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Projective and flat objects over rooted rings with several objects

by
Manuel Cortés-Izurdiaga, Blas Torrecillas

Dedicated to Professors Toma Albu and Constantin Năstăsescu on the occasion of their 70th birthdays

Abstract

In this paper we introduce the notion of rooted ring with several objects (rooted small preadditive category). Then, we obtain characterizations of projective and flat functors in the corresponding functor category, and use them to give new results concerning right perfect rooted rings with several objects and pure semisimple finitely presented additive categories. We conclude the paper applying the results to the module category over a rooted ring with enough idempotents.

Key Words: Functor category, ring with enough idempotents, projective, flat, rooted.
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1 Introduction

The starting point of this paper are two different situations. The first of them concerns representations of infinite quivers. Recall that a quiver Q is a pair (V, A) consisting of a set of vertices, V, and a set of arrows between them, A. Given R a ring, a representation X of the quiver by right R-modules assigns to each vertex v of Q a right R-module $X(v)$ and to each arrow $a : v \to w$ a R-morphism $X(a) : X(v) \to X(w)$. In the last thirty years, there have appeared many results concerning representations of quivers. We are interested in representations of infinite quivers, where the notion of rooted quiver plays an important role, because over this type of quivers there have been characterized some classes of representations, such as flat, projective and injective; see, for example, [4], [5], [6] and [7]. The second situation is relative to triangular matrix rings. Let R and S be unitary rings and M a (R, S)-bimodule. A well known result asserts that any right module over the triangular matrix ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is uniquely determined by a triple, (X, Y, f), where $X \in$ Mod-R, $Y \in$ Mod-S and $f : X \otimes M \to Y$ is a S-linear map, see [9, p. 17], [12, Theorems 1.5 and 1.10] and [8]. Using this description, we can study properties in Mod-T from properties in Mod-R and Mod-S. This have been done, for example, in [14], [13], [2] and [8].

In this paper we use and extend both ideas, the description of the module category over a triangular matrix ring and the study of representations of rooted quivers, to study functor categories over small preadditive categories (which are called rings with several objects in [16]; see [10], [15] and [20] too, for the theory of functor categories). More precisely, let $\mathbf{C}$ be a small preadditive category and denote by $(\mathbf{C}, \mathbf{Ab})$ the category whose objects are the additive covariant functors from $\mathbf{C}$ to the category of abelian groups, $\mathbf{Ab}$. Moreover, let R_a be the ring $\mathrm{End}_{\mathbf{C}}(a)$ and R_{ab} the (R_a, R_b)-bimodule $\mathrm{Hom}_{\mathbf{C}}(a, b)$ for each $a, b \in \mathbf{C}$. Then, extending the mentioned result for triangular matrix rings, any functor F is determined by a tuple $(M_a, s_{ab}^M)_{a,b\in\mathbf{C}}$ where $M_a \in$ Mod-R_a for each $a \in \mathbf{C}$ and s_{ab}^M is a R_b-morphism from $M_a \otimes_{R_a} R_{ab}$ to M_b making some diagrams commutative (see 2.2). Then we associate a quiver $Q(\mathbf{C})$ to the small preadditive category $\mathbf{C}$ and we study some classes objects in $(\mathbf{C}, \mathbf{Ab})$ in terms of the properties of the corresponding tuples when the quiver $Q(\mathbf{C})$ is left rooted. With these ideas, we are able to characterize projective functors (Theorem 1), right perfect small preadditive categories (Theorem 3) and pure semisimple locally finitely presented additive categories (Theorem 4).

These results are used to study the category of modules over (unitary and non-unitary) rings with enough idempotents. This is done by rewriting the previous results since, as it was essentially proved by Mitchell, [16, Theorem 7.1], there is a bijective correspondence between Morita equivalence classes of rings with enough idempotents and Morita equivalence classes of small preadditive categories (in the sense that two such categories are Morita equivalent if the corresponding functor categories are equivalent categories). Thus, the mentioned results actually characterize projective modules over rings with enough idempotents, right perfect rings with enough idempotents and right pure semisimple unitary rings (corollaries 2, 3 and 4 respectively). We note that we extend the corresponding results existing for triangular matrix rings too, see [14, Corollary 5.2], [14, Theorem 3.1] and [8, Proposition 2.1].

2 Preliminaries

Let $\mathbf{C}$ be a category. If c is an object of $\mathbf{C}$ we shall write $c \in \mathbf{C}$. Morphisms and functors will act on the right. Consequently, if $f : a \to b$ and $g : b \to c$ are morphisms in $\mathbf{C}$, then their composition will be fg; for any object c of $\mathbf{C}$, we shall denote 1_c the identity morphism of c. The category of abelian groups will be denoted by $\mathbf{Ab}$. If $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a functor between categories, then $\mathrm{Im}\,\mathbf{f}$ will denote the essential image of $\mathbf{f}$, that is, the class of all objects b of $\mathbf{B}$ for which there exists $a \in \mathbf{A}$ such that $(a)\mathbf{f} \cong b$. By a preadditive category we mean a category together with an abelian group structure on each of its hom sets such that composition is bilinear. An additive category is a preadditive category with finite products.

Fix, for the rest of the section, a small preadditive category $\mathbf{C}$. We shall denote by $(\mathbf{C}, \mathbf{Ab})$ the category whose objects are the additive covariant functors from $\mathbf{C}$ to $\mathbf{Ab}$ and whose morphisms are the natural transformations between functors. All functors between preadditive categories will be additive, even though we may neglect to say so explicitly each time. It is well known that the family of functors $\{\mathrm{Hom}_{\mathbf{C}}(a, \_) : a \in \mathbf{C}\}$ is a family of projective generators of

$(\mathbf{C}, \mathbf{Ab})$ (see [20, Corollary IV.7.5]) and, consequently, the functor

$$T_{\mathbf{C}} = \bigoplus_{a \in \mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(a, \_) \tag{2.1}$$

is a projective generator. We shall call it the *regular functor*.

For any object $c \in \mathbf{C}$, we shall denote by R_c the ring $End_{\mathbf{C}}(c)$. Composition in $\mathbf{C}$ gives to $\mathrm{Hom}_{\mathbf{C}}(a, b)$ the structure of (R_a, R_b)-bimodule for each $a, b \in \mathbf{C}$, and, for each triple $a, b, c \in \mathbf{C}$ defines a (R_a, R_c)-linear homomorphism from $\mathrm{Hom}_{\mathbf{C}}(a, b) \otimes_{R_b} \mathrm{Hom}_{\mathbf{C}}(b, c)$ to $\mathrm{Hom}_{\mathbf{C}}(a, c)$, which we shall denote by t_{abc}. In addition, we shall denote $\mathrm{Hom}_{\mathbf{C}}(a, b)$ by R_{ab}. Let $F : \mathbf{C} \to \mathbf{Ab}$ be any functor. Then, for each $a \in \mathbf{C}$, $(a)F$ is a right R_a-module with multiplication given by

$$xf = (x)(f)F \quad \forall x \in (a)F \text{ and } f \in R_a$$

and, for any other $b \in \mathbf{C}$, there is a R_b-linear map $s^F_{ab} : (a)F \otimes_{R_a} R_{ab} \to (b)F$ given by the same formula. This morphisms satisfy that, for any triple $a, b, c \in \mathbf{C}$, the following diagram is commutative:

$$\begin{array}{ccc} (a)F \otimes_{R_a} R_{ab} \otimes_{R_b} R_{bc} & \xrightarrow{s^F_{ab} \otimes 1} & (b)F \otimes_{R_b} R_{bc} \\ {\scriptstyle 1 \otimes t_{abc}} \downarrow & & \downarrow {\scriptstyle s^F_{bc}} \\ (a)F \otimes_{R_a} R_{ac} & \xrightarrow[s^F_{ac}]{} & (c)F \end{array} \tag{2.2}$$

