_c(\sigma))(h) \\ &= \big(\mathcal{S}_c(\phi_c(\sigma))^t \mathcal{N} \mathcal{S}_c(\phi_c(\sigma))\big)(1) \\ &= \big(\mathscr{T}_{\phi_c(\sigma), c}(\mathcal{N})\big)(1). \end{aligned}$$

This proves (7.2) and Theorem B. $\square$

8 The monodromy groupoid

Let $Z \subset \mathrm{int}(\mathbb{D})$ be a finite set and $v : Z \to S^1$ be a collection of unit tangent vectors $v(z) \in S^1$ for $z \in Z$. The pair (Z, v) is called **admissible** if no three elements lie on a straight line and $v(z) \neq |z' - z|^{-1}(z' - z)$ for all $z, z' \in Z$ with $z \neq z'$. We assume throughout that (Z, v) is an admissible pair. Associated to this pair is a groupoid whose objects are the elements of Z and the morphisms from z_0 to z_1 are homotopy classes of paths $\gamma : (0, 1) \to \mathbb{D} \setminus Z$ satisfying

$$\gamma(t) = z_0 + tv(z_0), \qquad \gamma(1 - t) = z_1 + tv(z_1), \qquad 0 \leq t \leq \delta, \tag{8.1}$$

for $\delta > 0$ sufficiently small. (This condition is required to hold for a uniform constant $\delta > 0$ along a homotopy.) Composition is given by catenation, pushed away from the intermediate point by the corresponding tangent vector. Denote the set of morphisms from z_0 to z_1 by $\mathcal{P}(z_0, z_1)$ and write

$$\mathcal{P} := \bigsqcup_{z_0, z_1 \in Z} \mathcal{P}(z_0, z_1).$$

It is convenient to present the groupoid $\mathcal{P}$ in terms of generators and relations. The generators are

$$s(z, z') \in \mathcal{P}(z', z), \qquad \varepsilon(z) \in \mathcal{P}(z, z)$$

for $z, z' \in Z$ with $z \neq z'$. Geometrically, $\varepsilon(z)$ represents a counterclockwise rotation about z and $s(z, z') \in \mathcal{P}(z', z)$ represents the straight line from z' to z, modified at each end by a counterclockwise turn to match the boundary conditions (see Figure 5). To describe the

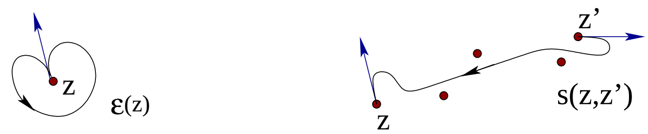

Figure 5: Generators of the groupoid $\mathcal{P}$.

relations we need some definitions. An **ordered triangle** is a triple z_0, w, z_1 of pairwise distinct elements of Z; it is called **local** if its convex hull contains no other elements of Z. Associated to every ordered triangle is an index

$$\mu(z_0, w, z_1) := \begin{cases} 0, & \text{if } v(w) \text{ points out of the triangle,} \\ \pm 1, & \text{if } v(w) \text{ points into the triangle.} \end{cases}$$

Here we choose $+1$ if z_0, w, z_1 are ordered counterclockwise around the boundary of their convex hull, and -1 if they are ordered clockwise. We have $\mu(z_1, w, z_0) = -\mu(z_0, w, z_1)$ and, when $w = e$ is an extremal point of Z, i.e. when e does not belong to the interior of the convex hull of Z, we have

$$\mu(z_0, e, z_1) + \mu(z_1, e, z_2) = \mu(z_0, e, z_2). \tag{8.2}$$

Theorem 5. *Let (Z, v) be an admissible pair. Then the groupoid $\mathcal{P}$ is generated by the morphisms $s(z, z')$ and $\varepsilon(z)$ subject to the relations*

$$s(z_0, z_1)s(z_1, z_0) = 1 \tag{8.3}$$

and, for every local triangle z_0, z_1, z_2,

$$s(z_0,z_1)\varepsilon(z_1)^{\mu_1}s(z_1,z_2)\varepsilon(z_2)^{\mu_2}s(z_2,z_0)\varepsilon(z_0)^{\mu_0} = 1, \tag{8.4}$$

where $\mu_1 := \mu(z_0,z_1,z_2)$, $\mu_2 := \mu(z_1,z_2,z_0)$, $\mu_0 := \mu(z_2,z_0,z_1)$.

Proof: That the generators satisfy these relations follows by inspection of local triangles (see Figure 6). To prove that there are no further relations and that every morphism is a composition

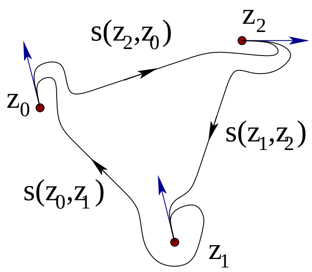

Figure 6: A positive local triangle.

