Bull. Math. Soc. Sci. Math. Roumanie
Tome 56(104) No. 3, 2013, 299–313

Hit-invariants and commutators for Hopf algebras *

by
Miriam Cohen and Sara Westreich

Dedicated to our colleagues and good friends Toma Albu and Constantin Năstăsescu upon their 70th birthday

Abstract

We continue studying normal left coideal subalgebras of a Hopf algebra H, realizing them as invariants of H under the left hit action of Hopf subalgebras of H^*. We apply this realization to test an equivalence relation on irreducible characters for two important examples. The commutator sublagebra of H, which is the analogue of the commutator subgroup of a group and the image of the Drinfeld map for quasitriangular Hopf algebras. We end with the example $H = D(kS_3)$ where commutators are computed.

Key Words: Normal left coideal subalgebras, left kernels, commutators, commutator algebra, Drinfeld map.
2010 Mathematics Subject Classification: Primary 16T05.

Introduction

We study natural analogues of normal subgroups, namely normal left coideal subalgebras of a Hopf algebra H. In §2 we consider H^T, invariants under the left hit action, and show:

Theorem 2.4: Let H be a finite dimensional Hopf algebra over a field k, then there exists a bijective correspondence between left coideal subalgebras T of H^* and left coideal subalgebras A of H. The maps

$$T \to H^T \quad A \to (H^*)^A$$

are inverses of each other, that is,

$$T = (H^*)^{H^T} \quad A = H^{(H^*)^A}$$

We apply this theorem to normal left coideal subalgebras LKer_V (defined in [2]) which are natural generalizations of kernel of group representations, and show:

*This research was supported by the Irael Science Foundation, 170-12.

Theorem 2.8: Let H be a Hopf algebra over k and V a finite dimensional representation of H with associated character χ_V. Let B_V be the bialgebra (and thus the Hopf subalgebra) of H^* generated by χ_V. Then

$$H^{B_V} = \mathrm{LKer}_V.$$

In §3 we study the special case of semisimple quasitriangular Hopf algebras. In this case $\mathrm{Im}f_Q$, where f_Q is the Drinfeld map, is a normal left coideal subalgebra of H. Let $\{\eta_j\}$ be the set of normalized class sums of H, defined in [5]. This set is the Hopf algebra analogue of the set of representatives of conjugacy classes for groups. The equivalence relation on the irreducible characters of H induced by the Hopf subalgebra $B = (H^*)^{\mathrm{Im}f_Q}$ satisfies:

Theorem 3.4: Let (H,R) be a quasitriangular semisimple Hopf algebra over a field k of characteristic 0, let $N_Q = \mathrm{Im}f_Q$ and $B = (H^*)^{N_Q}$. Then the map f_Q induces a bijective correspondence between the equivalence classes $\{[\chi_i]_B\}$ and the set of normalized class sums $\{\eta_j\} \cap N_Q$.

In §4 we study another natural example of a normal left coideal subalgebra, namely the commutator algebra, H', defined in [1]. It turns out that H' equals the invariants of H under the left hit action of $B = kG(H^*)$. We prove:

Theorem 4.2: Let H be a d-dimensional semisimple Hopf algebra over an algebraically closed field of characteristic 0 and let $B = kG(H^*)$. Then:

(i) $\chi_i \equiv_B \chi_j$ if and only if there exists $\sigma \in G(H^*)$ so that $\chi_i = \sigma\chi_j$. In this case $d_i = d_j$.

(ii) The cardinality of the equivalence class of χ_i equals $\frac{|G|}{|L_i|}$ where

$$L_i = \{\sigma \in G(H^*) | \sigma\chi_i = \chi_i\}.$$

In particular, the cardinality of each equivalence class divides d.

A special element of H', denoted by z_2 was introduced and studied in [8]. When the character algebra of H is commutative, we compute it from its generalized character table. In particular we compute various objects mentioned above for $H = D(kS_3)$. As a result we show that in this case $H' = z_2 \leftharpoonup H^*$.

1 Preliminaries

Throughout this paper, H is a finite-dimensional Hopf algebra over a field k. We denote by S and s the antipodes of H and H^* respectively and Λ and λ the left and right integrals of H and H^* respectively so that $\langle \lambda, \Lambda \rangle = 1$. Denote by $Z(H)$ the center of H.

Recall that any subbialgebra of H is necessarily a Hopf subalgebra.

The Hopf algebra H^* becomes a right and left H-module by the *hit* actions $\leftharpoonup$ and $\rightharpoonup$ defined for all $a \in H$, $p \in H^*$,

$$\langle p \leftharpoonup a, a' \rangle = \langle p, aa' \rangle \qquad \langle a \rightharpoonup p, a' \rangle = \langle p, a'a \rangle$$

H becomes a left and right H^*-module analogously.

Denote by $\underset{ad}{\cdot}$ the left adjoint action of H on itself, that is, for all $a, h \in H$,

$$h \underset{ad}{\cdot} a = \sum h_1 a S(h_2)$$

A left coideal subalgebra of H is called *normal* if it is stable under the left adjoint action of H.

Recall [16], any left coideal subalgebra A of H contains a left integral Λ_A. Moreover, if $A_1 \subset A_2$ are left coideal subalgebras then A_2 is free over A_1. This implies in particular that:

Remark 1.1. If $A \neq B$ are left coideal subalgebras of H then $\Lambda_A \neq \Lambda_B$. If H is semisimple, then A is semisimple and $\langle \Lambda_A, 1\rangle \neq 0$.

Denote by $R(H)$ the k-span of all irreducible characters. It is an algebra called the character algebra of H.

Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0 and let $\{V_0, \dots V_{n-1}\}$ be a complete set of non-isomorphic irreducible H-modules. Let $\{E_0, \dots E_{n-1}\}$ and $\mathrm{Irr}(H) = \{\chi_0, \dots, \chi_{n-1}\}$ be the associated central primitive idempotents and irreducible characters of H respectively, where $E_0 = \Lambda$, the idempotent integral of H and $\chi_0 = \varepsilon$. Let $\dim V_i = d_i = \langle \chi_i, 1\rangle$, then $\lambda = \chi_H = \sum_{i=0}^{n-1} d_i \chi_i$. One has (see e.g [15, Cor.4.6]):

$$\langle \chi_i, E_j\rangle = \delta_{ij} d_j, \qquad \Lambda \leftharpoonup \chi_j = \frac{1}{d_j} S(E_j). \tag{1}$$

In particular, $\{\chi_i\}$, $\{\frac{1}{d_j} E_j\}$ are dual bases of $R(H)$ and $Z(H)$ respectively.

Recall that H is a Frobenius algebra. One defines a Frobenius map $\Psi : H_{H^*} \to H^*_{H^*}$ by

$$\Psi(h) = \lambda \leftharpoonup S(h) \tag{2}$$

where H^* is a right H^*-module under multiplication and H is a right H^*-module under right *hit*. If H is semisimple then

$$\Psi(Z(H)) = R(H).$$

For a finite-dimensional Hopf algebra H we have for all $p \in H^*$,

$$\Psi^{-1}(p) = \Lambda \leftharpoonup p.$$

Any simple subcoalgebra B_i of H^* contains precisely one irreducible character that generates B_i as a coalgebra. Since $B = \bigoplus_{i \in I} B_i$, where each B_i is a simple subcoalgebra of H^*, it follows that B is the coalgebra generated by $B \cap \mathrm{Irr}(H)$. Also, if $\chi \in B$ then all its irreducible constituents belong to B as well.

In particular, if B is a Hopf subalgebra of H^* then

$$\lambda_B = \sum_{\chi_i \in \mathrm{Irr}(H) \cap B} d_i \chi_i$$

is a nonzero integral for B. In [14, Prop.18] the following equivalence relation was defined on simple subcoalgebras of H^* :

$$C_k \equiv_B C_{k'} \Leftrightarrow BC_k \supset C_{k'}$$

By the proof of [14, Prop.18] one can check that the above equivalence relation can be stated in terms of the following equivalence relation on $\mathrm{Irr}(H)$.

$$\chi_i \equiv_B \chi_j \Leftrightarrow \lambda_B \frac{\chi_i}{d_i} = \lambda_B \frac{\chi_j}{d_j} \tag{3}$$

When $R(H)$ is commutative, let $\frac{1}{d}\lambda = F_0, \ldots, F_{n-1}$ be the set of central primitive idempotents of $R(H)$. Then $\{F_j\}$ form another basis for $R(H)$. Define the **conjugacy class** $\mathcal{C}_i$ as:

$$\mathcal{C}_i = \Lambda \leftharpoonup F_i H^*.$$

We generalize also the notions of **Class sum** and of a representative of a conjugacy class as follows:

$$C_i = \Lambda \leftharpoonup dF_i \qquad \eta_i = \frac{C_i}{\dim(F_i H^*)}. \tag{4}$$

We refer to η_i as a **normalized class sum**. It follows (see e.g. [5]) that $\{\eta_i\}$ is also a basis of $Z(H)$ dual to the basis $\{F_i\}$ of $R(H)$.

We can define now a generalized character table (ξ_{ij}) for H where,

$$\xi_{ij} = \langle \chi_i, \eta_j \rangle ,$$

$0 \leq i, j \leq n-1$. Note that $\eta_0 = 1$ and so $\xi_{i0} = \langle \chi_i, 1 \rangle = d_i$. Moreover, (ξ_{ij}) is the change of bases matrix between $\{\chi_i\}$ and $\{F_i\}$.

2 Hit-invariants and normal left coideal subalgebras

In this section we relate Hopf subalgebras of H^* and normal left coideal subalgebras of H. We do in fact realize them as invariants under the left *hit* action.

For any subalgebra T of H^*, denote by H^T the set of T-invariants of H under the left *hit* action. That is,

$$H^T = \{h \in H \,|b \rightharpoonup h = \langle b, 1\rangle h, \ \forall b \in T\} \tag{5}$$

Remark 2.1. If H is semisimple and N is a left coideal subalgebra, then

$$(H^*)^N = \Lambda_N \rightharpoonup H^*$$

Proposition 2.2. *Let H be a finite dimensional Hopf algebra over any field k and T a subalgebra of H^*. Then:*

(i) *H^T is a left coideal of H.*

(ii) *It T is a left ot a right coideal subalgebra of H^* then H^T is an left coideal subalgebra of H.*

(iii) *If T is a normal left subalgebra in H^* then H^T is a Hopf subalgebra of H.*

(iv) *If T is a bialgebra then H^T is a normal left coideal subalgebra of H.*

(v). *If T is a normal left coideal subalgebra of H^* then H^T is a Hopf subalgebra of H.*

Proof: (i) We need to show that $H^T \leftharpoonup H^* \subset H^T$. Let $b \in T$, $h \in H^T$, $p \in H^*$, then

$$b \rightharpoonup (h \leftharpoonup p) = (b \rightharpoonup h) \leftharpoonup p = \langle b, 1\rangle (h \leftharpoonup p).$$

Hence H^T is a left coideal.