Moreover, any functor F in $(\mathbf{C}, \mathbf{Ab})$ is determined by this data, that is, a tuple $(M_a, s^M_{ab})_{a,b \in \mathbf{C}}$ where $M_a \in \text{Mod-}R_a$ for each $a \in \mathbf{C}$ and s^M_{ab} is a R_b-morphism from $M_a \otimes_{R_a} R_{ab}$ to M_b making diagram (2.2) commutative. With this tuple, the functor F is defined as $(a)F = M_a$ for each $a \in \mathbf{C}$ and $(f)F$ is the morphism given by $(x)(f)F = (x \otimes f)s^M_{ab}$ for each $f \in R_{ab}$ and $x \in (a)F$. With this identification, each natural transformation $\tau : F \to G$ in $(\mathbf{C}, \mathbf{Ab})$ is determined by a family of morphisms $(\tau_c)_{c \in \mathbf{C}}$ such that $\tau_c \in \mathrm{Hom}_{R_c}((c)F, (c)G)$ making the diagram

$$\begin{array}{ccc} (a)F \otimes_{R_a} R_{ab} & \xrightarrow{\tau_a \otimes 1_{ab}} & (a)G \otimes_{R_a} R_{ab} \\ {\scriptstyle s^F_{ab}} \downarrow & & \downarrow {\scriptstyle s^G_{ab}} \\ (b)F & \xrightarrow[\tau_b]{} & (b)G \end{array} \tag{2.3}$$

commutative for each $a, b \in \mathbf{C}$. Moreover, it is easy to see that a subfunctor G of a functor F is determined by a family of abelian groups $\{G_c : c \in \mathbf{C}\}$ such that G_c is a right R_c-submodule of $(c)F$ that verifies $(x \otimes f)s^F_{ab} \in G_b$ for each $x \in G_a$ and $f \in R_{ab}$. We shall call G the functor determined by the family of modules $\{G_c : c \in \mathbf{C}\}$.

Remark 1. *The identification of functors with tuples is an extension of some well known theorems concerning matrix rings, see [9, p. 17], [12, Theorems 1.5 and 1.10] and [8]. More precisely, define the category* $\mathbf{A}$ *with objects tuples* $(M_a, s^M_{ab})_{a,b \in \mathbf{C}}$ *such that* $M_a \in \text{Mod-}R_a$ *for each* $a \in \mathbf{C}$ *and* $s^M_{ab} \in Hom_{R_b}(M_a \otimes_{R_a} R_{ab}, M_b)$ *making diagram (2.2) commutative, and with* *morphisms tuples $(\tau_a)_{a\in\mathbf{C}}$ such that τ_a is a R_a-linear map making diagram (2.3) commutative for each $a \in \mathbf{C}$. Then, the categories $(\mathbf{C}, \mathbf{Ab})$ and $\mathbf{A}$ are equivalent.*

Moreover note that the category $\mathbf{A}$ is the category of representations of a generalized species, (in the sense that it is defined using rings with unit instead of the classical definition that uses division rings, see [11]) with a commutativity condition, see [18, §3]

We shall use the identification of functors with tuples freely along the paper. Summarising, we shall use the following notation and conventions.

Notation 1. *Let $F \in (\mathbf{C}, \mathbf{Ab})$ and $a, b, c \in \mathbf{C}$. Then:*

- *R_a and R_{ab} will be $Hom_{\mathbf{C}}(a,a)$ and $Hom_{\mathbf{C}}(a,b)$ respectively.*
- *$t_{abc} : R_{ab} \otimes_{R_b} R_{bc} \to R_{ac}$ is the morphism defined by composition.*
- *$s^F_{ab} : (a)F \otimes_{R_a} R_{ab} \to (b)$ is the morphism given by $(x \otimes f)s^F_{ab} = (x)(f)F$ for each $x \in (a)F$ and $f \in R_{ab}$.*
- *We shall omit the ring in the tensor product when it is clear from the context.*

Now we establish the relationship between the functor category $(\mathbf{C}, \mathbf{Ab})$ and the product category $\prod_{c\in\mathbf{C}} \text{Mod-}R_c$. In order to do this, we define the following functors:

- The functor $\mathbf{p} : \prod_{c\in\mathbf{C}} \text{Mod-}R_c \to (\mathbf{C}, \mathbf{Ab})$ is defined, for each object $(M_c)_{c\in\mathbf{C}}$ of $\prod_{c\in\mathbf{C}} \text{Mod-}R_c$, as $((M_c)_{c\in\mathbf{C}})\,\mathbf{p} = \bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} \text{Hom}_{\mathbf{C}}(c, \_)$; and for each morphism $\varphi = (\varphi_c)_{c\in\mathbf{C}}$, let $(\varphi)\mathbf{p}$ be the natural transformation such that, for each $a \in \mathbf{C}$, $(a)(\varphi)\mathbf{p} = \bigoplus_{c\in\mathbf{C}} \varphi_c \otimes_{R_c} 1_{\text{Hom}(c,a)}$.
- The functor $\mathbf{q} : (\mathbf{C}, \mathbf{Ab}) \to \prod_{c\in\mathbf{C}} \text{Mod-}R_c$ is defined, for each functor F of $(\mathbf{C}, \mathbf{Ab})$, as $(F)\mathbf{q} = \big((c)F\big)_{c\in\mathbf{C}}$; and, for each natural transformation $\tau : F \to G$, let $(\tau)\mathbf{q} = ((c)\tau)_{c\in\mathbf{C}}$.

Proposition 1. *Let $\mathbf{p}$ and $\mathbf{q}$ be as above. Then $\mathbf{p}$ is a left adjoint of $\mathbf{q}$ and $\mathbf{q}$ is exact.*

Proof: We shall denote by $\mathbf{P}$ the product category $\prod_{c\in\mathbf{C}} \text{Mod-}R_c$. We define the unit $\eta = (\eta^M)_{M\in\mathbf{P}} : 1_{\mathbf{P}} \to \mathbf{pq}$ and the counit $\varepsilon = (\varepsilon^F)_{F\in(\mathbf{C},\mathbf{Ab})} : \mathbf{qp} \to 1_{(\mathbf{C},\mathbf{Ab})}$ of the adjunction. Given any object $M = (M_c)_{c\in\mathbf{C}}$ in $\mathbf{P}$, let

$$\eta^M = \left(\eta^M_c\right)_{c\in\mathbf{C}} : (M_c)_{c\in\mathbf{C}} \to \left(\bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} Hom(c,a)\right)_{a\in\mathbf{C}}$$

be the morphism such that, for each $a \in \mathbf{C}$ and $m \in M_a$, $(m)\eta^M_a = m \otimes 1_a$.

For any functor $F : \mathbf{C} \to Ab$ let $\varepsilon^F : (F)\mathbf{qp} \to F$ be the natural transformation such that, for each $a \in \mathbf{C}$, $(a)\varepsilon^F : \bigoplus_{c\in\mathbf{C}}(c)F \otimes_{R_c} Hom(c,a) \to (a)F$ is the morphism $\oplus_{c\in\mathbf{C}} s^F_{ca}$.