of the generators, we choose a triangulation of the disc such that each element of Z is a vertex and all other vertices are on the boundary. Any path $\gamma : (0,1) \to \mathbb{D} \setminus Z$ satisfying (8.1) can then be approximated by a smooth path γ' intersecting the edges transversally and avoiding the vertices. Next one can homotop the path to a composition of the morphisms associated to the edges of the triangulation connecting two vertices in Z and suitable rotations $\varepsilon(z)$. To prove that there are no other relations one can study the combinatorial pattern of intersection points between the path γ' and the edges of the triangulation. First, the ambiguity in the choice of the word associated to the path γ' is governed by the local triangle relation. Second, every word representing γ can be obtained by a suitable choice of γ'. Third, in a generic homotopy of γ' there are two kinds of phenomena occuring at discrete times. Either two adjacent intersection points are created on the same edge or, conversely, two adjacent intersection points cancel. One can then check that these phenomena are again governed by the local triangle relation. This completes the sketch of the proof. □

A **monodromy character on** $\mathcal{P}$ is a map $\chi : \mathcal{P} \to \mathbb{Z}$ satisfying

$$\chi(\gamma^{-1}) = (-1)^n\chi(\gamma), \qquad \chi(1) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}, \end{cases} \tag{8.5}$$

for all $\gamma \in \mathcal{P}$ and

$$\chi(\gamma_{01}\varepsilon_1\gamma_{12}) = \chi(\gamma_{01}\gamma_{12}) - (-1)^{n(n+1)/2}\chi(\gamma_{01})\chi(\gamma_{12}) \tag{8.6}$$

for all $\gamma_{01} \in \mathcal{P}(z_1,z_0)$ and $\gamma_{12} \in \mathcal{P}(z_2,z_1)$, where $\varepsilon_1 := \varepsilon(z_1) \in \mathcal{P}(z_1,z_1)$ denotes the counterclockwise turn about z_1. As in Remark 1 one finds that every monodromy character satisfies

$$\chi(\gamma_{01}\varepsilon_1^{-1}\gamma_{12}) = \chi(\gamma_{01}\gamma_{12}) - (-1)^n(-1)^{n(n+1)/2}\chi(\gamma_{01})\chi(\gamma_{12}), \tag{8.7}$$

$$\chi(\varepsilon_0\gamma_{01}) = \chi(\gamma_{01}\varepsilon_1) = (-1)^{n+1}\chi(\gamma_{01}). \tag{8.8}$$

We refer to the equations (8.6) and (8.7) as the **reflection formulas**. The next theorem asserts that a monodromy character is uniquely determined by its values on the straight lines. Recall from the Introduction the definition of $\mathscr{N}_Z$ as the set of all matrices $Q : Z \times Z \to \mathbb{Z}$ satisfying (1.14).

Theorem 6. *For every $Q \in \mathscr{N}_Z$ there exists a unique monodromy character $\chi^Q : \mathcal{P} \to \mathbb{Z}$ such that*

$$\chi^Q(s(z, z')) = Q(z, z') \tag{8.9}$$

for all $z, z' \in Z$ with $z \neq z'$.

Remark 14. By the reflection formulas (8.6) and (8.7), the value of χ on any product of the form

$$\gamma = \gamma_{01}\varepsilon_1^{m_1}\gamma_{12}\cdots\varepsilon_{k-1}^{m_{k-1}}\gamma_{k-1,k}$$

is uniquely determined by the values of χ on the products $\gamma_{i,i+1}\cdots\gamma_{j-1,j}$ for $i < j$, regardless in which order we apply (8.6) and (8.7). An example is the identity

$$\begin{aligned}\chi(\gamma_{01}\varepsilon_1\gamma_{12}\varepsilon_2\gamma_{23}) \quad &= \quad \chi(\gamma_{01}\gamma_{12}\gamma_{23}) + \chi(\gamma_{01})\chi(\gamma_{12})\chi(\gamma_{23}) \\ &\quad - (-1)^{n(n+1)/2}\chi(\gamma_{01})\chi(\gamma_{12}\gamma_{23}) \\ &\quad - (-1)^{n(n+1)/2}\chi(\gamma_{01}\gamma_{12})\chi(\gamma_{23}).\end{aligned}$$

Remark 15. Equations (8.5) and (8.6) are consistent in the following sense. Suppose the values of χ on the morphisms γ_{01}, γ_{12}, $\gamma_{01}\gamma_{12}$ and their inverses all satisfy the first equation in (8.5) and that χ also satisfies the second equation in (8.5) on each identity morphism. Suppose further that the values of χ on $\gamma := \gamma_{01}\varepsilon_1\gamma_{12}$ and its inverse are both given by (8.6) and (8.7). Then we have $\chi(\gamma^{-1}) = (-1)^n\chi(\gamma)$.

Proof of Theorem 6: The proof is by induction on the number $N := \#Z$ of elements of Z. If $N = 1$ there is only one monodromy character χ given by $\chi(\varepsilon^m) = (-1)^{m(n+1)}\chi(1)$ and so the assertion is obvious. In the case $N = 2$ with $Z = \{z_0, z_1\}$ every morphism involving both vertices can be expressed as a composition of morphisms of the form

$$\alpha(k) := s(z_1, z_0)\varepsilon(z_0)^k, \qquad \beta(\ell) := s(z_0, z_1)\varepsilon(z_1)^\ell$$

where $k, l \in \mathbb{Z}$. Hence the map χ is uniquely determined by $\chi(s(z_0, z_1))$. Namely, the value of χ on the product $\alpha(k_1)\beta(\ell_1)\alpha(k_2)\beta(\ell_2)\cdots$ is obtained by applying equations (8.6) and (8.7) inductively. By Remark 14, the answer does not depend on the order in which we apply this formula, and hence the resulting function χ satisfies (8.6). That it also satisfies (8.5) follows from Remark 15. Thus χ is a monodromy character.

