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On the defining equations of the Hankel varieties $H(2, n)$

by
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Abstract

We investigate the problem to find the defining equations of the Hankel algebraic variety $H(r, n)$ of the Hankel r-planes of the projective space $\mathbb{P}^n$. We describe an algorithm which, provided of a positive output, gives the defining relations of $H(2, n)$, starting from the binomial relations of the toric deformation of $H(2, n)$.

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Introduction

The initial motivations of this note come from the papers [4] and [5] where we studied the Hankel variety $H(r, n)$ of r-planes of $\mathbb{P}^n$, introduced in [7], subvarieties and the singular locus. For the Grassmann variety of r-planes of $\mathbb{P}^n$ the defining equations are known, nevertheless not explicitly written [1]. For the defining equations of $H(r, n)$ the problem is open and more complicated. The defining equations of the toric deformation of $H(r, n)$, that are binomial relations, are found by Machado([8]). This approach allows to find definitive results in the case of $H(1, n)$ by using Sagbi bases([3]). In this paper we describe an algorithm, based on Sagbi basis theory, to determine the relations of $H(r, n)$ by lifting the binomial relations given by Machado. Theoretically it may happen that the lifting algorithm stops without giving the lifted relation. In all the cases considered this never occurred. In Section 1 we recall the Machado relations for $r = 2$, and in Section 2 we provide a Sagbi basis criterion which is a variation of the Robbiano-Sweedler criterion ([9]). Section 3 is devoted to describe the lifting algorithm for the Machado relations. Finally in Section 4 we provide partial liftings for $H(2, n)$. In the case $H(2, 5)$ the relations are determined. Some of the results of this paper have been conjectured after explicit computations performed by using the software CoCoA ([2]).[1]

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1 Machado's standard relations

Let R be a commutative ring. A matrix of the of the form

$$H_{r,n} = \begin{pmatrix} x_1 & x_2 & \cdots & \cdots & x_n \\ x_2 & x_3 & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{r-1} & x_r & \cdots & \cdots & x_{n+r-1} \\ x_r & x_{r+1} & \cdots & x_{n+r-1} & x_{n+r} \end{pmatrix}$$

with coefficients in R is called *Hankel matrix*. In this paper we consider generic Hankel matrices $H_{r,n}$, in other words, Hankel matrices whose entries are indeterminates. Let K be a field and $S = K[x_1, x_2, \ldots, x_{n+r}]$ the polynomial ring over K in $n + r$ indeterminates. We denote by $[i_1 i_2 \ldots i_r]$ the r-minor with columns $i_1 < i_2 < \ldots i_r$, $r \leq n$. Let $<$ be the lexicographical order induced by $x_1 > x_2 > \ldots > x_{n+r}$. Then

$$in_<[i_1 i_2 \ldots i_r] = x_{i_1} x_{i_2+1} \ldots x_{i_r+r-1}.$$

Notice that $x_{i_1} x_{i_2+1} \ldots x_{i_r+r-1}$ is the product of monomials corresponding to the main diagonal of the minor $[i_1 i_2 \ldots i_r]$.
In this section we study the K-algebra $A_{2,n}$ over K generated by the initial monomials $x_{i_1} x_{i_2+1} x_{i_3+2}$ with $1 \leq i_1 < i_2 < i_3 \leq n$ of the 3−minors of $H_{3,n}$. We will show that $A_{2,n}$ is the initial algebra of the coordinate ring of the Hankel variety $H(2, n)$ for $n = 5$.
Let $T = K[y_{i_1 i_2 i_3} : 1 \leq i_1 < i_2 < i_3 \leq n]$ be the polynomial ring in the variables $y_{i_1 i_2 i_3}$ and let $\psi : T \to A_{2,n}$ be the K-algebra homomorphism with $y_{i_1 i_2 i_3} \mapsto x_{i_1} x_{i_2+1} x_{i_3+2}$. Each monomial of degree d in T can be identified with a $d \times 3$ matrix