(ii) Assume T is a left coideal. Let $b \in T$, $h, h' \in H^T$, then

$$b \rightharpoonup (hh') = \sum (b_1 \rightharpoonup h)(b_2 \rightharpoonup h') = (b \rightharpoonup h)h' = \langle b, 1\rangle hh'.$$

The second equality follows from the fact that T is a left coideal. The same proof works if T is a right coideal subalgebra.

(iii). By (ii) all we need to show is that H^T is a right coideal, that is $H^* \rightharpoonup H^T \subset H^T$. Since T is stable under the left adjoint action, we have that

$$bp = \sum p_3(s^{-1}(p_2)bp_1) = \sum p_2(s^{-1}(p_1)_{\dot{ad}} b) \in H^*T$$

for all $b \in T$, $p \in H^*$. Hence we have for $b \in T$, $h \in H^T$, $p \in H^*$,

$$b \rightharpoonup (p \rightharpoonup h) = bp \rightharpoonup h = \sum p_3 s^{-1}(p_2 b p_1) \rightharpoonup h = \langle b, 1\rangle (p \rightharpoonup h).$$

So $(p \rightharpoonup h) \in H^T$.

(iv). Assume T is a coalgebra. We need to show that H^T is normal in H. Let $h \in H^T$, $x \in H$, $b \in T$, then

$$\begin{aligned} b \rightharpoonup \sum x_1 h S(x_2) = & \\ = & \sum (b_1 \rightharpoonup x_1)(b_2 \rightharpoonup h)(b_3 \rightharpoonup S(x_2)) \\ = & \sum (b_1 \rightharpoonup x_1) h (b_2 \rightharpoonup S(x_2)) \qquad \text{(since } B \text{ is a coalgebra)} \\ = & \sum \langle b_1, x_2\rangle \langle b_2, S(x_3)\rangle x_1 h S(x_4) \\ = & \sum \langle b, 1\rangle x_1 h S(x_2). \end{aligned}$$

Hence $\sum x_1 h S(x_2) \in H^T$.

(v). By (ii) and (iii) H^T is a bialgebra. Since H is finite dimensional it follows that it is a Hopf subalgebra. □

We have,

Lemma 2.3. *For any subalgebra T of H^* and a left coideal A of H we have:*

$$A \subset H^T \Leftrightarrow \langle p, a\rangle = \langle p, 1\rangle \langle \varepsilon, a\rangle \quad \forall a \in A, p \in T. \tag{6}$$

If T and A are left coideal subalgebras of H^* and H respectively, then

$$A \subset H^T \Leftrightarrow T \subset (H^*)^A. \tag{7}$$

In particular,

$$T \subset (H^*)^{H^T} \qquad A \subset H^{(H^*)^A} \tag{8}$$

Proof: By applying $\varepsilon \otimes \mathrm{Id}$ to both sides of (5) we get the right hand side of the equivalence in (6). Conversely, if A is a left coideal satisfying the right hand side of the equivalence, then $A \subset H^T$ by the definition of H^T.

(7) follows directly from (6).

The first inclusion in (8) follows by taking $A = H^T$ in (7). The second inclusion follows by replacing H with H^* and taking $T = (H^*)^A$. □

We can show now,

Theorem 2.4. *Let H be a finite dimensional Hopf algebra over a field k, then there exists a bijective correspondence between left coideal subalgebras T of H^* and left coideal subalgebras A of H. The maps*

$$T \to H^T \quad A \to (H^*)^A$$

are inverses of each other, that is,

$$T = (H^*)^{H^T} \quad A = H^{(H^*)^A}$$

Proof: Let T be a left coideal subalgebra of H^*. Set $A = H^T$, then A is a left coideal subalgebra of H by Proposition 2.2(ii). By (8), $T \subset (H^*)^A$. By Remark 1.1, equality will follow once we prove that every nonzero left integral of T is indeed a left integral of $(H^*)^A$. Let λ_T be a left integral for T. For any $p \in (H^*)^A$,

$$p\lambda_T \rightharpoonup \Lambda = p \rightharpoonup (\lambda_T \rightharpoonup \Lambda) = \langle p, 1\rangle(\lambda_T \rightharpoonup \Lambda)$$

The last equality follows from the fact that $(\lambda_T \rightharpoonup H) \subset H^T = A$, and $A \subset H^{(H^*)^A}$ by (8). hence $p\lambda_T = \langle p, 1\rangle\lambda_T$ and we are done. The proof that $A = H^{(H^*)^T}$ is identical replacing H by H^* and T by A in the above argument. □

It follows from the theorem above and Proposition 2.2(iii) that we can relate in particular normal left coideal subalgebras of H and Hopf subalgebras of H^*.

Corollary 2.5. Let H be a finite dimensional Hopf algebra. Then there exists a bijective correspondence between Hopf subalgebras B of H^* and normal left coideal subalgebras N of H given by:

$$B \to H^B, \quad N \to (H^*)^N.$$

Remark 2.6. The corollary above is in fact the bijective correspondence between normal left coideal subalgebras and Hopf quotients of H discussed in [17]. Explicitly, if B is a Hopf subalgebra of H^* and $\pi : H \to B^*$ is the corresponding Hopf projection, that is,

$$\langle \pi(h), b\rangle = \langle b, h\rangle\,,$$

for all $b \in B$, $h \in H$. Then

$$H^{co\pi} = \{h \in H | h_1 \otimes \pi(h_2) = h \otimes 1\}.$$

Now, for all $p \in H^*$, $b \in B$,

$$\left\langle \sum h_1 \otimes \pi(h_2), p \otimes b \right\rangle = \langle p, b \rightharpoonup h \rangle .$$

Hence it is easy to see that

$$H^B = H^{co\pi}. \tag{9}$$

Corollary 2.7. Let $N = H^B$. Since

$$H \cong H^{co\pi} \otimes H/HN^+ \cong H^B \otimes B^*,$$

we obtain:

$$\dim(H^B) = \frac{\dim H}{\dim B}.$$

Certain normal left coideal subalgebras appear in [2] as a natural generalization of kernels of group representations. For an H-module V define its left kernel as follows:

$$\mathrm{LKer}_V = \{h \in H \mid \sum h_1 \otimes h_2 \cdot v = h \otimes v, \forall v \in V\}. \tag{10}$$

As a corollary we suggest an additional description of left kernels. This can be proved directly by Proposition 2.2(iii) and Theorem 2.4. It follows however from results of [2] after adaptation.