Now let $N \geq 3$ and denote by $E \subset Z$ the set of extremal points of the convex hull of Z. Then E has at least three elements, because $N \geq 3$ and Z is admissible. For every $e \in E$ denote $Z_e := Z \setminus \{e\}$ and let $\mathcal{P}_e$ be the space of all morphisms $\gamma \in \mathcal{P}$ that can be expressed as compositions of generators involving only vertices in Z_e. Then the induction hypothesis takes the following form.

Induction hypothesis. *For every $e \in E$ there is a unique monodromy character $\chi_e : \mathcal{P}_e \to \mathbb{Z}$ that satisfies* (8.9) *for all $z, z' \in Z_e$ with $z \neq z'$. Moreover, for all $e_1, e_2 \in E$, the functions χ_{e_1} and χ_{e_2} agree on $\mathcal{P}_{e_1} \cap \mathcal{P}_{e_2}$.*

Assuming this we wish to prove that there exists a unique monodromy character $\chi : \mathcal{P} \to \mathbb{Z}$ satisfying (8.9). This monodromy character will necessarily restrict to χ_e on $\mathcal{P}_e$ for every $e \in E$, so that the uniqueness of χ is granted. To prove the existence of χ it is convenient to introduce another category with the same set Z of objects, and whose morphisms from a to z are sequences

$$\widetilde{\gamma} = (z_0, m_0, z_1, m_1, \dots, z_k, m_k)$$

with $m_i \in \mathbb{Z}$, $z_i \in Z$ such that $z_i \neq z_{i+1}$ for all i, and $z_k = a$, $z_0 = z$. In this category the space of morphisms of length at most k will be denoted by $\widetilde{\mathcal{P}}^k(a, z)$ and we write

$$\widetilde{\mathcal{P}}(a, z) := \bigcup_k \widetilde{\mathcal{P}}^k(a, z), \qquad \widetilde{\mathcal{P}} := \bigsqcup_{a,z \in Z} \widetilde{\mathcal{P}}(a, z).$$

The notion of a monodromy character extends to the category $\widetilde{\mathcal{P}}$ (with γ^{-1} in (8.5) replaced by the sequence $\widetilde{\gamma}^{\#} := (z_k, -m_k, \dots, z_0, -m_0)$). There is a functor $\pi : \widetilde{\mathcal{P}} \to \mathcal{P}$ given by

$$\pi(\widetilde{\gamma}) := \varepsilon(z_0)^{m_0} s(z_0, z_1) \varepsilon(z_1)^{m_1} \cdots s(z_{k-1}, z_k) \varepsilon(z_k)^{m_k}. \tag{8.10}$$

By Theorem 5, this functor is surjective and two morphisms in $\widetilde{\mathcal{P}}(a, z)$ give the same morphism in $\mathcal{P}(a, z)$ if and only if they are related by a sequence of elementary moves corresponding to the relations (8.3) and (8.4). If $\chi : \mathcal{P} \to \mathbb{Z}$ is a monodromy character then so is the composition $\widetilde{\chi} := \chi \circ \pi : \widetilde{\mathcal{P}} \to \mathbb{Z}$. We shall construct a monodromy character $\widetilde{\chi} : \widetilde{\mathcal{P}} \to \mathbb{Z}$ and then show that it descends to $\mathcal{P}$. The proof has three steps.

Step 1. *If e is an extremal point and $a, z \in Z_e$, then we have $s(z, e)\varepsilon(e)^{\mu} s(e, a) \in \mathcal{P}_e$ where $\mu := \mu(z, e, a) \in \{-1, 0, 1\}$.*

When z, e, a is a local triangle then it follows from the local triangle relation (8.4) that, for suitable integers $\mu_a, \mu_z \in \{-1, 0, 1\}$, we have

$$s(z, e)\varepsilon(e)^{\mu} s(e, a) = \varepsilon(z)^{\mu_z} s(z, a) \varepsilon(a)^{\mu_a} \in \mathcal{P}_e(a, z).$$

In general we can find a sequence $z = z_0, \dots, z_k = a$ in Z_e such that z_{i-1}, e, z_i is a local triangle and hence

$$\gamma_{i-1,i} := s(z_{i-1}, e)\varepsilon(e)^{\mu_i} s(e, z_i) \in \mathcal{P}_e(z_i, z_{i-1})$$

for $\mu_i := \mu(z_{i-1}, e, z_i)$. Composing these morphisms we obtain

$$s(z, e)\varepsilon(e)^{\mu'} s(e, a) = \gamma_{01}\gamma_{12} \cdots \gamma_{k-1,k} \in \mathcal{P}_e(a, z), \qquad \mu' := \mu_1 + \cdots + \mu_k.$$

Hence the assertion of Step 1 follows from (8.2) which implies that $\mu = \mu'$.