$$\begin{pmatrix} i_{11} & i_{12} & i_{13} \\ i_{21} & i_{22} & i_{23} \\ \vdots & \vdots & \vdots \\ i_{d1} & i_{d2} & i_{d3} \end{pmatrix}$$

such that $(i_{11} i_{12} i_{13}) \geq (i_{21} i_{22} i_{23}) \geq \cdots \geq (i_{d1} i_{d2} i_{d3})$ in the lexicographical order. In particular a monomial of degree two in d corresponds to a matrix of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

with $a < b < c$, $d < e < f$ and $(a, b, c) \geq (d, e, f)$.
The kernel $J = \ker \psi$ has been determined by Machado([8]), even for generalized Hankel matrices of arbitrary size. In our case J is generated by the following type of relations

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & e & c \\ d & b & f \end{pmatrix} \quad \text{with} \quad e < b,\, c \leq f,$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & e & f \\ d & b & c \end{pmatrix} \quad \text{with} \quad e < b,\, f < c,$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & b & f \\ d & e & c \end{pmatrix} \quad \text{with} \quad b \leq e,\ f < c,$$

and assuming that $a \leq d$, $b \leq e$, $c \leq f$ one has the following relations:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & d-1 & c \\ b+1 & e & f \end{pmatrix} \qquad \text{with} \quad b << d,\ e-c \leq 1,\ d-1 < c$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & b & e-1 \\ d & c+1 & f \end{pmatrix} \qquad \text{with} \quad d-b \leq 1,\ c << e,\ c+1 < f$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{pmatrix} \quad \text{with} \quad b << d,\ c << e.$$

Here we set $i << j$ if $j - i \geq 2$.

2 Sagbi basis criterion

This section contains a new approach to the problem to determine the defining equations of $H(2,n)$, "via" the Sagbi bases theory. We start by considering a formulation of a Sagbi basis criterion which will be used later to determine Hankel relations. Let K be a field, $S = K[x_1, \cdots, x_n]$ the polynomial ring in n variables and $A \subseteq S$ a finitely generated K-subalgebra of S. We fix a monomial order $<$ on S, then the K-algebra $K[in_<(a) : a \in A]$ is called the initial algebra of A and denoted by $in_<(A)$. In general this algebra is not finitely generated. A finite set of generators $a_1, \ldots, a_m$ of A is called Sagbi basis of A if $in_<(A) = K[in_<(a_1), \ldots, in_<(a_m)]$. Sagbi basis does not always exist. In the following theorem we give a criterion for the existence of a Sagbi basis which is a variation of the known criterion ([9]).

Theorem 2.1. *Let $T = K[y_1, \ldots, y_m]$ be the polynomial ring over K in the variables $y_1, \ldots, y_m$, and let $\varphi : T \to A$ the K-algebra homomorphism with $y_i \mapsto a_i$ and $\psi : T \to in_<(A)$ the K-algebra homomorphism with $y_i \mapsto in_<(a_i)$ for $i = 1, \ldots, m$. Let $I = Ker\ \varphi$ and $f_1, \ldots, f_r$ be a set of binomial generators of $J = Ker\ \psi$. Then the following conditions are equivalent:*

(*a*) *$a_1, \ldots, a_m$ is a Sagbi basis of A.*

(*b*) *For each j, there exist monomials $m_1, \ldots, m_s \in T$ and $c_1, \ldots, c_s \in K$ such that*

 (i) $f_j + \sum_i^s c_i m_i \in I$.

 (ii) $in_<(\varphi(m_{i+1})) = in_<(\varphi(f_j + \sum_{k=1}^i c_k m_k)) < in_<(\varphi(m_i))$,
 $in_<(\varphi(f_j + c_1 m_1)) < in_<(\varphi(f_j))$.