It is easy, but tedious, to check that η and ε are natural transformations such that $\eta^{(F)\mathbf{q}} \circ \left(\varepsilon^F\right)\mathbf{q} = 1_{(F)\mathbf{q}}$ and $\left(\eta^M\right)\mathbf{p} \circ \varepsilon^{(M)\mathbf{p}} = 1_{(M)\mathbf{p}}$ for any pair of objects $F \in (\mathbf{C}, \mathbf{Ab})$ and $M \in \mathbf{P}$. By [15, Theorem IV.2], these natural transformation determine an adjunction between $\mathbf{p}$ and $\mathbf{q}$ in which $\mathbf{p}$ is the left adjoint of $\mathbf{q}$.

The functor $\mathbf{q}$ is exact since kernels and cokernels are computed componentwise in $\mathbf{P}$ and in $(\mathbf{C}, \mathbf{Ab})$. $\square$

Remark 2. *Let $M \in \prod_{c\in\mathbf{C}} Mod\text{-}R_c$ and $a, b \in \mathbf{C}$. Then the morphism $s_{ab}^{(M)\mathbf{p}}$ is given by:*

$$((m_c \otimes f_{ca})_{c\in\mathbf{C}} \otimes f_{ab})\, s_{ab}^{(M)\mathbf{p}} = (m_c \otimes f_{ca} f_{ab})_{c\in\mathbf{C}}$$

for each $(m_c \otimes f_{ca})_{c\in\mathbf{C}} \otimes f_{ab} \in \left(\bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} Hom_{\mathbf{C}}(c, a)\right) \otimes_{R_a} Hom_{\mathbf{C}}(a, b)$.

3 Projective and flat objects over rooted rings with several objects

Recall that a quiver Q is a directed graph, i. e., a pair (V, A) consisting of a set of vertices, V, and a set of arrows between them, A. Given an arrow $a : v_1 \to v_2$ in Q, we shall write $i(a) = v_1$ and $t(a) = v_2$. A path in Q is a sequence of arrows $a_1 a_2 \cdots a_n$ with $t(a_i) = i(a_{i+1})$ for each $i \in \{1, \ldots, n-1\}$. A cycle in Q is a path $a_1 \cdots a_n$ with $t(a_n) = i(a_1)$. It is easy to associate a quiver to any small category.

Definition 1. *Let $\mathbf{C}$ be a small category. We define the quiver associated to $\mathbf{C}$, $Q(\mathbf{C})$, as follows: the set of vertices is the set of objects of $\mathbf{C}$ and there is an arrow $a \to b$ precisely when $a \neq b$ and $Hom_{\mathbf{C}}(a, b) \neq 0$.*

Let $Q = (V, E)$ be a quiver. We define a set of vertices V_α for each ordinal α as follows. If $\alpha = 0$, define

$$V_0 = \{v \in V : \text{ there exists no arrow } a \text{ of } Q \text{ with } t(a) = v\}.$$

If α is a successor ordinal, say $\alpha = \gamma + 1$, we define

$$V_\alpha = \{v \in V : \text{ there exists no arrow } a \text{ of } Q^\gamma \text{ with } t(a) = v\},$$

where $Q^\gamma = (V^\gamma, E^\gamma)$ is the subquiver of Q with $V^\gamma = V - V_\gamma$ and $E^\gamma = E - \{a \in E : i(a) \in V_\gamma\}$. Finally, if α is a limit ordinal we define $V_\alpha = \bigcup_{\gamma<\alpha} V_\gamma$ and $Q^\alpha = (V^\alpha, E^\alpha)$ the subquiver of Q given by $V^\alpha = V - \bigcup_{\gamma<\alpha} V_\gamma$ and $E^\alpha = E - \{a \in E : i(a) \in V_\alpha\}$. The quiver Q is said to be *left rooted* if there exists an ordinal α such that $V = \bigcup_{\gamma<\alpha} V_\gamma$, see [7].

Definition 2. *A small category $\mathbf{C}$ is said to be left rooted if the quiver associated to $\mathbf{C}$ is left rooted.*

The dual procedure gives the definition of right rooted category. If $\mathbf{C}$ is a left rooted small category, we shall denote by $V(\mathbf{C})_\alpha$ the corresponding subsets of vertices of the quiver $Q(\mathbf{C})$. Now we characterize projective functors. We need the following notation and a preliminary lemma.

Notation 2. *Let $\mathbf{C}$ be a small preadditive category and $F : \mathbf{C} \to \mathbf{Ab}$ a functor. If, for each object $a \in \mathbf{C}$, we denote by $S_a^F = \sum_{c\neq a} \operatorname{Im} s_{ca}^F$, then the family of submodules $\{S_c^F : c \in \mathbf{C}\}$ induces a subfunctor of F. We shall denote it by S^F.*

Lemma 1. *Let $\mathbf{C}$ be a small preadditive category such that $Q(\mathbf{C})$ has no cycles. Then, for any $M = (M_c)_{c\in\mathbf{C}} \in \prod_{c\in\mathbf{C}} Mod\text{-}R_c$, $(b)S^{(M)\mathbf{p}} = \bigoplus_{a\neq b} M_a \otimes_{R_a} R_{ab}$ $\forall b \in \mathbf{C}$. In particular, $(b)S^{(M)\mathbf{p}} \bigoplus (M_b \otimes R_b) = (b)(M)\mathbf{p}$ for each $b \in \mathbf{C}$ and, if $\overline{s}_{ab}^{(M)\mathbf{p}}$ denotes the restriction of $s_{ab}^{(M)\mathbf{p}}$ to $M_a \otimes R_a \otimes R_{ab}$ for each $a, b \in \mathbf{C}$, then $\bigoplus_{a\neq b} \overline{s}_{ab}^{(M)\mathbf{p}}$ is monic for each $b \in \mathbf{C}$.*

Proof: Fix $b \in \mathbf{C}$ and let $a \neq b$ be an object of $\mathbf{C}$ and $(m_c \otimes r_{ca})_{c\in\mathbf{C}} \otimes r_{ab} \in (a)(M)\mathbf{p} \otimes_{R_a} R_{ab}$. By Remark 2, $((m_c \otimes r_{ca})_{c\in\mathbf{C}} \otimes r_{ab})\, s_{ab}^{(M)\mathbf{p}} = (m_c \otimes (r_{ca} \otimes r_{ab})t_{cab})_{c\in\mathbf{C}}$, and the coordinate in position b of this element is zero since, if $c = b$, then $r_{ab} = 0$ or $r_{ba} = 0$ (otherwise there would be a cycle $a \to b \to a$ and this is not true by hyphotesis). This proves the inclusion $\sum_{a\neq b} \operatorname{Im} s_{ab}^{(M)\mathbf{p}} \leq \bigoplus_{a\neq b} M_a \otimes \operatorname{Hom}_{\mathbf{C}}(a, b)$. The other inclusion follows from the fact that, for every $(m_a \otimes r_{ab})_{a\neq b} \in \bigoplus_{a\neq b} M_a \otimes_{R_a} R_{ab}$, we have the identity

$$(m_a \otimes r_{ab})_{a\neq b} = \sum_{a\neq b} ((m_a \otimes 1_a)\iota_a \otimes r_{ab})\, s_{ab}^{(M)\mathbf{p}}$$

where ι_a is the inclusion of $M_a \otimes_{R_a} R_a$ in $\bigoplus_{c\in\mathbf{C}} M_c \otimes_{R_c} R_{ca}$.