Step 2. *There is a unique monodromy character $\widetilde{\chi} : \widetilde{\mathcal{P}} \to \mathbb{Z}$ satisfying the following conditions.*

(i) *If $z_0 \neq z_1$ and $\widetilde{s}(z_0, z_1) := (z_0, 0, z_1, 0)$ then $\widetilde{\chi}(\widetilde{s}(z_0, z_1)) = Q(z_0, z_1)$.*

(ii) *If* $a, z \in Z$, $e \in E \setminus \{a, z\}$, *and*

$$\widetilde{\gamma} = (z_0, m_0, \dots, z_k, m_k) \in \widetilde{\mathcal{P}}(a, z), \qquad z_k = a, \qquad z_0 = z,$$

such that

$$z_i = e \quad \Longrightarrow \quad m_i = \mu_i := \begin{cases} \mu(z_{i-1}, z_i, z_{i+1}), & \text{if } z_{i-1} \neq z_{i+1}, \\ 0, & \text{if } z_{i-1} = z_{i+1}, \end{cases} \tag{8.11}$$

then $\widetilde{\chi}(\widetilde{\gamma}) := \chi_e(\pi(\widetilde{\gamma}))$. *(In this case* $\pi(\widetilde{\gamma}) \in \mathcal{P}_e$ *by Step 1.)*

We prove uniqueness of the value $\widetilde{\chi}(\widetilde{\gamma})$ by induction on the length k. For $k = 0$ we must have

$$\widetilde{\chi}(z_0, m_0) = (-1)^{m_0(n+1)} \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}, \end{cases}$$

and for $k = 1$ we can argue as in the case $N = 2$ above. Now let $k \geq 2$ and suppose, by induction, that uniqueness of $\widetilde{\chi}(\widetilde{\alpha})$ has been verified for every morphism of length at most $k - 1$. Fix two elements $a, z \in Z$, an extremal point $e \in E \setminus \{a, z\}$, and a morphism

$$\widetilde{\gamma} = (z_0, m_0, \dots, z_k, m_k) \in \widetilde{\mathcal{P}}(a, z), \qquad z_k = a, \qquad z_0 = z.$$

If $\widetilde{\gamma}$ and e satisfy (8.11) then $\widetilde{\chi}(\widetilde{\gamma})$ is determined by condition (ii). If there is precisely one value i with $z_i = e$ and $m_i \neq \mu_i$ then the value $\widetilde{\chi}(\widetilde{\gamma})$ is determined inductively by (ii) and equation (8.6) for $\widetilde{\chi}$. Namely, with

$$\widetilde{\alpha} := (z_0, m_0, \dots, z_i, m_i), \qquad \widetilde{\beta} := (z_i, 0, z_{i+1}, m_{i+1}, \dots, z_k, m_k),$$

and $\widetilde{\varepsilon}_i := (z_i, 1)$ we have $\widetilde{\gamma} = \widetilde{\alpha}\widetilde{\beta}$ and

$$\widetilde{\gamma}' := \widetilde{\alpha}\widetilde{\varepsilon}_i\widetilde{\beta} = (z_0, m_0, \dots, z_i, m_i + 1, \dots, z_k, m_k),$$

$$\widetilde{\chi}(\widetilde{\gamma}') = \widetilde{\chi}(\widetilde{\gamma}) - (-1)^{n(n+1)/2}\widetilde{\chi}(\widetilde{\alpha})\widetilde{\chi}(\widetilde{\beta}). \tag{8.12}$$

This proves uniqueness when (8.11) fails for precisely one value of i. In general one can repeat this argument inductively for all i.

The same argument can be used to construct the value $\widetilde{\chi}(\widetilde{\gamma})$ by induction on the length k. The induction hypothesis is that $\widetilde{\chi}^{k-1} : \widetilde{\mathcal{P}}^{k-1} \to \mathbb{Z}$ has been constructed as to satisfy (i) and (ii) as well as (8.5) and (8.6) (the latter for compositions of two morphisms with total length at most $k - 1$). We must extend this map to one on $\widetilde{\mathcal{P}}^k$ with the same properties. For this we fix a morphism $\widetilde{\gamma} \in \widetilde{\mathcal{P}}(a, z)$ of length k and use the above induction argument with the auxiliary choice of an extremal point e to define $\widetilde{\chi}^k(\widetilde{\gamma})$. Assume there are two such extremal points $e, e' \in E \setminus \{a, z\}$. If

$$z_i \in \{e, e'\} \qquad \Longrightarrow \qquad m_i = \mu_i \tag{8.13}$$

then $\pi(\widetilde{\gamma}) \in \mathcal{P}_e \cap \mathcal{P}_{e'}$ and our value $\widetilde{\chi}(\widetilde{\gamma})$ is independent of e by (ii). In general, the same induction argument as above, using (8.6) repeatedly for those values of i for which (8.13) fails, shows that the two definitions of $\widetilde{\chi}(\widetilde{\gamma})$ (associated to e and e') agree for all values of the multiplicities m_i. Thus our definition of $\widetilde{\chi}^k(\widetilde{\gamma})$ is independent of e.

It remains to show that $\widetilde{\chi}^k$ satisfies the conditions (8.5) and (8.6) (for compositions of two morphisms with total length equal to k). Let

$$\widetilde{\alpha} \in \widetilde{\mathcal{P}}^i(a,b), \qquad \widetilde{\beta} \in \widetilde{\mathcal{P}}^{k-i}(b,c)$$

be two such morphisms. We must prove that (8.12) holds. If there is an extremal point $e \in E \setminus \{a,b,c\}$ we can argue as above and verify (8.12) for this pair by using Remark 14 and reducing the discussion to the case where both $\widetilde{\alpha}$ and $\widetilde{\beta}$ satisfy (8.11). In this case equation (8.12) follows from the fact that χ_e is a monodromy character. If $E = \{a,b,c\}$ we may choose $e = b$ and use the definition of $\widetilde{\chi}(\widetilde{\alpha}\widetilde{\beta})$ and $\widetilde{\chi}(\widetilde{\alpha}\widetilde{\varepsilon}(b)\widetilde{\beta})$ given above to establish the reflection formula. This shows that $\widetilde{\chi}^k$ satisfies (8.6). That it also satisfies (8.5) follows from Remark 15. This proves Step 2.