Proof: $(a) \Rightarrow (b)$ By the criterion of Robbiano-Sweedler ([9]), a set $a_1, \ldots, a_m$ of generators of A is a Sagbi basis if and only if for each binomial generator $f_j(y_1, \ldots, y_m)$ of $Ker\ \psi$ we have:

$$(1) \qquad f_j(a_1, \ldots, a_m) = \sum \lambda_\nu^{(j)} a^\nu$$

for all $\lambda_\nu^j \neq 0$, where, as usual, $a^\nu = a_1^{\nu_1} \ldots a_1^{\nu_n}$ and

$$(2) \qquad in_<(a^\nu) < in_< f_j(a_1, \ldots, a_m),\ in_<(a^\nu) \neq in_<(a^\mu) \text{ for } \nu \neq \mu.$$

From (1), $f_j(a_1, \ldots, a_m) - \sum \lambda_\nu^{(j)} a^\nu = 0$ and it follows that there exist monomials $m_1, \ldots, m_l$ in the variables $y_1, \ldots, y_m$ such that

$$(3) \qquad \varphi\left(f_j(y_1, \ldots, y_m) + \sum_{r=1,\ldots,l} c_r m_r\right) = 0.$$

Hence

$$(4) \qquad f_j(y_1, \ldots, y_m) + \sum_{r=1,,l} c_r m_r \in I.$$

This is the first assertion. Now, we can order the monomials $m_1, \ldots, m_l$ such that $in_<(\varphi(m_{i+1})) < in_<(\varphi(m_i))$. Then, in particular by (3) we obtain:

$$in_<(\varphi(m_{i+1})) = in_<\varphi\left(f_j(y_1, \ldots, y_m) + \sum_{r=1,\ldots,l,r\neq i+1} c_r m_r\right) =$$

$$in_<\varphi\left(f_j(y_1, \ldots, y_m) + \sum_{r=1,\ldots,i} c_r m_r\right)$$

because all monomials m_l, with $l > i+1$, are such that $in_<\varphi(m_l) < in_<\varphi(m_{i+1})$.

$(b) \Rightarrow (a)$:we have to prove (1) and (2). The assertion (1) follows by (4). To obtain (2), by (b) we see that

$$in_<(\varphi(m_{i+1})) = in_<(\varphi(f_j(y_1, \ldots, y_m) + \sum_{r=1,\ldots,i} c_r m_r)) \leq in_<(\varphi(f_j(y_1, \ldots, y_m))),$$

being $in_<(\varphi(m_{i+1})) < in_<(\varphi(m_i))$.

□

If the equivalent conditions are satisfied, we call $f_j + \sum_i^s c_i m_i$ a lifting of f_j.

Remark 2.2. In the criterion of Robbiano-Sweedler([9], or [3], Prop.1.3) only the equivalent conditions (a) and $(b)(i)$ appear. In our criterion we prove, in addition, the condition $(b)(ii)$, that concerns a constraint on the initial terms of elements of the ring S with respect to the fixed monomial order $<$. We need the previous condition (ii) in a crucial way in the algorithm that we shall introduce in the next section to find the partial and the complete liftings of Machado relations.

3 An algorithm to compute the lifting of Machado relations

We shall make essential use of Theorem 2.1 in order to describe when the maximal minors of the Hankel matrix $H_{3,n}$ form a Sagbi basis of the coordinate ring of the Hankel variety $H(2,n)$. According to Theorem 2.1, we describe an algorithm to determine when a Machado relation is liftable.

The imput of our algorithm is a Machado relation, given by the generators of $A_{2,n}$.

Step 1: Take one of the binomial Machado's relations and replace in the relation the initial terms by the corresponding minors to obtain the element $m_1 \in A_{2,n}$, and determine its initial term.

Step 2: If $in_< m_1$ is a product of the initial terms of two minors M_1, M_2, then m_1 is partially liftable and we add a suitable multiple of M_1M_2 to m_1 to obtain m_2 with the property that $in_< m_2 < in_< m_1$ or $m_2 = 0$. Consider all the possible partial lifting of m_1 and go to Step 3.

If $in_< m_1$ is not a product of the initial terms of two minors of $H_{3,n}$, then the relation m_1 is not liftable and STOP (In this case the minors of Hankel matrix do not form a Sagbi basis).