Now we prove that $\bigoplus_{a\neq b} \overline{s}_{ab}^{(M)\mathbf{p}}$ is monic. As a consequence of Remark 2 we have, for each pair of objects of $\mathcal{C}$, $a \neq b$, that $\overline{s}_{ab}^{(M)\mathbf{p}} = (1_{M_a} \otimes t_{aab})\iota_{ab}$, where ι_{ab} is the inclusion of $M_a \otimes_{R_a} R_{ab}$ in $(b)(M)\mathbf{p}$. Then, $\bigoplus_{a\neq b} \overline{s}_{ab}^{(M)\mathbf{p}}$ can be viewed as the direct sum of the homomorphisms

$$1_{M_a} \otimes t_{aab} : M_a \otimes_{R_a} R_a \otimes_{R_a} R_{ab} \to M_a \otimes_{R_a} R_{ab}$$

with $a \neq b$. Consequently, $\bigoplus_{a\neq b} \overline{s}_{ab}^{(M)\mathbf{p}}$ is a monomorphism, since t_{aab} is just the canonical isomorphism $R_a \otimes_{R_a} R_{ab} \cong R_{ab}$ for each $a, b \in \mathbf{C}$. □

Now we obtain the characterization of projective functors over left rooted small preadditive categories:

Theorem 1. *Let $\mathbf{C}$ be a left rooted small preadditive category and $P \in (\mathbf{C}, \mathbf{Ab})$. The following assertions are equivalent:*

1. *P is projective.*
2. *$P \cong \left(\left(\frac{(c)P}{(c)S^P}\right)_{c\in\mathbf{C}}\right)\mathbf{p}$ and $\frac{(c)P}{(c)S^P}$ is projective in Mod-R_c for each $c \in \mathbf{C}$.*
3. *There exists a projective object $(P_c)_{c\in\mathbf{C}}$ of $\prod_{c\in\mathbf{C}} Mod\text{-}R_c$ such that $P = \big((P_c)_{c\in\mathbf{C}}\big)\mathbf{p}$.*
4. *They are verified:*
   - *(a) $\frac{(b)P}{(b)S^P}$ is projective for each $b \in \mathbf{C}$.*
   - *(b) There exists a decomposition $(b)P = (b)S^P \oplus K_b$ for each $b \in \mathbf{C}$ such that, if $\overline{s}_{ab}^P$ denotes the restriction of the morphism s_{ab}^P to $K_a \otimes R_{ab}$ for each $a, b \in \mathbf{C}$, then $\bigoplus_{a\neq b} \overline{s}_{ab}^P$ is monic for each $b \in \mathbf{C}$.*

Proof: (1) $\Rightarrow$ (2). First of all we prove that $\frac{(c)P}{(c)S^P}$ is projective for each $c \in \mathbf{C}$. For each $c \in \mathbf{C}$ let $f_c : M_c \to \frac{(c)P}{(c)S^P}$ be an epimorphism in Mod-R_c. Let M be the functor such that $(c)M = M_c$ for each object $c \in \mathbf{C}$, and for each morphism in $\mathbf{C}$, $f : a \to b$, $(f)M = 0$ if $a \neq b$ and $(f)M$ is right multiplication by f if $a = b$. It is easy to see that, since $Q(\mathbf{C})$ has no cycles, M is a functor and f_c defines a natural transformation $f : M \to \frac{P}{S^P}$ which is an epimorphism in $(\mathbf{C}, \mathbf{Ab})$ (this is due to, for each morphism $h : a \to b$ in $\mathbf{C}$, $(h)\frac{P}{S^P} = 0$). Applying that P is projective in $(\mathbf{C}, \mathbf{Ab})$, there exists a natural transformation $g : P \to M$ such $gf = \pi$, where π is the canonical projection from P to $\frac{P}{S^P}$. Now, using that g is natural, it is easily seen that $\left((c)S^P\right) g_c = 0$ for each $c \in \mathbf{C}$ and, by the factor theorem, there exists $h_c : \frac{(c)P}{(c)S^P} \to M_c$ with $\pi_c h_c = g_c$ for each $c \in \mathbf{C}$. This morphism verifies that $h_c f_c$ is the identity for each $c \in \mathbf{C}$. That is, f_c is split and $\frac{(c)P}{(c)S^P}$ is projective in Mod-R_c for each $c \in \mathbf{C}$.

Now using that P is a direct summand of a direct sum of copies of the regular functor $T_{\mathbf{C}}$, that $T_{\mathbf{C}}$ belongs to the image of $\mathbf{p}$ and that $\mathbf{p}$ commutes with direct sums (since it commutes with direct limits as it is a left adjoint), we can find a projective object $(N_c)_{c\in\mathbf{C}}$ of $\prod_{c\in\mathbf{C}} \text{Mod-}R_c$ such that P is a direct summand of $((N_c)_{c\in\mathbf{C}})\,\mathbf{p}$. Write $P \oplus Q = ((N_c)_{c\in\mathbf{C}})\,\mathbf{p}$ for some projective functor Q and denote $M = ((N_c)_{c\in\mathbf{C}})\,\mathbf{p}$. By Lemma 1, it is verified that $(b)S^M = \bigoplus_{a\neq b} N_a \otimes R_{ab}$, that $(b)S^M \oplus (N_b \otimes R_b) = (b)M$ and, if $\overline{s}^M_{ab}$ denotes the restriction of s^M_{ab} to $N_a \otimes_{R_a} R_a \otimes_{R_a} R_{ab}$ for each pair of objects $a \neq b$, then $\bigoplus_{a\neq b} \overline{s}^M_{ab}$ is monic for each $b \in \mathbf{C}$. Moreover, since s^M_{ab} is the direct sum of s^P_{ab} and s^Q_{ab} for each pair $a, b \in \mathbf{C}$, it is verified $(b)S^M = (b)S^P \oplus (b)S^Q$, $\forall b \in \mathbf{C}$.

We are going to find a subfunctor P' of M such that $P \cong P'$ and P' belongs to $\operatorname{Im}\mathbf{p}$. By the first part of this proof there exists, for each $b \in \mathbf{C}$, submodules $K_b \leq (b)P$ and $L_b \leq (b)Q$ such that $(b)S^P \oplus K_b = (b)P$ and $(b)S^Q \oplus L_b = (b)Q$. Now, since $(b)S^P \oplus (b)S^Q \oplus (N_b \otimes R_b) = (b)M$ and $(b)S^P \oplus (b)S^Q \oplus K_b \oplus L_b = (b)M$ for each $b \in \mathbf{C}$, $K_b \oplus L_b \cong (N_b \otimes R_b)$, and there exists a decomposition $N_b = P'_b \oplus Q'_b$ with $K_b \cong P'_b$ and $L_b \cong Q'_b$. Finally, consider the subfunctor P' of M given by

$$(b)P' = (P'_b \otimes_{R_b} R_b) \bigoplus \left(\bigoplus_{a\in\mathbf{C}} P'_a \otimes_{R_a} R_{ab} \right)$$

Then it is easy to see that $P' \cong P$. Moreover, note that $P' \cong ((P'_c)_{c\in\mathbf{C}})\mathbf{p}$ and since $P'_b \cong K_b \cong \frac{(b)P}{(b)S^P}$ for each $b \in \mathbf{C}$, we get that $P' \cong \left(\left(\frac{(b)P}{(b)S^P} \right)_{b\in\mathbf{C}} \right) \mathbf{p}$ and the proof is finished.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are trivial.

(3) $\Rightarrow$ (4). By Lemma 1.

(4) $\Rightarrow$ (3). Consider the natural transformation $\overline{s} : ((K_b)_{b\in\mathbf{C}})\mathbf{p} \to P$ given by $(b)\overline{s} = \bigoplus_{a\in\mathbf{C}} \overline{s}^P_{ab}$ for each $b \in \mathbf{C}$. It is easy to see that $\overline{s}$ is actually a natural transformation. We prove that it is an equivalence.