Theorem 2.8. *Let H be a Hopf algebra over k and V a finite dimensional representation of H with associated character χ_V. Let B_V be the bialgebra (and thus the Hopf subalgebra) of H^* generated by χ_V. Then*

$$H^{B_V} = \mathrm{LKer}_V.$$

Proof: Observe that $B_V^{\perp} = \bigcap_m ann_H V^{\otimes m}$. By [2, Th. 2.3.6], $H^{co\pi} = \mathrm{LKer}_V$, where $\pi : H \to H/(\cap_m ann_H V^{\otimes m})$ is the canonical projection. The result follows now from (9). □

3 The special case - quasitriangular Hopf algebras

Recall, if (H, R) is a finite dimensional quasitriangular Hopf algebra then the maps $f_R : H^{*cop} \to H$, defined by $f_R(p) = \langle p, R^1 \rangle R^2$ and $f_R^* : H^{*op} \mapsto H$, defined by $f_R^*(p) = \langle p, R^2 \rangle R^1$ are Hopf algebra maps. Then $Q = R^{21}R$ and the Drinfeld map f_Q is given by $f_Q = f_R^* * f_R$. When (H, R) is a quasitriangular Hopf algebra then $R(H)$ is necessarily commutative. If H is also semisimple then the Drinfeld map f_Q is an algebra map from $R(H)$ to $Z(H)$.

The S-matrix for (H, R) is defined by:

$$s_{ij} = \langle \chi_i, f_Q(\chi_j) \rangle. \tag{11}$$

The quasitriangular Hopf algebra (H, R) is *factorizable* if f_Q is a monomorphism, or equivalently, if its S-matrix is invertible.

Assume (H,R) is factorizable. Reorder the set $\{F_j\}$ so that

$$f_Q(F_j) = E_j$$

for all $1 \le j \le m$. It follows that $\dim(F_jH^*) = d_j^2$, where $d_j = \langle \chi_j, 1\rangle$. Also (see e.g [4, (15)]),

$$f_Q(\chi_j) = \frac{1}{d_j} C_j = d_j \eta_j. \tag{12}$$

The last equality follows since $\dim(F_jH^*) = \dim(E_jH) = d_j^2$. It follows that the S-matrix satisfies

$$s_{ij} = d_j \xi_{ij}. \tag{13}$$

It was shown in [6] that:

Lemma 3.1. *Let (H,R) be a quasitriangular semisimple Hopf algebra over k. Then f_Q maps subcoalgebras of H^* to left coideals of H stable under the adjoint action, and Hopf subalgebras of H^* to normal left coideal subalgebras of H.*

We study next how quasitriangularity affects the equivalence relation defined in (3). We start with the following more general situation:

Lemma 3.2. *Let H be a semismple Hopf algebra so that $R(H)$ is commutative, B be a Hopf subalgebra of H^* and $N = H^B$. Consider the equivalence relation defined in* (3). *Then $\chi_i \equiv_B \chi_i'$ if and only if $\langle \frac{\chi_i}{d_i}, \eta_j\rangle = \langle \frac{\chi_i'}{d_i'}, \eta_j\rangle$ for all $\eta_j \in N$.*

Proof: Assume $\lambda_B \frac{\chi_i}{d_i} = \lambda_B \frac{\chi_i'}{d_i'}$. Take λ_B so that $\langle \lambda_B, 1\rangle = 1$, then we have:

$$\langle \frac{\chi_i}{d_i}, \eta_j\rangle = \langle \frac{\chi_i}{d_i}\lambda_B, \eta_j\rangle = \langle \frac{\chi_i'}{d_i'}\lambda_B, \eta_j\rangle = \langle \frac{\chi_i'}{d_i'}, \eta_j\rangle$$

Conversely, assume $< \frac{\chi_i}{d_i}, \eta_j >=< \frac{\chi_i'}{d_i'}, \eta_j >$ for all $\eta_j \in N$. Since $R(H)$ is commutative and $N = \bigoplus_j \mathcal{C}_j$, we have for each $n \in N$, the central element $\Lambda_{\dot{a}d} n$ is a linear combination of $\{\eta_j\} \cap N$. Since characters are cocommutative it follows that:

$$\langle \frac{\chi_i}{d_i}, n\rangle = \langle \frac{\chi_i}{d_i}, \Lambda_{\dot{a}d} n\rangle = \langle \frac{\chi_i'}{d_i'}, \Lambda_{\dot{a}d} n\rangle = \langle \frac{\chi_i'}{d_i'}, n\rangle$$

for all $n \in N$. By Remark 2.1, (with H replacing H^*), $N = \lambda_B \rightharpoonup H$. Hence we have for all $h \in H$,

$$\langle \frac{\chi_i}{d_i}\lambda_B, h\rangle = \langle \frac{\chi_i}{d_i}, \lambda_B \rightharpoonup h\rangle = \langle \frac{\chi_i'}{d_i'}, \lambda_B \rightharpoonup h\rangle = \langle \frac{\chi_i'}{d_i'}\lambda_B, h\rangle.$$