Step 3: Apply the same procedure of Step 2 until the complete lifting of the Machado relation is determined for at least one of partial liftings computed OR stop when you find an initial term that is not partially liftable.

The following example demonstrates our method. We denote the product of two minors $[abc][def]$ and we write

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

We begin with the relation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}.$$

Replacing the monomials in this relation by the corresponding minors

$$m_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} = -x_1x_3x_4x_5x_7^2 + \cdots .$$

Here we see that the initial term $x_1x_3x_4x_5x_7^2$ can be written as the product of the minors $[135]$ and $[345]$. This is the first partial lifting. Next we consider

$$m_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 5 \end{bmatrix} = -x_2^2x_4x_5x_6x_7 + \cdots .$$

This new initial term can be decomposed. Proceeding in this way we obtain the following lifting of our binomial:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

We observe that in this case each initial term computed admits only one partial lifting and then the complete lifting of Machado relation is unique. In the next example we see that a initial term can admit more than one partial lifting. Starting by a Machado relation and replacing it with the corresponding minors we have:

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} = -x_1x_3x_4x_6x_8^2 + \cdots$$

that we can write

$$\begin{bmatrix} 1 & 3 & 6 \\ 3 & 5 & 6 \end{bmatrix} \text{or} \begin{bmatrix} 1 & 2 & 6 \\ 4 & 5 & 6 \end{bmatrix}.$$

Then we have two partial liftings. Applying the algorithm to each of them we will obtain always one partial lifting in the other steps. Then we have two complete liftings for this Machado relation:

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 6 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 6 \\ 2 & 5 & 6 \end{bmatrix}$$

$$+ \begin{bmatrix} 2 & 4 & 5 \\ 2 & 4 & 7 \end{bmatrix} - 2\begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

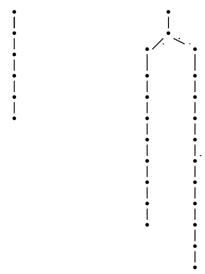
or

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 6 & 7 \end{bmatrix} +$$

$$- \begin{bmatrix} 2 & 3 & 6 \\ 2 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix} +$$

$$- \begin{bmatrix} 2 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}.$$

It is useful to give a scheme of the algorithm in a graphic form. We take a tree where the binomial relation is the root and the set of the partial liftings, that we find during the research, appears in the interior nodes. In the figure we have the trees for the liftings described before:



Now we want to lift the binomial relations in general which is of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} g & h & k \\ l & m & n \end{pmatrix}$$

where each of the entries in the second matrix coincide up to constant with one of the entries of the first matrix. All Machado's relations are of this type. The monomials in the support of $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ are all of the form $x_{a+i_1}x_{b+i_2}x_{c+i_3}x_{d+j_1}x_{e+j_2}x_{f+j_3}$. A similar statement for the second matrix. Therefore all monomials in the support of the difference are product of variables whose indices are the entries of the first matrix up to constant. Therefore if this difference can be lifted, we will obtain

$$f_2 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} - \begin{pmatrix} g & h & k \\ l & m & n \end{pmatrix} \pm \begin{pmatrix} o & p & q \\ r & s & t \end{pmatrix},$$

where the entries of $\begin{pmatrix} o & p & q \\ r & s & t \end{pmatrix}$ coincide with those of $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ up to position and up a difference by one.

4 On the relations of $H(2,n)$

In the following theorem we apply the algorithm and we find the general expression of the first and second lifting of the binomial relations of Machado.