Since $\operatorname{Im} \bigoplus_{a\neq b} \overline{s}^P_{ab} \leq (b)S^P$, the morphism $(b)\overline{s}$ is the direct sum of the morphisms $\bigoplus_{a\neq b} \overline{s}^P_{ab} : \bigoplus_{a\neq b} K_a \otimes R_{ab} \to (b)S^P$ and $\overline{s}^P_{bb} : K_b \otimes R_b \to K_b$. For each $b \in \mathbf{C}$, note that $\overline{s}^P_{bb}$ is just the canonical isomorphism from $K_b \otimes R_b$ to K_b and, in order to see that $(b)\overline{s}$ is an isomorphism we just have to prove that $\oplus_{a\neq b}\overline{s}^P_{ab}$ is. Since this map is, by hyphotesis, monic, we have to see that is is epic.

Let $b \in \mathbf{C}$ and α be an ordinal number such that $b \in V(\mathbf{C})_\alpha$. We prove that $\operatorname{Im} \bigoplus_{a \neq b} \overline{s}^P_{ab} = (b)S^P$ by transfinite induction on α. Case $\alpha = 0$ is trivial. Suppose that α is greater than 0 and that the result is true for each object in $V(\mathbf{C})_\gamma$ for each $\gamma < \alpha$. Let $a \neq b$ with $R_{ab} \neq 0$, $r_{ab} \in R_{ab}$ and $m \in (a)P$. Write $m = x + y$ with $x \in (a)S^P$ and $y \in K_a$ and we have to see that $(x \otimes r_{ab})s^P_{ab} \in \operatorname{Im} \oplus_{a \neq b} \overline{s}^P_{ab}$. Since $a \in V(\mathbf{C})_\gamma$ for some $\gamma < \alpha$, we can use induction hyphotesis to write $x = \sum_{c \neq a} \sum_{k=1}^{n_c} (p^k_c \otimes r^k_{ca}) \overline{s}^P_{ca}$ for elements $p^k_c \otimes r^k_{ca} \in K_c \otimes_{R_c} R_{ca}$ for each $k \in \{1, \dots, n_c\}$ and $c \neq a$. Then

$$\begin{aligned}(x \otimes r_{ab})s^P_{ab} &= \textstyle\sum_{c \neq a} \sum_{k=1}^{n_c} (p^k_c \otimes r^k_{ca} \otimes r_{ab})(\overline{s}^P_{ca} \otimes 1_{ab}) s^P_{ab} \\ &= \textstyle\sum_{c \neq a} \sum_{k=1}^{n_c} (p^k_c \otimes r^k_{ca} \otimes r_{ab})(1 \otimes t_{cab}) \overline{s}^P_{cb}\end{aligned}$$

which concludes the proof. ∎

Remark 3. *If $\mathbf{C}$ is a small preadditive category such that $Q(\mathbf{C})$ has no cycles, then any projective functor $P \in (\mathbf{C}, \mathbf{Ab})$ satisfies that $\frac{(b)P}{(b)S^P}$ is projective. This is not true if $Q(\mathbf{C})$ has cycles, as it is proved in the following example.*

Example 1. *Let R be a unitary ring and M a non-zero right R-module such that the trace ideal of M in R is not a direct summand (recall that the trace ideal of M in R is $t_M(R) = \sum_{f \in Hom_R(M,R)} \operatorname{Im} f$). Consider the small preadditive category $\mathbf{C}$ whose set of objects is $\{a, b\}$, whose set of morphisms are $End_{\mathbf{C}}(a) = R$, $End_{\mathbf{C}}(B) = End_R(M)$, $Hom_{\mathbf{C}}(a, b) = Hom_R(M, R)$ and $Hom_{\mathbf{C}}(b, a) = M$ and composition is defined in an obvious way. Let T be the regular functor. Then $(a)T = R \oplus M$ and $(a)S^T = \operatorname{Im} s^T_{21} = t_M(R) \oplus M$. Since, by hyphotesis, $t_M(R)$ is not a direct summand, $\frac{(a)T}{(a)S^T}$ cannot be projective. Then T is a projective functor that does not satisfy the previous remark. Note, in addition, that the quiver associated to $\mathbf{C}$ is*

$$1 \rightleftarrows 2$$

and that $(\mathbf{C}, \mathbf{Ab})$ is equivalent to the module category associated to the matrix rings of the derived Morita context of M.

Remark 4. *In the proof of (2) ⇒ (3) of the preceding theorem it is actually proved the following more general result: Let $P, Q \in (\mathbf{C}, \mathbf{Ab})$ be such that $P \oplus Q \in \operatorname{Im} \mathbf{p}$ and $(b)S^P$ and $(b)S^Q$ are direct summands of $(b)P$ and $(b)Q$ for each $b \in \mathbf{C}$. Then $P, Q \in \operatorname{Im} \mathbf{p}$.*

Now we characterize flat functors that belong to $\operatorname{Im} \mathbf{p}$. If $\mathbf{C}$ is a small preadditive category, a functor $F : \mathbf{C} \to \mathbf{Ab}$ is flat each short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow F \longrightarrow 0$$

is pure. By [19, Theorem 3], this condition is equivalent to the existence of a family of projective functors such that F is its direct limit.

Proposition 2. *Let $\mathbf{C}$ be a small preadditive category such that $Q(\mathbf{C})$ has no cycles and let $F \in (\mathbf{C}, \mathbf{Ab})$ be a flat. Then $\frac{(c)F}{(c)S^F}$ is a flat R_c-module for each $c \in \mathbf{C}$.*

Proof: First of all note that for each natural transformation $\tau : M \to N$ in $(\mathbf{C}, \mathbf{Ab})$, there are natural transformations $k : S^M \to S^N$ and $p : \frac{M}{S^M} \to \frac{N}{S^N}$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^M & \xrightarrow{u_M} & M & \xrightarrow{v_M} & \frac{M}{S^M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & S^N & \xrightarrow[u_N]{} & N & \xrightarrow[v_N]{} & \frac{N}{S^N} & \longrightarrow & 0 \end{array}$$

commutative with u_M and u_N inclusions, and v_M and v_N projections. Moreover, k and p are uniquely determined by this property.

Let $(P^i, \iota^{ij})_{i \leq j \in I}$ be a direct system of projective functors whose direct limit is F. Using the preceding fact there exists, for each $i \leq j$, a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^i & \longrightarrow & P^i & \longrightarrow & \frac{P}{S^i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k^{ij}} & & \downarrow{\scriptstyle \iota^{ij}} & & \downarrow{\scriptstyle p^{ij}} & & \\ 0 & \longrightarrow & S^j & \longrightarrow & P^j & \longrightarrow & \frac{P^j}{S^j} & \longrightarrow & 0 \end{array}$$

where $S^i = S^{P^i}$ for each $i \in I$. Using the uniqueness of k^{ij} and p^{ij} it is easy to see that these diagrams define direct systems of short exact sequences whose direct limit is precisely the short exact sequence

$$0 \longrightarrow S^F \longrightarrow F \longrightarrow \frac{F}{S^F} \longrightarrow 0$$

Using that P^i is projective for each $i \in I$, it is easy to see that, actually, $\dfrac{(c)F}{(c)S^F} = \varinjlim_{i \in I} \dfrac{(c)P^i}{(c)S^i}$ for each $c \in \mathbf{C}$. Now since, by Theorem 1, $\frac{(c)P^i}{(c)S^i}$ is projective in Mod-R_c, we conclude that $\frac{(c)F}{(c)S^F}$ is a flat R_c-module for each $c \in \mathbf{C}$. □

Now the characterization of flat functors belonging to $\operatorname{Im} \mathbf{p}$ is an easy consequence of Theorem 1.

