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On the irreducibility of bivariate polynomials

by
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Dedicated to Professors Toma Albu and Constantin Năstăsescu
on the occasion of their 70th birthdays

Abstract

We study the irreducibility of general bivariate polynomials over algebraically closed fields of characteristic zero. We obtain factorization conditions in terms of the degree index and we deduce the irreducibility for classes of polynomials that include that of quasi–generalized difference polynomials.

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Introduction

There exist many consistent results on the irreducibility of bivariate polynomials. Many of them study the irreducibility of difference polynomials $f(X) - g(Y)$ and of their generalizations. One of the largest families of such polynomials is the class of quasi–generalized difference polynomials. We remind that, if k is a field, $F(X,Y) \in k[X,Y]$ is called a *quasi–generalized difference polynomial* if

$$F(X,Y) \,=\, cY^d + \sum_{i=1}^{d} P_i(X) Y^{d-i}\,,$$

with $c \in k \setminus \{0\}$, $d \in \mathbb{N}^*$, $P_i(X) \in k[X]$.
If

$$\frac{\deg(P_i)}{i} \,<\, \frac{\deg(P_d)}{d}, \quad 1 \leq i \leq d-1\,,$$

then F is called a *generalized difference polynomial*.

Various authors obtained irreducibility conditions for such polynomials, for example Abhyankar–Rubel [1], Angermüller [2], Ayad [3], Bhatia–Khanduja [4], Bishnoi–Khanduja–Sudesh [5], Cohen–Movahhedi–Salinier [6], Panaitopol–Ştefănescu [8], Rubel–Schinzel–Tverberg [9].

In this paper we shall study the irreducibility for some classes of bivariate polynomials of the form

$$F(X,Y) \;=\; \sum_{i=0}^{d} P_i(X)Y^{d-i} \in k[X,Y]\,,$$

where $P_0(X)$ can be a non–constant polynomial and k is an algebraically closed field of characteristic zero. This will be done using properties of the degree index associated to a general bivariate polynomial and factorization properties derived from them.

Irreducibility conditions

To the polynomial

$$F(X,Y) \;=\; \sum_{i=0}^{d} P_i(X)Y^{d-i} \in k[X,Y]$$

we associate the *degree-index*

$$p_Y(F) \;=\; \max_{1\le i\le n} \frac{\deg(P_i)-\deg(P_0)}{i}.$$

We have

Proposition 1. *If $F = F_1F_2$ is a nontrivial factorization of the polynomial $F \in k[X,Y]$ we have*

$$p_Y(F) \;=\; \max\{p_Y(F_1), p_Y(F_2)\}\,.$$

Proof: We suppose that

$$F(X,Y) \;=\; P_0(X)Y^d + P_1(X)Y^{d-1} + \cdots + P_{d-1}Y + P_d(X)$$

and we consider the formal power series $G(X,Y) = Y^dF(X^{-1},Y^{-1}) \in k((X))[Y]$. We use the representation

$$G(X,Y) \;=\; \sum_{i=0}^{d} H_i(X)Y^i\,,$$

where $H_i(X) = P_{d-i}(X^{-1}) \in k((X))$. We consider $r_i = \mathrm{ord}H_i(X)$ and

$$e(G) \;=\; \max_{1\le i\le d} \frac{r_0-r_i}{i}.$$

We put $d_1 = \deg(F_1)$, $d_2 = \deg(F_2)$ and we consider

$$\begin{aligned} G_1(X,Y) &= Y^{d_1}F_1(X^{-1},Y^{-1}), \quad \text{and}\\ G_2(X,Y) &= Y^{d_2}F_2(X^{-1},Y^{-1}). \end{aligned}$$

We have $G = G_1G_2$ in $k((X))[Y]$ and by results of G. Dumas [7] on the Newton polygon of a product of two polynomials we have

$$e(G) \;=\; \max\{e(G_1), e(G_2)\}\,.$$

On the other hand $r_i = \mathrm{ord} H_i(X) = \mathrm{ord} P_i(X^{-1}) = -\deg(P_i)$, so

$$\frac{r_0 - r_i}{i} = \frac{\deg(P_i) - \deg(P_0)}{i},$$

which gives the result. ∎

In the case P_0 is a nonzero constant it was proved that the polynomial $F(X,Y)$ is irreducible provided the degree index satisfies suitable conditions (see for example [2] and [8]).