Theorem 4.1. *The binomial relations of the $K-$algebra $A_{2,n}$, have lifting polynomials of length ≥ 3 if they are Plücker relations and of length ≥ 4 if they are Hankel relations. More precisely, depending on the Machado inequalities, we have the following liftings and partial liftings:*

(I) $\quad e < b,\ c \leq f$

(I_A) $c = f, \quad a < d < e < b < c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & c \\ d & b & c \end{bmatrix} + \begin{bmatrix} a & d & c \\ e & b & c \end{bmatrix}$$

(I_B) $c < f, \quad a < d < e < b < c < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & c \\ d & b & f \end{bmatrix} + \begin{bmatrix} a & e & b \\ d & c & f \end{bmatrix} + \begin{bmatrix} a & d & c \\ e & b & f \end{bmatrix} + \cdots$$

(II) $e<b,\ f<c, \quad a<d<e<b\le f<c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & e & f \\ d & b & c \end{bmatrix} + \begin{bmatrix} a & d & f \\ e & b & c \end{bmatrix} + \begin{bmatrix} a & d & e \\ b & f & c \end{bmatrix} + \cdots$$

(III) $b\le e,\ f<c$

(III_A) $a=d, \quad a<b<e<f<c$

$$\begin{bmatrix} a & b & c \\ a & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ a & e & c \end{bmatrix} + \begin{bmatrix} a & b & e \\ a & f & c \end{bmatrix}$$

(III_B) $b=e, \quad a<d<b<f<c$

$$\begin{bmatrix} a & b & c \\ d & b & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & b & c \end{bmatrix} + \begin{bmatrix} a & d & b \\ b & f & c \end{bmatrix}$$

(III_C) $b\le d, \quad a<b\le d<e<f<c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & e & c \end{bmatrix} + \begin{bmatrix} a & b & e \\ d & f & c \end{bmatrix} + \begin{bmatrix} a & b & f-2 \\ d+1 & e+1 & c \end{bmatrix} + \cdots$$

(III_D) $b>d, \quad a<d<b<e<f<c$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & f \\ d & e & c \end{bmatrix} + \begin{bmatrix} a & b & e \\ d & f & c \end{bmatrix} + \begin{bmatrix} a & d & b \\ e & f & c \end{bmatrix} + \ldots$$

(IV) $2\le d-b,\ e-c\le 1,\ d-1<c$

(IV_A) $a<b<<d<e-1\le c<e<f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & c \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & d & e-1 \\ b+1 & c & f \end{bmatrix} + \begin{bmatrix} a & d & e \\ b+1 & c & f-1 \end{bmatrix} + \cdots$$

(IV_B) $d = c = e-1, \quad a < b << e-1 < e < f-1 < f$

$$\begin{bmatrix} a & b & e-1 \\ e-1 & e & f \end{bmatrix} - \begin{bmatrix} a & e-2 & e-1 \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & f-1 \end{bmatrix} + \\ + \begin{bmatrix} a & e-2 & e \\ b+2 & e & f-1 \end{bmatrix} + \cdots$$

(IV_{B1}) $d = c = e-1, \, e = f-1, \quad a < b << e-1 < e < e+1$

$$\begin{bmatrix} a & b & e-1 \\ e-1 & e & e+1 \end{bmatrix} - \begin{bmatrix} a & e-2 & e-1 \\ b+1 & e & e+1 \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & e \end{bmatrix} + \\ + \begin{bmatrix} a+1 & e-2 & e-1 \\ b & e & e+1 \end{bmatrix} + \cdots$$

(IV_C) $d < e-1, \, c = e, \quad a < b << d < e-1 < e < f$

$$\begin{bmatrix} a & b & e \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & d & e \\ b+1 & e-1 & f \end{bmatrix} + \\ - \begin{bmatrix} a & d & e-1 \\ b+1 & e & f \end{bmatrix} + \cdots$$

(IV_D) $d = e-1, \, c = e, \quad a < b << e-1 < e < f-1 < f$

$$\begin{bmatrix} a & b & e \\ e-1 & e & f \end{bmatrix} - \begin{bmatrix} a & e-2 & e \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & f \end{bmatrix} + \\ + \begin{bmatrix} a & e-1 & e \\ b+1 & e & f-1 \end{bmatrix} + \cdots$$