Theorem 2. *Let $\mathbf{C}$ be a left rooted small preadditive category and $F \in (\mathbf{C}, \mathbf{Ab})$. The following assertions are equivalent:*

1. *F is flat and belongs to* $\operatorname{Im} \mathbf{p}$.
2. *$F \cong \left(\left(\frac{(c)F}{(c)S^F} \right)_{c \in \mathbf{C}} \right) \mathbf{p}$ and $\frac{(c)F}{(c)S^F}$ is flat in Mod-R_c for each $c \in \mathbf{C}$.*
3. *There exists a flat object $(F_c)_{c \in \mathbf{C}}$ of $\prod_{c \in \mathbf{C}}$ Mod-R_c such that $F = \big((F_c)_{c \in \mathbf{C}} \big) \mathbf{p}$.*
4. *They are verified:*

(a) $\frac{(b)F}{(b)S^F}$ *is flat for each* $b \in \mathbf{C}$.

(b) *There exists a decomposition* $(b)F = (b)S^F \oplus K_b$ *for each* $b \in \mathbf{C}$ *such that, if* $\overline{s}^F_{ab}$ *denotes the restriction of the morphism* s^F_{ab} *to* $K_a \otimes R_{ab}$ *for each* $a, b \in \mathbf{C}$*, then* $\bigoplus_{a \neq b} \overline{s}^F_{ab}$ *is monic for each* $b \in \mathbf{C}$.

Proof: (1) $\Rightarrow$ (2). If $F \in \operatorname{Im} \mathbf{p}$, say $F \cong ((N_c)_{c \in \mathbf{c}})\mathbf{p}$, then $\frac{(c)F}{(c)S^F} \cong N_c$ for each $c \in \mathbf{C}$ by Lemma 1. Then $F \cong \left(\left(\frac{(c)F}{(c)S^F} \right)_{c \in \mathbf{C}} \right) \mathbf{p}$. Moreover $\frac{(b)F}{(b)S^F}$ is flat for each $b \in \mathbf{C}$ by the previous proposition.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Since $\mathbf{p}$ is a left adjoint it preserves projective objects and direct limits. Consequently it preserves flat objects.

(1) $\Rightarrow$ (4). (a) Follows from Proposition 2. (b) is a consequence of Lemma 1.

(4) $\Rightarrow$ (3). The same proof of (4) $\Rightarrow$ (3) of Theorem 1. ∎

4 Applications to functor categories

In this section we apply the results of the previous one in order to characterize right perfect functor categories and pure semisimple additive categories. Recall that a small preadditive category $\mathbf{C}$ is called right perfect (see [17]) if each flat functor of $(\mathbf{C}, \mathbf{Ab})$ is projective. If $\mathbf{C}$ is left rooted and α is an ordinal number, we shall denote by $\mathbf{q}_\alpha$ the subfunctor of $\mathbf{q}$ such that, for each functor $F \in (\mathbf{C}, \mathbf{Ab})$, $(F)\mathbf{q}_\alpha$ is the object $(X_c)_{c \in \mathbf{C}}$ of $\prod_{c \in \mathbf{C}} \text{Mod-}R_c$ with $X_c = (c)F$ if $c \in V(\mathbf{C})_\alpha$ and $X_c = 0$ otherwise. We shall use the following technical result.

Lemma 2. *Let* $\mathbf{C}$ *be a left rooted small preadditive category and* $F \in (\mathbf{C}, \mathbf{Ab})$ *a flat functor such that* $(b)F = 0$ *for each* $b \in \bigcup_{\gamma < \alpha} V(\mathbf{C})_\gamma$ *for some ordinal* α*. Then there exists a direct system* $(P^i, \iota^{ij})_{i \leq j}$ *consisting of projective functors satisfying* $(b)P^i = 0$ *for each* $b \in \bigcup_{\gamma < \alpha} V(\mathbf{C})_\gamma$ *and whose limit is* F.

Proof: Straightforward by transfinite induction on α. ∎

Theorem 3. *Let* $\mathbf{C}$ *be a left rooted small preadditive category. Then the following assertions are equivalent:*

1. $\mathbf{C}$ *is right perfect.*

2. *Each flat functor belonging to* $\operatorname{Im} \mathbf{p}$ *is projective.*

3. R_c *is a right perfect ring for each* $c \in \mathbf{C}$.

4. *For each finitely generated projective functor* $P \in (\mathbf{C}, \mathbf{Ab})$, $End_{(\mathbf{C},\mathbf{Ab})}(P)$ *is a right perfect ring.*

Proof: (1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). Fix $c \in \mathbf{C}$ and F_c a flat module in Mod-R_c. Since $F = (F_c)\mathbf{p}$ is flat, it is projective by hyphotesis. Now, by Theorem 1, $\frac{(c)F}{(c)S^F}$ is projective; but, since $(c)S^F = 0$, $\frac{(c)F}{(c)S^F} = F_c$ is actually projective. As F_c was arbitrary, we conclude that R_c is right perfect.

(3) $\Rightarrow$ (1). Let F be a flat functor. We are going to construct, for each ordinal γ, a short exact sequence

$$0 \longrightarrow \bigoplus_{\delta<\gamma} P_\delta \xrightarrow{s_\gamma} F \xrightarrow{r_\gamma} F_\gamma \longrightarrow 0$$

with P_δ a projective submodule of F for each $\delta < \gamma$, s_γ the inclusion, F_γ flat and $(b)F_\gamma = 0$ for each $b \in \bigcup_{\delta<\gamma} V(\mathbf{C})_\delta$. From this construction and using that $\mathbf{C}$ is left rooted it follows that $F = \bigoplus_{\delta<\gamma} P_\delta$ for some ordinal number γ and, consequently, it is projective.

We shall make the construction using transfinite recursion on γ. If $\gamma = 0$, simply take $P_0 = 0$ and $F_0 = F$.

Assume that we have made the construction for each $\delta < \gamma$ and that γ is successor, say $\gamma = \mu + 1$. Using the previous lemma, take $(P^i, \iota^{ij})_{i \leq j}$ a direct system of projective functors whose limit is F_μ and such that $(b)P^i = 0$ for each $b \in \bigcup_{\delta<\mu} V(\mathbf{C})_\delta$. Denote by $S^i = S^{P^i}$ and $s^i_{ab} = s^{P^i}_{ab}$ for each $a, b \in \mathbf{C}$ and $i \in I$. Since, by Theorem 1, $(b)P^i$ is projective for each $b \in V(\mathbf{C})_\mu$ and $i \in I$ (as $(b)S^i = 0$), and $\bigoplus_{a \in V(\mathbf{C})_\mu} s^i_{ab}$ is monic for each $b \in \mathbf{C}$, we conclude that $(b)F$ is flat and $\bigoplus\limits_{a \in V(\mathbf{C})_\mu} s^F_{ab}$ is monic for each $b \in \mathbf{C}$ (note that $(b)F = \varinjlim\limits_{i \in I} (b)P^i$ and $\bigoplus\limits_{a \in V(\mathbf{C})_\mu} s^F_{ab} = \varinjlim\limits_{i \in I} \bigoplus\limits_{a \in V(\mathbf{C})_\mu} s^i_{ab}$). Thus, as in the proof of Theorem 1, $\bigoplus\limits_{a \in V(\mathbf{C})_\mu} s^F_{ab}$ and $\bigoplus\limits_{a \in V(\mathbf{C})_\mu} s^i_{ab}$ induce natural transformations $\overline{s}_{\mu+1} : (F_\mu)\mathbf{q}_\mu\mathbf{p} \to F_\mu$ and $\overline{s}^i : (P^i)\mathbf{q}_\mu\mathbf{p} \to P^i$ respectively, which actually are monic by the previous comments. Moreover, we have a direct system of splitting short exact sequences

$$0 \longrightarrow (P^i)\mathbf{q}_\mu\mathbf{p} \xrightarrow{s^i} P^i \longrightarrow \overline{P}^i \longrightarrow 0$$

whose direct limit is precisely

$$0 \longrightarrow (F)\mathbf{q}_\mu\mathbf{p} \xrightarrow{\overline{s}_{\mu+1}} F_\mu \xrightarrow{\overline{r}_{\mu+1}} F_{\mu+1} \longrightarrow 0$$

In particular, $F_{\mu+1}$ is a flat functor with $(b)F_{\mu+1} = 0$ for each $b \in \bigcup_{\delta<\mu+1} V(\mathbf{C})_\delta$.