Thus $\chi_i \equiv_B \chi_i'$. □

When H is quasitriangular more can be said:

Proposition 3.3. *Let (H, R) be a quasitriangular semisimple Hopf algebra over a field k of characteristic 0, let $N_Q = \mathrm{Im} f_Q$ and $B = (H^*)^{N_Q}$. Then:*

(i) *The Hopf subalgebra B satisfies:*

$$B = \{b \in H \mid \sum b_1 \otimes f_Q^*(b_2) = b \otimes 1\} = \{b \in H \mid \sum f_Q(b_1) \otimes b_2 = 1 \otimes b\}.$$

In particular, $f_Q^*(b) = \langle b, 1\rangle 1 = f_Q(b)$ for all $b \in B$.

(ii) $\chi_i \equiv_B \chi_i'$ *if and only if* $f_Q(\frac{\chi_i}{d_i}) = f_Q(\frac{\chi_i'}{d_i'})$.

Proof: (i) B is a Hopf algebra since N_Q is a left normal coideal subalgebra by Lemma 3.1. Let $b \in B$, $x \in H^*$. By Remark 2.1, $b = \Lambda_{N_Q} \rightharpoonup p$, thus,

$$\begin{aligned} &\langle f_Q^*(\Lambda_{N_Q} \rightharpoonup p), x\rangle = \\ &= \langle \Lambda_{N_Q} \rightharpoonup p, f_Q(x)\rangle = \langle p, f_Q(x)\Lambda_{N_Q}\rangle \\ &= \langle p, \Lambda_{N_Q}\rangle\langle f_Q(x), 1\rangle = \langle b, 1\rangle\langle x, 1\rangle. \end{aligned}$$

Hence $f_Q^*(b) = \langle b, 1\rangle 1$. Since B is a Hopf subalgebra, $s(b) \in B$, thus by above $f_Q^* s(b) = \langle b, 1\rangle 1$. Hence $S f_Q^* s(b) = \langle b, 1\rangle 1$ as well. Since $S f_Q^* s = f_Q$ we have also $f_Q(b) = \langle b, 1\rangle 1$ for all $b \in B$.

Since B is in particular a coalgebra, it follows now that for all $b \in B$, $\sum b_1 \otimes f_Q^*(b_2) = b \otimes 1$.

Conversely, assume $\sum b_1 \otimes f_Q^*(b_2) = b \otimes 1$. Then for all $n = f_Q(y)$,

$$\sum \langle n, b_2\rangle b_1 = \sum \langle y, f_Q^*(b_2)\rangle b_1 = \langle y, 1\rangle\langle b, 1\rangle = \langle \varepsilon, n\rangle\langle b, 1\rangle.$$

Hence $b \in (H^*)^{N_Q}$.

(ii). If $\chi_i \equiv_B \chi_i'$ then since f_Q is multiplicative on $R(H)$ and $f_Q(\lambda_B) = 1$ by part (i), we have:

$$f_Q(\frac{\chi_i}{d_i}) = f_Q(\lambda_B \frac{\chi_i}{d_i}) = f_Q(\lambda_B \frac{\chi_i'}{d_i'}) = f_Q(\frac{\chi_i'}{d_i'})$$

Conversely, assume $f_Q(\frac{\chi_i}{d_i}) = f_Q(\frac{\chi_i'}{d_i'})$. Note first that

$$f_Q^* s(\frac{\chi_i}{d_i}) = s f_Q(\frac{\chi_i}{d_i}) = s f_Q(\frac{\chi_i'}{d_i'}) = f_Q^* s(\frac{\chi_i'}{d_i'}).$$

Hence we have for all $n = f_Q(p) \in N_Q$,

$$\begin{aligned} &\langle s(\frac{\chi_i}{d_i}), n\rangle = \\ &= \langle s(\frac{\chi_i}{d_i}), f_Q(p)\rangle = \langle f_Q^* s(\frac{\chi_i}{d_i}), p\rangle = \langle f_Q^* s(\frac{\chi_i'}{d_i'}), p\rangle = \langle s(\frac{\chi_i'}{d_i'}), f_Q(p)\rangle \\ &= \langle s(\frac{\chi_i'}{d_i'}), n\rangle. \end{aligned}$$

By Lemma 3.2, we have $s(\chi_i) \equiv_B s(\chi_i')$. Since $\lambda_B = s(\lambda_B)$ and $R(H)$ is commutative we have:

$$\lambda_B \frac{\chi_i}{d_i} = s(\lambda_B s(\frac{\chi_i}{d_i})) = s(\lambda_B s(\frac{\chi_i'}{d_i'})) = \lambda_B \frac{\chi_i'}{d_i'}.$$

Hence $\chi_i \equiv_B \chi_i'$. □

Recall the Hopf algebra surjection $\Phi : D(H) \to H$ given by [9]:

$$\Phi(p \bowtie h) = f_R(p)h.$$

A direct computation shows that

$$\Phi f_{Q_{D(H)}} \Phi^* = f_Q \quad \text{and} \quad \Phi \Psi_{D(H)} \Phi^* = \Psi_H, \tag{14}$$

where Ψ is the Frobenius map. Combining the above we have:

Theorem 3.4. *Let (H,R) be a quasitriangular semisimple Hopf algebra over a field k of characteristic 0, let $N_Q = \mathrm{Im} f_Q$ and $B = (H^*)^{N_Q}$. Then the map f_Q induces a bijective correspondence between the equivalence classes $\{[\chi_i]_B\}$ and the set of normalized class sums $\{\eta_j\} \cap N_Q$.*