(IV_{D1}) $d = e-1, \, c = e, \, f = e+1, \quad a < b << e-1 < e < e+1$

$$\begin{bmatrix} a & b & e \\ e-1 & e & e+1 \end{bmatrix} - \begin{bmatrix} a & e-2 & e \\ b+1 & e & e+1 \end{bmatrix} + \begin{bmatrix} a & e-1 & e \\ b+1 & e-1 & e+1 \end{bmatrix} + \\ + \begin{bmatrix} a+1 & e-2 & e \\ b & e & e+1 \end{bmatrix} + \cdots$$

$(IV_E)\quad a < b << d < e < c < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & c \\ b+1 & e & f \end{bmatrix} + \begin{bmatrix} a & d & c \\ b+1 & e-1 & f \end{bmatrix} + \\ + \begin{bmatrix} a & d & c-1 \\ b+1 & e & f \end{bmatrix} + \cdots$$

$(V)\quad d - b \leq 1,\ 2 \leq e - c,\ c + 1 < f$

$(V_A)\ b = d-1,\ d < c,\ c+3 < f,\quad a < b < d < c << e < f$

$$\begin{bmatrix} a & d-1 & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ d & c+1 & f \end{bmatrix} + \begin{bmatrix} a & d-1 & e \\ d & c+1 & f-1 \end{bmatrix} + \\ - \begin{bmatrix} a & d-1 & e-1 \\ d & c+2 & f-1 \end{bmatrix} + \cdots$$

$(V_{A1})\ b = d-1,\ d = c,\ d = f-3,\quad a < d-1 < d << d+2 < d+3$

$$\begin{bmatrix} a & d-1 & d \\ d & d+2 & d+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & d+1 \\ d & d+1 & d+3 \end{bmatrix} + \begin{bmatrix} a & d-1 & d+2 \\ d & d+1 & d+2 \end{bmatrix} + \\ - \begin{bmatrix} a+1 & d & d+1 \\ d-1 & d & d+3 \end{bmatrix} + \cdots$$

$(V_B)\ b = d-1,\ d < c,\ c+3 = f\quad a < b < d < c << c+2 < c+3$

$$\begin{bmatrix} a & d-1 & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & c+1 \\ d & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & d-1 & c+2 \\ d & c+1 & c+2 \end{bmatrix} + \\ - \begin{bmatrix} a & d & c+2 \\ d & c & c+2 \end{bmatrix} + \cdots$$

$(V_C)\ d \leq b,\ c+3 < f\quad a < d \leq b < c << e < f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & b & e-1 \\ d & c+1 & f \end{bmatrix} + \begin{bmatrix} a & b & e \\ d & c+1 & f-1 \end{bmatrix} + \\ + \begin{bmatrix} a & b & c+1 \\ d & e & f-1 \end{bmatrix} + \cdots$$

(V_{C1}) $d \leq b$, $c+3=f$, $\quad a<d\leq b<c<<e<f$

$$\begin{bmatrix} a & b & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & b & c+1 \\ d & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & b & c+2 \\ d & c+1 & c+2 \end{bmatrix} + \\ - \begin{bmatrix} a & b+1 & c+1 \\ d & c & c+3 \end{bmatrix} + \cdots$$

(VI) $\quad 2 \leq d-b$, $2 \leq e-c$

(VI_A) $c<d$, $\quad a<b<c<d<e<f$, $c+3<f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{bmatrix} + \begin{bmatrix} a & d-1 & e \\ b+1 & c+1 & f-1 \end{bmatrix} + \\ + \begin{bmatrix} a & d & e \\ b+1 & c+1 & f-2 \end{bmatrix} + \cdots$$

(VI_{A1}) $c<d$, $c+3=f$, $\quad a<b<c<c+1<c+2<c+3$

$$\begin{bmatrix} a & b & c \\ c+1 & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & c & c+1 \\ b+1 & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & c & c+2 \\ b+1 & c+1 & c+2 \end{bmatrix} + \\ + \begin{bmatrix} a+1 & c & c+1 \\ b & c+1 & c+3 \end{bmatrix} + \cdots$$