Now use that R_b is right perfect for each $b \in \mathbf{C}$ to get that $(b)F$ is projective for each $b \in V(\mathbf{C})_\mu$ and, consequently, that $(F)\mathbf{q}_\mu\mathbf{p}$ is projective in $(\mathbf{C}, \mathbf{Ab})$. Then, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\delta<\mu} P_\delta & \xrightarrow{s_\mu} & F & \xrightarrow{r_\mu} & F_\mu & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle \overline{s}_{\mu+1}} & & \\ & & & & & & (F)\mathbf{q}_\mu\mathbf{p} & & \end{array}$$

can be completed with $t_{\mu+1} : (F)\mathbf{q}_\mu\mathbf{p} \to F$ satisfying $t_{\mu+1} r_\mu = \overline{s}_{\mu+1}$. Note that, since $\overline{s}_{\mu+1}$ is monic, $t_{\mu+1}$ is monic too. Now, case $\gamma = \mu + 1$ finishes setting $P_\mu = \operatorname{Im} t_{\mu+1}$, $s_{\mu+1} = s_\mu \oplus t_{\mu+1}$

and $r_{\mu+1} = r_\mu \overline{r}_{\mu+1}$, because we get the short exact sequence

$$0 \longrightarrow \bigoplus_{\delta<\mu+1} P_\delta \xrightarrow{s_{\mu+1}} F \xrightarrow{r_{\mu+1}} F_{\mu+1} \longrightarrow 0$$

In order to conclude the construction, suppose that γ is a limit ordinal. Then, we have a direct system of exact sequences,

$$0 \longrightarrow \bigoplus_{\delta<\mu} P_\delta \xrightarrow{s_\mu} F \xrightarrow{r_\mu} F_\mu \longrightarrow 0,$$

which are pure since F_μ is flat for each $\mu < \delta$. It is very easy to see that its direct limit is the short exact sequence

$$0 \longrightarrow \bigoplus_{\delta<\gamma} P_\delta \xrightarrow{s_\gamma} F \xrightarrow{r_\gamma} \varinjlim_{\delta<\gamma} F_\delta \longrightarrow 0$$

in which s_γ is the inclusion and r_γ is the direct limit of $\{r_\delta : \delta < \gamma\}$. By [19, Theorem 3], $F_\gamma = \varinjlim_{\delta<\gamma} F_\delta$ is flat and trivially has the property that $(b)F_\gamma = 0$ for each $b \in \bigcup_{\delta<\gamma} V(\mathbf{C})_\delta$. This concludes the proof.

(3) $\Rightarrow$ (4). By Yoneda's lemma, for each $c \in \mathbf{C}$, $\mathrm{End}_{(\mathbf{C},\mathbf{Ab})}(\mathrm{Hom}_{\mathbf{C}}(c, \_))$ is isomorphic to R_c and, consequently, it is a right perfect ring. Now, if P is a finitely generated projective functor in $(\mathbf{C}, \mathbf{Ab})$, then P is a direct summand of a finite direct sum of representable functors. From this it follows that $\mathrm{End}_{(\mathbf{C},\mathbf{Ab})}(P)$ is right perfect. □

This result adds new equivalent conditions to the general characterization of right perfect functor categories established in [17, Theorem 2.4], in the special case when the category $\mathbf{C}$ is left rooted. With respect to (4) note that, by [17, Theorem 2.4], each right perfect functor category $(\mathbf{C}, \mathbf{Ab})$ satisfy that R_c is a right perfect ring for each $c \in \mathbf{C}$. We have proved that the converse of this result is true for left rooted small preadditive categories.

When the small preadditive category $\mathbf{C}$ is actually additive, we can improve Theorem 3. First of all we need the notion of strong generating family of objects, see [3].

Definition 3. *Let $\mathbf{C}$ be a small additive category. A subset $\mathbf{N}$ of $\mathbf{C}$ is said to be a strong generating family in case each object in $\mathbf{C}$ is isomorphic to a direct summand of a finite direct sum of objects of $\mathbf{N}$.*

The following result establish the relationship between the categories of functors defined over a small additive category and a strong generating family on it.

Proposition 3. *Let $\mathbf{C}$ be a small additive category and $\mathbf{N}$ a strong generating family. Then the categories $(\mathbf{C}, \mathbf{Ab})$ and $(\mathbf{N}, \mathbf{Ab})$ are equivalent.*

Proof: Denote by $S(\mathbf{N})$ the full subcategory of $\mathbf{C}$ whose class of objects are all finite direct sums of objects in $\mathbf{N}$. Since each object in $\mathbf{C}$ is a retract of an object in $S(\mathbf{N})$, then $(\mathbf{C}, \mathbf{Ab})$ and $(S(\mathbf{N}), \mathbf{Ab})$ are equivalent by see [16, p. 12]. Consequently, we only have to prove that

$(\mathbf{N}, \mathbf{Ab})$ and $(S(\mathbf{N}), \mathbf{Ab})$ are equivalent. But this is easily deduced from [16, Lemma 1.1], since any functor $F : \mathbf{N} \to \mathbf{Ab}$ can be extended to a functor $\overline{F} : S(\mathbf{N}) \to \mathbf{Ab}$; and if $\tau : F \to G$ is a natural transformation in $(\mathbf{N}, \mathbf{Ab})$ and $\overline{F}$ and $\overline{G}$ extend F and G to $S(\mathbf{N})$ respectively, there exists a unique extension $\overline{\tau} : \overline{F} \to \overline{G}$ of τ. □

We shall use the following weak version of rooted category:

Definition 4. *Let* $\mathbf{C}$ *be a small additive category. We shall say that* $\mathbf{C}$ *is weak left rooted if there exists a strong generating family* $\mathbf{N}$ *of* $\mathbf{C}$ *such that the full subcategory of* $\mathbf{C}$ *whose class of objects is* $\mathbf{N}$ *is left rooted.*

Corollary 1. *Let* $\mathbf{C}$ *be a weak left rooted small additive category. Then the following assertions are equivalent:*

1. $\mathbf{C}$ *is right perfect.*
2. *There exists a strong generating set* $\mathbf{N}$ *of* $\mathbf{C}$ *which is left rooted and such that* $End_{\mathbf{C}}(n)$ *is right perfect for each* $n \in \mathbf{N}$*.*
3. *For each* $c \in \mathbf{C}$*,* $End_{\mathbf{C}}(c)$ *is a right perfect ring.*
4. *For each finitely generated projective functor* $P \in (\mathbf{C}, \mathbf{Ab})$*,* $End_{(\mathbf{C},\mathbf{Ab})}(P)$ *is a right perfect ring.*

Proof: (1) ⇔ (2). $\mathbf{C}$ is right perfect if and only if $\mathbf{N}$ is right perfect by Proposition 3. But, since $\mathbf{N}$ is left rooted, $\mathbf{N}$ is right perfect if and only if $\mathrm{End}_{\mathbf{C}}(n)$ is right perfect for each $n \in \mathbf{N}$ by Theorem 3.