Proof: Since $\Phi^*(\chi_i) \in D(H)$ is the character of V_j considered as a $D(H)$-module (see [5]), it follows from (12) that:

$$f_{Q_{D(H)}} \Phi^*(\chi_i) = d_i \widehat{\eta}_i$$

where $\widehat{\eta}_i$ is the corresponding normalized class sum in $D(H)$. By [5, Lemma 2.4], $\Phi(\widehat{\eta}_i) = \eta_{s_i}$ for some $s_i \geq 0$, hence by (14) we have for all i,

$$f_Q(\frac{\chi_i}{d_i}) = \Phi f_{Q_{D(H)}} \Phi^*(\frac{\chi_i}{d_i}) = \eta_{s_i} \tag{15}$$

By Proposition 3.3(ii) this map is injective. We show surjectivity. Let $\eta_j = f_Q(p)$. Since $\eta_j \in Z(H)$ we have by [3, Prop.2.5.5],

$$\eta_j = \sum \Lambda_1 \eta_j S(\Lambda_2) = f_Q(\Lambda_2 \rightharpoonup p \leftharpoonup S(\Lambda_1))$$

But $\Lambda_2 \rightharpoonup p \leftharpoonup S(\Lambda_1)$ is a cocommutative element in H^*, and since H is semisimple it follows that it is an element of $R(H)$. Thus $f_Q^{-1}(\eta_j) = \sum \alpha_i \chi_i$, hence by (15),

$$\eta_j = \sum_i \alpha_i f_Q(\chi_i) = \sum_s \beta_s \eta_s.$$

Since $\{\eta_k\}$ are linearly independent, it follows that $\eta_s = \eta_j$ for all s, and thus $f_Q(\frac{\chi_i}{d_i}) = \eta_j$ for all i in the above sum. □

4 The commutator algebra - a distinguished normal left coideal subalgebra

In this section we focus on a specific normal left coideal subalgebra, the commutator algebra, first defined in [1]. It is a normal left coideal subalgebra of H for which $H/(HH'^{+})$ is commutative and it is minimal with respect to this property. For $H = kG$ one has $H' = kG'$, where G' is the commutator subgroup of G.

Based on [1, §6] and [6, Prop.1.14], it is not hard to see that if $\mathcal{S}$ is the set of all 1-dimensional H-representations then

$$H' = \bigcap_{V \in \mathcal{S}} \mathrm{LKer}_V = \{h \in H \mid \sigma \rightharpoonup h = h \ \forall \sigma \in G(H^*)\}. \tag{16}$$

Generalizing from groups we describe H' in terms of Hopf algebraic commutators. These commutators were defined and discussed in [8]. Let H be any Hopf algebra over k. Define commutators in H as:

$$\{a, b\} = \sum a_1 b_1 S a_2 S b_2 \qquad \mathit{Com} = \mathrm{span}_k\{\{a, b\} \mid a, b \in H\}. \tag{17}$$

It was shown in [8] that:

Proposition 4.1. *Let H be a Hopf algebra over k, then Com is a left coideal of H and H' is the algebra generated by Com.*

When considering the normal left coideal subalgebra $N = H'$, we have $B = H^N = kG(H^*)$. The equivalence relation $\equiv_B$ defined in (3) satisfies the following:

Theorem 4.2. *Let H be a d-dimensional semisimple Hopf algebra over an algebraically closed field of characteristic 0 and let $B = kG(H^*)$. Then:*

(i) $\chi_i \equiv_B \chi_j$ if and only if there exists $\sigma \in G(H^*)$ so that $\chi_i = \sigma\chi_j$. In this case $d_i = d_j$.

(ii) The cardinality of the equivalence class of χ_i equals $\frac{|G|}{|L_i|}$ where

$$L_i = \{\sigma \in G(H^*) | \sigma\chi_i = \chi_i\}.$$

In particular, the cardinality of each equivalence class divides d.

Proof: (i). Note that $\sigma\chi_i$ is an irreducible character for all $\sigma \in G(H^*)$. This follows since if $\sigma\chi_i = \sum n_k\chi_k$ then $\chi_i = \sum n_k\sigma^{-1}\chi_k$. Since χ_i is irreducible this is possible only if $\chi_i = \sigma^{-1}\chi_k$ for some k. Now, if $\chi_i \equiv_B \chi_j$ then $\chi_j \in kG(H^*)\chi_i$.

(ii) Follows directly from part (i). □

The commutator algebra is related to a specific element defined as follows:

$$z_2 = \sum \Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2, \tag{18}$$

where Λ^1 and Λ^2 are copies of the idempotent integral of H. In particular, for a finite group G,

$$z_{2,G} = \frac{1}{|G|^2} \sum_{a,b\in G} aba^{-1}b^{-1}$$

It was proved in [8] that for any semisimple Hopf algebra over a field of characteristic 0,

$$z_2 = \sum_i \frac{1}{d_i^2} E_i. \tag{19}$$

When $R(H)$ is commutative then z_2 can be computed directly from the character table as follows:

Lemma 4.3. *Assume $R(H)$ is commutative. Then*

$$z_2 = \frac{1}{d} \sum_k \left(\sum_t \frac{1}{d_t} \xi_{tk} \dim(F_k H^*) \right) \eta_k.$$

If H is also factorizable then

$$z_2 = \frac{1}{d} \sum_k \left(\sum_t \xi_{kt} d_k \right) \eta_k.$$

Proof: Recall the character table is the change of bases matrix (written as rows) between $\{\chi_i\}$ and $\{F_i\}$. Applying Ψ^{-1} yields that it is the change of bases matrix between $\{\frac{1}{d_i} SE_i\}$ and $\{\frac{dim(F_i H^*)}{d}\eta_i\}$. The first formula follows now by using the coordinates of z_2 given in (19).