Step 3. *Let $\widetilde{\chi} : \widetilde{\mathcal{P}} \to \mathbb{Z}$ be as in Step 2. Then there is a unique monodromy character $\chi : \mathcal{P} \to \mathbb{Z}$ such that $\widetilde{\chi} = \chi \circ \pi$.*

We must prove that $\widetilde{\chi}$ descends to $\mathcal{P}$. This means that for all $a, z \in Z$ and all $\widetilde{\gamma}, \widetilde{\gamma}' \in \widetilde{\mathcal{P}}(a,z)$ we have

$$\pi(\widetilde{\gamma}) = \pi(\widetilde{\gamma}') \qquad \Longrightarrow \qquad \widetilde{\chi}(\widetilde{\gamma}) = \widetilde{\chi}(\widetilde{\gamma}'). \tag{8.14}$$

To see this fix an extremal point $e \in Z \setminus \{a,z\}$. If both $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ satisfy condition (8.11) then (8.14) follows immediately from (ii) in Step 2. In general, we must lower or raise the indices m_i with $z_i = e$. Inserting a term $s(z_i,w)s(w,z_i)$ into the word associated to $\widetilde{\gamma}$ does not affect this induction. When $w = e$ this is obvious and when $z_i = e$ the total number of induction steps on the left and right of the inserted term agrees with the number of steps at z_i before inserting, because

$$\mu(z_{i-1}, e, z_{i+1}) = \mu(z_{i-1}, e, w) + \mu(w, e, z_{i+1}).$$

Replacing a term $s(z_i, z_{i+1})$ with the product

$$s(z_i, z_{i+1}) = \varepsilon(z_i)^{\mu(z_{i+1}, z_i, w)} s(z_i, w) \varepsilon^{\mu(z_i, w, z_{i+1})} s(w, z_{i+1}) \varepsilon(z_{i+1})^{\mu(w, z_{i+1}, z_i)}$$

for a local triangle z_i, w, z_{i+1} also does not affect this induction for a similar reason. Here at most one of the terms z_i, w, z_{i+1} can be equal to e. If $w = e$ then the exponent $\mu(z_i, w, z_{i+1})$ already satisfies (8.11). If $z_i = e$ we use the fact that $\mu(z_{i-1}, e, z_{i+1}) + \mu(z_{i+1}, e, w) = \mu(z_{i-1}, e, w)$ and similarly in the case $z_{i+1} = e$.

Thus we have proved that $\widetilde{\chi}$ descends to $\mathcal{P}$. That the resulting map $\chi : \mathcal{P} \to \mathbb{Z}$ is a monodromy character is obvious and that it satisfies (8.9) follows from the fact that we have $\pi(\widetilde{s}(z_0, z_1)) = s(z_0, z_1)$ for all $z_0, z_1 \in Z$ with $z_0 \neq z_1$. Uniqueness follows immediately from the uniqueness statement in Step 2. This completes the proof of Theorem 6. $\square$

9 Proof of Theorem C

Fix an admissible pair (Z, v) and let $\mathcal{P}$ be the groupoid of Section 8. Recall from Section 4 the notion of marked distinguished configurations $c = (c_1, \dots, c_m) \in \widetilde{\mathcal{C}}$ with $m = \#Z$ such that $\{c_1(1), \dots, c_m(1)\} = Z$ and $\dot{c}_i(1) = -v(c_i(1))$ for all $i = 1, \dots, m$.

Lemma 5. **(i)** *For every $c \in \widetilde{\mathcal{C}}$ and every $N = (n_{ij})_{i,j=1}^m \in \mathscr{N}_m$ there is a unique monodromy character $\chi_{c,N} : \mathcal{P} \to \mathbb{Z}$ such that*

$$\chi_{c,N}(c_i \cdot c_j^{-1}) = n_{ij}$$

for all i and j.
(ii) *For all $c \in \widetilde{\mathcal{C}}$, $N \in \mathscr{N}_m$, and $\sigma \in \widetilde{\mathcal{B}}_m$ we have*

$$\chi_{c,N} = \chi_{\sigma^* c, \sigma^* N}.$$

Proof: We prove (i). Let $z_1, \dots, z_m$ be the ordering of Z determined by c. Uniqueness follows from the definition of a monodromy character and from the fact that any element $\gamma \in \mathcal{P}(z_i, z_j)$ can be expressed uniquely as $c_i \cdot g \cdot c_j^{-1}$ for some $g \in \Gamma_m \cong \Gamma$, and g can be expressed uniquely in reduced form as a product of the generators $g_1, \dots, g_m$ and their inverses. To prove existence, let $c \in \widetilde{\mathcal{C}}$ and $N = (n_{ij})_{i,j=1}^m \in \mathscr{N}_m$ be given and define