(2) ⇒ (3). Any c is a direct summand of a direct sum of objects belonging to $\mathbf{N}$. From this follows that $\mathrm{End}_{\mathbf{C}}(c)$ is right perfect.

(3) ⇒ (4) ⇒ (2). By Yoneda's lemma. □

Let $\mathbf{A}$ be a locally finitely presented additive category. We shall denote by $\mathrm{fp}(\mathbf{A})$ the full subcategory of $\mathbf{A}$ whose class of objects are the finitely presented objects of $\mathbf{A}$.

Definition 5. *Let* $\mathbf{A}$ *be a locally finitely presented additive category. We shall say that* $\mathbf{A}$ *is left fp-rooted (resp. weak left fp-rooted) if* $fp(\mathbf{A})$ *is a left rooted small category (resp. weak left rooted small category).*

At the end of the paper we shall give an example of a weak fp-rooted category. The following result characterizes pure semisimple weak left fp-rooted locally finitely presented additive categories (compare with [17, Theorem 3.1]).

Theorem 4. *Let* $\mathbf{A}$ *be a weak left fp-rooted locally finitely presented additive category. The following assertions are equivalent:*

1. $\mathbf{A}$ *is pure semisimple.*

2. *There exists a strong generating set* $\mathbf{N}$ *of* $fp(\mathbf{A})$ *which is left rooted and such that* $End_{\mathbf{C}}(N)$ *is right perfect for each* $N \in \mathbf{N}$.

3. *For each finitely presented object* N, $End_{\mathbf{A}}(N)$ *is a right perfect ring.*

4. *For each finitely generated projective functor* $P \in (fp(\mathbf{A}), \mathbf{Ab})$, $End_{(fp(\mathbf{A}),\mathbf{Ab})}(P)$ *is a right perfect ring.*

Proof: It is well known (see, for example [3, Proposition 1.5]) that $\mathbf{A}$ is pure semisimple if and only if $(\mathrm{fp}(\mathbf{A}), \mathbf{Ab})$ is right perfect. Then the result follows from Theorem 1. □

5 Applications to rings and modules

In this section we apply the results of the paper to the category of unitary modules over a ring with enough idempotents. Moreover, we show how some results concerning triangular matrix rings are particular cases of our results. Recall that a ring (not necessarily unitary) R is said to have enough idempotents if it contains a set $\{e_i : i \in I\}$ of pairwise orthogonal idempotent elements of such that $R = \bigoplus_{i\in I} Re_i = \bigoplus_{i\in I} e_iR$. In this case, $\{e_i : i \in I\}$ is called a complete family of idempotents in R. When talking about modules, we shall refer to right R-modules and we shall denote by Mod-R the category whose class of objects are the unitary right R-modules, i. e. modules M verifying $MR = M$.

The applications of this section are direct consequences of the fact that there is a bijective correspondence between Morita equivalence classes of rings with enough idempotents and Morita equivalence classes of small preadditive categories (in the sense that two such categories are Morita equivalent if the corresponding functor categories are equivalent categories), see [16, Theorem 7.1] and the remark after this theorem. Under this correspondence, any ring R with a complete family of idempotents $\{e_i : i \in I\}$ corresponds with the small category whose set of objects is I and, for each $i, j \in I$, $\mathrm{Hom}_{\mathbf{C}}(i, j) = \mathrm{Hom}_R(Re_i, Re_j)$. Then, using Remark 1, any unitary right R-module is determined by a tuple $(M_i, s^M_{ij})_{i,j\in I}$ with $M_i \in e_iRe_i$-Mod and $s^M_{ij} : M_i \otimes e_iRe_j \to M_j$ is the morphism in e_jRe_j-Mod given by multiplication for each $i, j \in I$. Then, we can describe properties in Mod-R in terms of properties of modules over the unitary rings e_iRe_i $(i \in I)$. Moreover, as it was mentioned in Remark 1, this fact contains, as a particular case, the well known result that describes the module category over certain matrix rings, see, for example, [9, p. 17], [12, Theorems 1.5 and 1.10] and [8].

Of course, we have the notion of rooted ring with enough idempotents too. Following Definition 2, a ring R with enough idempotents is left rooted if there exists a complete family of idempotents, $\{e_i : i \in I\}$, in R such that the corresponding preadditive category is left rooted. In such case, we shall call $\{e_i : i \in I\}$ a left rooted complete family of idempotents in R. Triangular matrix rings are examples of rooted rings with enough idempotents.

Remark 5. *Not every complete family of idempotents in a rooted ring with enough idempotents is rooted. For example, let* R *be a ring with a non-trivial idempotent* e *and consider the triangular matrix ring* $\Lambda = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. *Then* Λ *is left and right rooted but*

$$\left\{ \begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-e & -e \\ 0 & 1 \end{pmatrix} \right\}$$
is a complete set of idempotent in Λ *which is not left nor right rooted.*

Using these ideas, Theorem 1 actually gives the characterization of projective modules over ring with enough idempotents. This is an extension of [14, Theorem 3.1] and [8, Proposition 2.1], where the characterization is obtained for generalized triangular matrix rings in which the structural morphisms are monic (the t_{abc} in Notation 1).

Corollary 2. *Let* R *be a left rooted ring with enough idempotents and* $\{e_i : i \in I\}$ *a left rooted complete set of idempotents in* R*. Then, for a right* R*-module* M*, the following assertions are equivalent:*

1. M *is projective.*
2. *There exists a projective object* $(P_i)_{i\in I} \in \prod_{i\in I}$ *Mod-*e_iRe_i *such that*

$$M = \bigoplus_{j\in I} \bigoplus_{i\in I} P_i \otimes e_iRe_j$$

Analogously, Theorem 3 gives a characterization of when a left rooted ring with enough idempotents is right perfect. This is an extension of the corresponding result for unitary rings, see [1, Proposition 28.11], and for triangular matrix rings, see [14, Corollary 5.2] and [8, Corollary 2.10].

Corollary 3. *Let* R *be a left rooted ring with enough idempotents and* $\{e_i : i \in I\}$ *a left rooted complete family of idempotents in* R*. Then* R *is right perfect if and only if* e_iRe_i *is right perfect for each* $i \in I$*.*

Finally, Theorem 4 allow us to characterize when a fp-rooted (unitary) ring is pure semisimple. Of course, a unitary ring R is called left weakly fp-rooted if the category Mod-R is left weakly fp-rooted.

Corollary 4. *Let* R *be a left weakly fp-rooted ring. Then* R *is right pure semisimple if and only if* $End_R(M)$ *is a left perfect ring for each finitely presented module* M*.*

To conclude the paper, we shall give the announced example of a weakly fp-rooted category.

Example 2. *Let* R *be a left rooted ring with enough idempotents (for example, a triangular matrix ring) and consider the full subcategory of Mod-*R *whose class of objects are the flat modules, Fl*(R)*. Then, by [3, Theorem 1.1], Fl*(R) *is a locally finitely presented additive category whose class of finitely presented objects coincides with the class of finitely generated projective* R*-modules. Since* R *is left rooted, fp*$($*Fl*$(R))$ *is left weak rooted and Fl*(R) *is weak left fp-rooted.*

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Department of Mathematics,
University of Almería
E-mail: `mizurdia@ual.es`