If H is factorizable, then $dim(F_k H^*) = d_k^2$. By (13) $s_{ij} = d_i \xi_{ij}$ and the matrix S is known to be symmetric. Hence we obtain $\frac{d_i}{d_j}\xi_{ij} = \xi_{ji}$. □

The following is a special case in which we test the ideas mentioned above. Let G be a finite group and $D(G)$ its Yetter Drinfeld double, which is always a factorizable Hopf algebra. The product and the integral inside $D(G)$ are given by:

$$(\varepsilon \bowtie g)(p_h \bowtie 1) = p_{ghg^{-1}} \bowtie g, \qquad \Lambda = p_1 \bowtie \frac{1}{|G|} \sum_{g\in G} g. \tag{20}$$

Thus we have:

Proposition 4.4. *Let G be a finite group, k an algebraically closed field of characteristic 0 and $H = D(kG)$. For $x \in G$, let $z_{2,C_G(x)}$ denote the element z_2 of $C_G(x)$. Then the element z_2 of H satisfies:*

$$z_2 = \frac{1}{|G|^2} \sum_{x\in G} p_x \bowtie \sum_{g,h\in C_G(x)} ghg^{-1}h^{-1} = \frac{1}{|G|^2} \sum_{x\in G} p_x \bowtie |C_G(x)|^2 z_{2,C_G(x)}.$$

Proof: 1. The first equality follows from (18) by using (20),

$$
\begin{aligned}
&\sum \Lambda_1^1\Lambda_2^1 S(\Lambda_2^1)S(\Lambda_2^2) = \\
&= \frac{1}{|G|^2}\sum_{g,h,x,y\in G}(p_x \bowtie g)(p_y \bowtie h)(p_{g^{-1}xg}\bowtie g^{-1})(p_{h^{-1}yh}\bowtie h^{-1}) \\
&= \frac{1}{|G|^2}\sum_{g,h,x,y\in G} p_x p_{gyg^{-1}} p_{ghg^{-1}xgh^{-1}g^{-1}} p_{ghg^{-1}h^{-1}yhgh^{-1}g^{-1}} \bowtie ghg^{-1}h^{-1} \\
&= \frac{1}{|G|^2}\sum_{x\in G} p_x \bowtie \sum_{g,h\in C_G(x)} ghg^{-1}h^{-1}.
\end{aligned}
$$

The second equality follows from the definition of $z_{2,C_G(x)}$. ∎

We focus now on the specific example $H = D(kS_3)$. Based on the representations of the centralizers (see details in [5, §3]), the characters for $D(kS_3)$ are computed as follows: $\chi_0 = 1\otimes\varepsilon$

$$
\begin{aligned}
\chi_1 &= 1\otimes(p_1 + p_{(123)} + p_{(132)} - p_{(12)} - p_{(13)} - p_{(23)}) \\
\chi_2 &= 1\otimes(2p_1 - p_{(123)} - p_{(132)}) \\
\chi_3 &= (12)\otimes(p_1 + p_{(12)}) + (13)\otimes(p_1 + p_{(13)}) + (23)\otimes(p_1 + p_{(23)}) \\
\chi_4 &= (12)\otimes(p_1 - p_{(12)}) + (13)\otimes(p_1 - p_{(13)}) + (23)\otimes(p_1 - p_{(23)}) \\
\chi_5 &= (123)\otimes(p_1 + p_{(123)} + p_{(132)}) + (132)\otimes(p_1 + p_{(123)} + p_{(132)}) \\
\chi_6 &= (123)\otimes(p_1 + \omega p_{(123)} + \omega^2 p_{(132)}) + (132)\otimes(p_1 + \omega^2 p_{(123)} + \omega p_{(132)}) \\
\chi_7 &= (123)\otimes(p_1 + \omega^2 p_{(123)} + \omega p_{(132)}) + (132)\otimes(p_1 + \omega p_{(123)} + +\omega^2 p_{(132)}).
\end{aligned}
$$

Since $\eta_i = \frac{1}{d_i} f_Q(\chi_i)$, we obtain that $\eta_0 = \varepsilon \bowtie 1$ and

$$
\begin{aligned}
\eta_1 &= (p_1 + p_{(123)} + p_{(132)} - p_{(12)} - p_{(13)} - p_{(23)}) \bowtie 1 \\
\eta_2 &= \frac{1}{2}(2p_1 - p_{(123)} - p_{(132)}) \bowtie 1 \\
\eta_3 &= \frac{1}{3}(p_1 + p_{(12)}) \bowtie (12) + (p_1 + p_{(13)}) \bowtie (13) + (p_1 + p_{(23)}) \bowtie (23) \\
\eta_4 &= \frac{1}{3}(p_1 - p_{(12)}) \bowtie (12) + (p_1 - p_{(13)}) \bowtie (13) + (p_1 - p_{(23)}) \bowtie (23) \\
\eta_5 &= \frac{1}{2}(p_1 + p_{(123)} + p_{(132)}) \bowtie (123) + (p_1 + p_{(123)} + p_{(132)}) \bowtie (132) \\
\eta_6 &= \frac{1}{2}(p_1 + \omega p_{(123)} + \omega^2 p_{(132)}) \bowtie (123) + (p_1 + \omega^2 p_{(123)} + \omega p_{(132)}) \bowtie (132) \\
\eta_7 &= \frac{1}{2}(p_1 + \omega^2 p_{(123)} + \omega p_{(132)}) \bowtie (123) + (p_1 + \omega p_{(123)} + \omega^2 p_{(132)}) \bowtie (132).
\end{aligned}
$$