$$\chi_{c,N}(\gamma) = \left(N\rho_N(\iota_c^{-1}(c_i^{-1} \cdot \gamma \cdot c_j))\right)_{ij} \tag{9.1}$$

for $\gamma \in \mathcal{P}(z_i, z_j)$. Denote $n_{ij}(g) := \left(N\rho_N(\iota_c^{-1}(g))\right)_{ij}$ for $g \in \Gamma$. These functions define a monodromy character on Γ in the sense of Definition 1 (see Example 1). Let $\gamma_{ij} \in \mathcal{P}(z_j, z_i)$ and $\gamma_{jk} \in \mathcal{P}(z_k, z_j)$ and denote

$$g := c_i^{-1} \cdot \gamma_{ij} \cdot c_j, \qquad h := c_j^{-1} \cdot \gamma_{jk} \cdot c_k.$$

Abbreviate $\chi := \chi_{c,N}$. Then

$$\begin{aligned}
\chi(\gamma_{ij}\varepsilon_j\gamma_{jk}) &= n_{ik}(c_i^{-1} \cdot \gamma_{ij}\varepsilon_j\gamma_{jk} \cdot c_k) \\
&= n_{ik}((c_i^{-1} \cdot \gamma_{ij} \cdot c_j) \cdot (c_j^{-1} \cdot \varepsilon_j \cdot c_j) \cdot (c_j^{-1} \cdot \gamma_{jk} \cdot c_k)) \\
&= n_{ik}(gg_jh) \\
&= n_{ik}(gh) - (-1)^{n(n+1)/2} n_{ij}(g) n_{jk}(h) \\
&= \chi(\gamma_{ij}\gamma_{jk}) - (-1)^{n(n+1)/2} \chi(\gamma_{ij})\chi(\gamma_{jk}).
\end{aligned}$$

Hence χ is a monodromy character. Moreover, by (9.1) we have

$$\chi(c_i \cdot c_j^{-1}) = \left(N\rho_N(\iota_c^{-1}(1))\right)_{ij} = n_{ij}$$

for all i and j. This proves (i).

We prove (ii). Abbreviate $\tau := \phi_c(\sigma)$ and

$$n_{i'j'}(g) := \left(N\rho_N(\iota_c^{-1}(g))\right)_{i'j'}, \qquad \widetilde{n}_{ij}(g) := \left((\sigma^*N)\rho_{\sigma^*N}(\iota_{\sigma^*c}^{-1}(g))\right)_{ij}.$$

Let $i, j \in \{1, \dots, m\}$ and $i' = \pi_{\tau,c}(i)$, $j' := \pi_{\tau,c}(j)$. Let $\gamma \in \mathcal{P}(z_{j'}, z_{i'})$. Then

$$\begin{aligned}
\chi_{\sigma^* c, \sigma^* N}(\gamma) &= \widetilde{n}_{ij}((\sigma^* c_i)^{-1} \cdot \gamma \cdot (\sigma^* c_j)) \\
&= \widetilde{n}_{ij}((\tau_* c_i)^{-1} \cdot c_{i'} \cdot c_{i'}^{-1} \cdot \gamma \cdot c_{j'} \cdot c_{j'}^{-1} \cdot (\tau_* c_j)) \\
&= \widetilde{n}_{ij}(s_{i,c}(\tau)^{-1} \cdot c_{i'}^{-1} \cdot \gamma \cdot c_{j'} \cdot s_{j,c}(\tau)) \\
&= n_{i'j'}(c_{i'}^{-1} \cdot \gamma \cdot c_{j'}) \\
&= \chi_{c,N}(\gamma).
\end{aligned}$$

Here the third equation follows from (5.2) and the fourth equation is a consequence of (5.11). This proves the lemma. □

Proof of Theorem C: The invariance of the map $(c, N) \mapsto Q_{c,N}$ follows from Lemma 5 (ii) and the fact that

$$Q_{c,N}(z, z') = \chi_{c,N}(s(z, z')).$$

Given $Q \in \mathscr{N}_Z$, the existence of an equivariant map $c \mapsto N_c$ with $Q_{c,N_c} = Q$ follows from Lemma 5 and Theorem 6. We define

$$N_c := (n_{ij})_{i,j=1}^m, \qquad n_{ij} := \chi^Q(c_i \cdot c_j^{-1}).$$

Then it follows from uniqueness in Lemma 5 that $\chi_{c,N_c} = \chi^Q$ and hence

$$Q_{c,N_c}(z_i, z_j) = \chi_{c,N_c}(s(z_i, z_j)) = \chi^Q(s(z_i, z_j)) = Q(z_i, z_j).$$

Here the first equation follows from the definition of Q_{c,N_c} in the Introduction, and the last equation follows from the definition of χ^Q in Theorem 6. This proves existence. To prove uniqueness, let $c \mapsto N'_c = (n'_{ij})_{i,j=1}^m$ be another equivariant family satisfying $Q_{c,N'_c} = Q$, so that

$$\chi_{c,N'_c}(s(z, z')) = Q_{c,N'_c}(z, z') = Q_{c,N_c}(z, z') = \chi_{c,N_c}(s(z, z')).$$