(VI_B) $d \leq c$, $c+3<f$, $\quad a<b<<d\leq c<<e<f$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+1 & f \end{bmatrix} + \begin{bmatrix} a & d-1 & e \\ b+1 & c+1 & f-1 \end{bmatrix} + \\ + \begin{bmatrix} a & d-1 & e-1 \\ b+1 & c+2 & f-1 \end{bmatrix} + \cdots$$

(VI_{B1}) $d \leq c$, $\quad a<b<<d\leq c<<c+2<c+3$

$$\begin{bmatrix} a & b & c \\ d & c+2 & c+3 \end{bmatrix} - \begin{bmatrix} a & d-1 & c+1 \\ b+1 & c+1 & c+3 \end{bmatrix} + \begin{bmatrix} a & d-1 & c+2 \\ b+1 & c+1 & c+2 \end{bmatrix} + \\ + \begin{bmatrix} a & d & c+1 \\ b+1 & c+1 & c+2 \end{bmatrix} + \cdots$$

Proof: : In the following we describe how to find the liftings according to our algorithm. The binomial relation (I_A)

$$\begin{pmatrix} a & b & c \\ d & e & c \end{pmatrix} - \begin{pmatrix} a & e & c \\ d & b & c \end{pmatrix}$$

is replaced by the difference

$$\begin{bmatrix} a & b & c \\ d & e & c \end{bmatrix} - \begin{bmatrix} a & e & c \\ d & b & c \end{bmatrix}$$

of the products of the corresponding minors. Here

$$\begin{bmatrix} a & b & c \\ d & e & c \end{bmatrix} = \begin{vmatrix} x_a & x_b & x_c \\ x_{a+1} & x_{b+1} & x_{c+1} \\ x_{a+2} & x_{b+2} & x_{c+2} \end{vmatrix} \begin{vmatrix} x_d & x_e & x_c \\ x_{d+1} & x_{e+1} & x_{c+1} \\ x_{d+2} & x_{e+2} & x_{c+2} \end{vmatrix} \qquad (1)$$

$$\begin{bmatrix} a & e & c \\ d & b & c \end{bmatrix} = \begin{vmatrix} x_a & x_e & x_c \\ x_{a+1} & x_{e+1} & x_{c+1} \\ x_{a+2} & x_{e+2} & x_{c+2} \end{vmatrix} \begin{vmatrix} x_d & x_b & x_c \\ x_{d+1} & x_{b+1} & x_{c+1} \\ x_{d+2} & x_{b+2} & x_{c+2} \end{vmatrix} \qquad (2).$$

All monomials of (1) divisible by $x_a x_d x_{e+1}$ or $x_a x_d x_{e+2}$ cancel against monomials in the support of (2). Next in the lexicographical order in (1) consider the monomials divisible by $x_a x_{d+1} x_e$, $m_1 = x_a x_{d+1} x_e x_{b+1} x_{c+2} x_{c+2} > x_a x_{d+1} x_e x_{b+2} x_{c+1} x_{c+2}$. Since m_1 does not appear in (2) it follows that $in_<((1)-(2)) = m_1$ that gives

$$\begin{bmatrix} a & d & c \\ e & b & c \end{bmatrix}$$

which is:

$$\begin{vmatrix} x_a & x_d & x_c \\ x_{a+1} & x_{d+1} & x_{c+1} \\ x_{a+2} & x_{d+2} & x_{c+2} \end{vmatrix} \begin{vmatrix} x_e & x_b & x_c \\ x_{e+1} & x_{b+1} & x_{c+1} \\ x_{e+2} & x_{b+2} & x_{c+2} \end{vmatrix} \qquad (3).$$

It is easy to verify that the remaining terms in the sum of $(1)-(2)+(3)$ vanish. Therefore (I_A) is the desired lifting. For the following we always adopt this procedure in the proof of the remaining cases. The employed monomial order is the lexicographic order and the order of the variables is the usual $x_1 > x_2 > \ldots > x_{n+2}$. Moreover, we will identify the symbol $\begin{bmatrix} a & d & c \\ e & b & f \end{bmatrix}$ with the corresponding product of minors and the difference between two symbols by the difference $(i)-(j)$ on the right of the difference of the two symbols. We will show only another lifting procedure that is not yet a complete lifting. For more details on the computation of all partial liftings see [6].