Hence the character table is given as follows:

$$\begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \\ \chi_6 \\ \chi_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} \eta_0 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & -1 & -1 & -1 \\ 3 & -3 & 0 & 1 & -1 & 0 & 0 & 0 \\ 3 & -3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 2 & -1 & 0 & 0 & 2 & -1 & -1 \\ 2 & 2 & -1 & 0 & 0 & -1 & -1 & 2 \\ 2 & 2 & -1 & 0 & 0 & -1 & 2 & -1 \end{pmatrix} \end{array}$$

One can see that $G(H^*) = \{\chi_0, \chi_1\}$. Hence the equivalence classes described in Theorem 4.2 are as follows:

$$\{[\chi_0, \chi_1], [\chi_2], [\chi_3, \chi_4], [\chi_5], [\chi_6], [\chi_7]\}$$

Since V_1 is the only non-trivial 1-dimensional representation, it follows from (16) that $H' = \mathrm{Lker}_{V_1}$. By using the character table it follows from [6, Cor.1.10]) that

$$\mathrm{Lker}_{\mathrm{V}_1} = \mathfrak{C}_0 \oplus \mathfrak{C}_1 \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_5 \oplus \mathfrak{C}_6 \oplus \mathfrak{C}_7$$

Now, z_2 can be computed either directly from the character table by using Lemma 4.3 or by using Theorem 4.4.1. Both ways imply that:

$$\begin{aligned} z_2 &= \\ &= \frac{1}{36}(8\eta_0 + 4\eta_1 + 6\eta_2 + 6\eta_5 + 6\eta_6 + 6\eta_7) \\ &= \frac{1}{36}p_1 \bowtie (18 \cdot 1 + 9 \cdot (123) + 9 \cdot (132)) + \\ &+ \frac{1}{9}(p_{12} + p_{13} + p_{23}) \bowtie 1 + \frac{1}{4}(p_{123} + p_{132}) \bowtie 1 \end{aligned}$$

Since $\mathfrak{C}_i = \eta_i \leftharpoonup H_i^*$ for all i, it follows that

$$H' = \mathfrak{C}_0 + \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_5 + \mathfrak{C}_6 + \mathfrak{C}_7 = z_2 \leftharpoonup H^* \subset \mathrm{Com}_H \subset H'.$$

Thus $D(kS_3)$ exhibits an affirmative answer to the question raised in [8, Qu. 2.11] whether $z_2 \leftharpoonup H^* = \mathrm{Com}$.

References

[1] S. Burciu, Normal coideal subalgebras of semisimple Hopf algebras, Algebra, Geometry, and Mathematical Physics, Journal of Physics: Conference Series 346 (2012),1-10.

[2] S. Burciu, Kernel of representations and coideal subalgebras for Hopf algebras, Glasgow Math. J. 54 (2012) 107-119.

[3] M. Cohen and S. Westreich, Some interrelations between Hopf algebras and their duals, J. Algebra 283 (2005), no. 1, 42–62.

[4] M. Cohen and S. Westreich, Structure constants related to symmetric Hopf algebras, J. Alg. 324 (2010), pp. 3219-3240.

[5] M. Cohen and S. Westreich, Conjugacy Classes, Class Sums and Character Tables for Hopf Algebras, Communications in Algebra, 39, (2011), 4618-4633.

[6] M.Cohen and S. Westreich, Recovering information from character tables of Hopf algebras, Hopf algebras and tensor categories, Contemporary Math (585), Amer. Math. Soc , (2013), 213-227.

[7] M. Cohen and S. Westreich, Character tables and normal left coideal subalgebra, arXiv:1212.5785 [math.QA].

[8] M. Cohen and S. Westreich, Are we counting or measuring something? arXiv:1304.0968 [math.QA].

[9] V. G. Drinfeld, On Almost Cocommutative Hopf Algebras, Leningrad Math. J. 1 (1990), 321-342.

[10] R. Dijkgraaf, V. Pasquier, P. Roche, QuasiHopf algebras, group cohomology and orbifold models, Nucl. Phys. B. Proc. Suppl. 18B(1990), 60-72.

[11] G.I. Kac, Certain arithmetic properties of ring groups, Funct. Anal. Appl. 6 (1972) 158-160.

[12] R. G. Larson, Characters of Hopf algebras, J. Alg. 17 (1971), 352-368.

[13] G. Mason, The quantum double of a finite group and its role in conformal field theory, Groups 93 Galway/St. Andrews, LMS Lecture notes 212, Cambridge University Press, 1995 (2) 405-417.

[14] W. D. Nichols and M. B. Richmond, The Grothendieck algebra of a Hopf algebra I, Comm. Algebra 26 (1998), no. 4, 1081-1095.

[15] H-J Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. of Math., 72, (1990), 167-195.

[16] S. Skryabin, Projectivity and freeness over comodule algebras, Trans. Amer. Math. Soc. 359 (2007), 2597-2623.

[17] Takeuchi. Quotient Spaces for Hopf Algebras. Comm. Alg., 22(7):2503-2523, 1995.

[18] Y. Zhu, Hopf algebras of prime dimension, Int. Math. Res. Not. 91994), 53-59.

Received: 10.04.2013,

Accepted: 14.06.2013.

Department of Mathematics, Ben Gurion University, Beer Sheva, Israel
E-mail: mia@math.bgu.ac.il

Department of Management, Bar-Ilan University, Ramat-Gan, Israel
E-mail: swestric@biu.ac.il