By uniqueness in Theorem 6 we have $\chi_{c,N'_c} = \chi_{c,N_c} = \chi^Q$, and therefore

$$n'_{ij} = \chi_{c,N'_c}(c_i \cdot c_j^{-1}) = \chi_{c,N_c}(c_i \cdot c_j^{-1}) = n_{ij}$$

for all i, j. This completes the proof of Theorem C. □

10 An Example

Let $f : \Sigma \to \mathbb{T}^2$ be a degree 3 branched cover of a genus 3 surface over the torus with 4 branch points of branching order 2 (see Figure 7). The map f is a covering over the complement of the two slits from z_1 to z_3 and z_2 to z_4 depicted in the figure. The surface Σ is obtained by gluing the three sheets labeled 1, 2 and 3 along the slits as indicated. We identify the fiber M over z_0 with the set $\{1, 2, 3\}$. Each vanishing cycle is an oriented 0-sphere, i.e. consists of two points labeled by opposite signs (which can be chosen arbitrarily). The marked distinguished

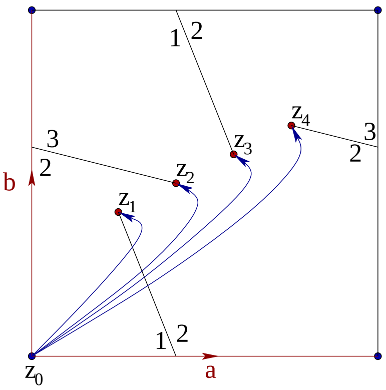

Figure 7: A 3-sheeted branched cover over the torus.

configuration indicated in the figure determines the vanishing cycles up to orientation. We choose the orientations as follows:

$$L_1 = \{1^-, 2^+\}, \qquad L_2 = L_4 = \{1^+, 3^-\}, \qquad L_3 = \{1^+, 2^-\}.$$

The fundamental group Γ of the 4-punctured torus is generated by the elements a, b, g_1, g_2, g_3, g_4 where the g_i are determined by the distinguished configuration and a and b are the horizontal and vertical loops indicated in Figure 7. These generators are subject to a single relation $b^{-1}a^{-1}ba = g_1g_2g_3g_4$. Understood as permutations, the generators g_1, g_3 and a act by the transposition $(1,2)$, while g_2 and g_4 act by the transposition $(1,3)$, and b acts by the transposition $(2,3)$. Note that the commutator

$$b^{-1}a^{-1}ba = g_1g_2g_3g_4 = (231)$$

is not the identity.

The intersection matrix of the L_i is

$$\mathcal{N}_c(1) = \begin{pmatrix} 2 & -1 & -2 & -1 \\ -1 & 2 & 1 & 2 \\ -2 & 1 & 2 & 1 \\ -1 & 2 & 1 & 2 \end{pmatrix}.$$

Moreover, the images of the L_i under a are

$$a(L_1) = \{1^+, 2^-\}, \qquad a(L_2) = a(L_4) = \{2^+, 3^-\}, \qquad a(L_3) = \{1^-, 2^+\}$$

and hence

$$\mathcal{N}_c(a) = \begin{pmatrix} -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & -2 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Similarly,

$$\mathcal{N}_c(g_2) = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -2 & -1 & -2 \\ -1 & -1 & 1 & -1 \\ 1 & -2 & -1 & -2 \end{pmatrix}, \ \mathcal{N}_c(b) = \begin{pmatrix} 1 & -2 & -1 & -2 \\ -2 & 1 & 2 & 1 \\ -1 & 2 & 1 & 2 \\ -2 & 1 & 2 & 1 \end{pmatrix},$$

$$\mathcal{N}_c(ab) = \begin{pmatrix} -1 & 2 & 1 & 2 \\ -1 & -1 & 1 & -1 \\ 1 & -2 & -1 & -2 \\ -1 & -1 & 1 & -1 \end{pmatrix}, \ \mathcal{N}_c(ba) = \begin{pmatrix} -1 & -1 & 1 & -1 \\ 2 & -1 & -2 & -1 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & -2 & -1 \end{pmatrix}.$$

Note that $\mathcal{N}_c(g_1) = \mathcal{N}_c(g_3) = \mathcal{N}_c(a)$ and $\mathcal{N}_c(g_4) = \mathcal{N}_c(g_2)$. The matrix $\mathcal{N}_c(g_i)$ can also be obtained from $\mathcal{N}_c(1)$ via the gluing formula

$$\mathcal{N}_c(g_i) = \mathcal{N}_c(1) - \mathcal{N}_c(1)E_i\mathcal{N}_c(1).$$

The action of the fundamental group on the homology of the fiber $M = \{1, 2, 3\}$ obviously factors through the permutation group $\mathfrak{S}_3$ and hence the function $\mathcal{N}_c : \Gamma \to \mathbb{Z}^{4\times 4}$ is uniquely determined by its values on 1, g_2, a, b, ab and ba. One can verify directly that $\mathcal{N}_c$ satisfies the relations (2.1-2.3), for example $\mathcal{N}_c(ba) = \mathcal{N}_c(ab)^T$ and

$$\mathcal{N}_c(b) = \mathcal{N}_c(ag_1b) = \mathcal{N}_c(ab) - \mathcal{N}_c(a)E_1\mathcal{N}_c(b).$$