Consider the binomial relation (IV_A)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} a & d-1 & c \\ b+1 & e & f \end{bmatrix} = (4) - (5).$$

All monomials in (4) divisible by $x_a x_{b+1} x_d$ cancel against monomials in the support of (5). Consider all monomials in (4) and in (5) divisible by $x_a x_{b+1} x_{d+1}$: $x_a x_{b+1} x_{d+1} x_{c+2} x_{e+2} x_f$,

$x_a x_{b+1} x_{d+1} x_{c+2}\ x_e x_{f+2}$ in (4), $m_1 = x_a x_{b+1} x_{d+1} x_{c+1} x_{e+1} x_{f+2}$, $x_a x_{b+1} x_{d+1} x_{c+1} x_{e+2} x_{f+1}$ in (5). All of them do not vanish. Then $in_<((4)-(5)) = m_1$ that gives

$$\begin{bmatrix} a & d & e-1 \\ b+1 & c & f \end{bmatrix} = (6).$$

Now in (6) there are not monomials divisible by x_a, x_{b+1}, x_d. Consider in (4), (5), (6) the monomials divisible by x_a, x_{b+1}, x_{d+1}: $x_a x_{b+1} x_{d+1} x_{c+1} x_{e+1} x_{f+2}$ cancels against a monomial of (5), but $x_a x_{b+1} x_{d+1} x_{c+2} x_{e+2} x_f$, $x_a x_{b+1} x_{d+1} x_{c+2} x_e x_{f+2}$ in (4), $m_2 = x_a x_{b+1} x_{d+1} x_{c+1} x_{e+2} x_{f+1}$ in (5) and $x_a x_{b+1} x_{d+1} x_{c+2} x_{e+1} x_{f+1}$ in (6) do not vanish. Then $in_<((4)-(5)+(6)) = m_2$ that gives:

$$\begin{bmatrix} a & d & e \\ b+1 & c & f-1 \end{bmatrix}.$$

□

Remark 4.2. The lifting relations from (I_A) to (III_D), are just the Plücker relations of the Grassmann variety as we can verify applying the result of [1] (Lemma 7.2.3).

Corollary 4.3. *The variety $H(2,5)$ has the following relations:*

(I_A)

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 6 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 6 \\ 3 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 6 \\ 4 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 5 & 6 \\ 3 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 6 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 5 & 6 \\ 3 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 6 \\ 4 & 5 & 6 \end{bmatrix}$$

(I_B)

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(II)

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(III_A)

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 6 \\ 1 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 3 & 6 \\ 1 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$

(III_B)

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 6 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 6 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 4 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

(III$_C$)

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 3 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(III$_D$)

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(IV$_{B1}$)

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 5 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 5 \end{bmatrix}$$

(IV$_{D1}$)

$$\begin{bmatrix} 1 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$
$$- \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \end{bmatrix}$$

(V$_{A1}$)

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

(V_{C1})

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

(VI_{A1})

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Corollary 4.4. *The set* $\{[i_1 i_2 i_3]\}$ *of* $3\times 3-$*minors of Hankel matrix* 3×6, $1 \leq i_1 < i_2 < i_3 \leq 6$, *is a Sagbi basis for the* $K-$*algebra* $K[[i_1 i_2 i_3], 1 \leq i_1 < i_2 < i_3 \leq 6]$, *coordinate ring of the Hankel variety* $H(2,5)$ *of* $2-$*planes in* $\mathbb{P}^5$.

Remark 4.5. We observe that in $H(2,n)$, for $n > 6$, we can have relations of length ≥ 13 and also different complete liftings for a fixed Machado relation with different length, as showed in Section 2. Instead, in $H(2,5)$ we have only one complete lifting for each binomial relation.

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