11 Appendix: Lefschetz fibrations

In this section we review some basic facts about Lefschetz fibrations (see [2, Chapter I]). Let X be a compact Kähler manifold of complex dimension $n + 1$ and $f : X \to \Sigma$ be a Lefschetz fibration with a finite set $Z \subset \Sigma$ of critical values. Let $m := \#Z$ be the number of critical values. We assume that each critical fiber $f^{-1}(z)$ contains precisely one (Morse) critical point x_z. Choose a regular value $z_0 \in \Sigma$ and denote $M := f^{-1}(z_0)$. Then the fundamental group

$$\Gamma := \pi_1(\Sigma \setminus Z, z_0)$$

acts on the middle dimensional homology

$$H_n(M) := H_n(M; \mathbb{Z})/\text{torsion}$$

via parallel transport. We denote the action by

$$\rho : \Gamma \to \mathrm{Aut}(H_n(M)).$$

Each path $c : [0, 1] \to \Sigma$ connecting $z_0 = c(0)$ to a critical value $z = c(1)$, avoiding critical values for $t < 1$, and satisfying $\dot{c}(1) \neq 0$, determines a vanishing cycle $L \subset M$ and an element $g \in \Gamma$. Geometrically, L is the set of all points in M that converge to the critical point $x_z \in f^{-1}(z)$ under parallel transport along c. The orientation of L is not determined by the path and can be chosen independently. The element g is the homotopy class of the path obtained by following

c, encircling z once counterclockwise, and then following c^{-1}. Thus $\rho(g)$ acts on $H_n(M)$ by the Dehn–Arnold–Seidel twist ψ_L about L with its framing as a vanishing cycle. (The framing is a choice of a diffeomorphism from L to the n-sphere, see Seidel [20].) Now choose m such paths c_i, $i = 1, \dots, m$, connecting z_0 to the critical values, denote by $L_i \in H_n(M)$ the homology classes of the resulting vanishing cycles with fixed choices of orientations, and by $g_i \in \Gamma$ the resulting elements of the fundamental group.

Remark 16. For $i = 1, \dots, m$ the action of g_i on $H_n(M)$ is given by

$$\rho(g_i)\alpha = \alpha - (-1)^{n(n+1)/2} \langle L_i, \alpha \rangle L_i, \tag{11.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the intersection form. The intersection numbers

$$n_{ij}(g) := \langle L_i, \rho(g) L_j \rangle, \qquad i, j = 1, \dots, m,$$

satisfy (2.1-2.3). Equation (2.1) follows from the (skew-)symmetry of the intersection form, equation (2.2) is a general fact about the self-intersection numbers of Lagrangian spheres, and (2.3) follows from (11.1). Equation (11.1) follows from Example 2 below.

Remark 17. The intersection number $n_{ij}(g) = \langle L_i, \rho(g) L_j \rangle$ can be interpreted as the algebraic number of horizontal lifts of the path $c_j g^{-1} c_i^{-1}$ (first c_i^{-1} then g^{-1} then c_j) connecting the critical points x_i and x_j. That these numbers continue to be meaningful (at least conjecturally) in certain infinite dimensional settings (where the vanishing cycles only exist in some heuristic sense) is one of the key ideas in Donaldson–Thomas theory [9].

Example 2. The archetypal example of a Lefschetz fibration is the function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ given by

$$f(z_0, \dots, z_n) := z_0^2 + z_1^2 + \dots + z_n^2.$$

Denote the fiber over 1 by

$$M := f^{-1}(1) = \left\{ x + iy \in \mathbb{C}^{n+1} \mid |x|^2 - |y|^2 = 1,\ \langle x, y \rangle = 0 \right\}.$$

The vanishing cycle obtained by parallel transport along the real axis from 0 to 1 is the unit sphere $L := M \cap \mathbb{R}^{n+1}$. The manifold M is symplectomorphic to the cotangent bundle of the n-sphere

$$T^*S^n = \left\{ (\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |\xi| = 1,\ \langle \xi, \eta \rangle = 0 \right\}$$

via $M \to T^*S^n : (x, y) \mapsto (x/|x|, |x|\, y)$. Then the monodromy around the loop $t \mapsto e^{2\pi i t}$ is the symplectomorphism $\psi : T^*S^n \to T^*S^n$ given by

$$\psi(\xi, \eta) = \left(-\cos(\theta)\xi - \sin(\theta)\frac{\eta}{|\eta|}, \sin(\theta)\,|\eta|\,\xi - \cos(\theta)\eta \right) \tag{11.2}$$

where $\theta := 2\pi|\eta|/\sqrt{1 + 4|\eta|^2}$. Let $\bar{M}$ be the sphere bundle over S^n obtained by the one point compactification of each fiber of the cotangent bundle. Its middle dimensional homology

$H_n(\bar{M};\mathbb{Z})$ has two generators, namely the homology class of the zero section and the homology class of the fiber, and ψ acts on $H_n(\bar{M};\mathbb{Z})$ by the formula

$$\psi_*\alpha = \alpha - (-1)^{n(n+1)/2}\,\langle L,\alpha\rangle\, L.$$

To see this fix ξ in (11.2) and project to S^n to get a map of degree $(-1)^{n+1}$. An additional factor $(-1)^{n(n-1)/2}$ arises by comparing the orientation of the cotangent bundle T^*L with the complex orientation of M. In the even case the value of the factor follows from the observation that the self-intersection number of the vanishing cycle is $\langle L, L\rangle = 2(-1)^{n/2}$.

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