Proposition 2. *If $F = F_1F_2$ is a factorization of the polynomial $F(X,Y) = \sum_{i=0}^{d} P_i(X)Y^{d-i} \in k[X,Y]$ with $F_1, F_2 \in k[X,Y] \setminus k$ and*

$$p_Y(F) = \frac{\deg(P_d) - \deg(P_0)}{d},$$

then $p_Y(F) = p_Y(F_1) = p_Y(F_2)$.

Proof: We consider the polynomial $G(X,Y) \in k[X,Y]$ defined by

$$G(X,Y) = P_0(X)^{d-1} F\left(X, \frac{Y}{P_0(X)}\right) = Y^d + \sum_{i=1}^{d} P_0(X)^{i-1} P_i(X) Y^{d-i}$$

Next, we consider the factorization $F = F_1F_2$ and we assume that

$$\begin{aligned} F_1(X,Y) &= Q_0(X)Y^{d_1} + \sum_{i=1}^{d_1} Q_i(X)Y^{d_1-i}, \\ F_2(X,Y) &= R_0(X)Y^{d_2} + \sum_{i=1}^{d_2} R_i(X)Y^{d_2-i}. \end{aligned}$$

We have $P_0 = Q_0R_0$ and we put

$$\begin{aligned} G_1(X,Y) &= R_0(X)P_0(X)^{d_1-1} F_1\left(X, \frac{Y}{P_0(X)}\right), \\ G_2(X,Y) &= Q_0(X)P_0(X)^{d_2-1} F_2\left(X, \frac{Y}{P_0(X)}\right). \end{aligned}$$

We observe that G, G_1 and G_2 are monic with respect to Y, and that

$$G(X,Y) = G_1(X,Y)G_2(X,Y).$$

For the computation of the degree index of the polynomial G we note that

$$\frac{\deg(P_0^{i-1}P_i)}{i} = \frac{(i-1)\deg(P_0) + \deg(P_i)}{i} = \deg(P_0) + \frac{\deg(P_i) - \deg(P_0)}{i},$$

which yields

$$p_Y(G) = \deg(P_0) + \max_{1\le i\le d} \frac{\deg(P_i) - \deg(P_0)}{i} = \deg(P_0) + p_Y(F).$$

On the other hand,

$$G_1(X,Y) = Y^{d_1} + \sum_{i=1}^{d_1} R_0 P_0^{i-1} Q_i Y^{d_1-i},$$

so for the computation of the degree index of G_1 we notice that

$$\begin{aligned}\frac{\deg(R_0P_0^{i-1}Q_i)}{i} &= \frac{\deg(R_0) + (i-1)\deg(P_0) + \deg(Q_i)}{i},\\ &= \deg(P_0) + \frac{\deg(Q_i) - \deg(Q_0)}{i}.\end{aligned}$$

Similarly, we have

$$G_2(X,Y) = Y^{d_2} + \sum_{i=1}^{d_2} Q_0 P_0^{i-1} R_i Y^{d_2-i},$$

and we obtain

$$\begin{aligned}\frac{\deg(Q_0P_0^{i-1}R_i)}{i} &= \frac{\deg(Q_0) + (i-1)\deg(P_0) + \deg(R_i)}{i},\\ &= \deg(P_0) + \frac{\deg(R_i) - \deg(R_0)}{i}.\end{aligned}$$

Therefore

$$\begin{aligned}p_Y(G_1) &= \deg(P_0) + \max_{1\le i\le d_1} \frac{\deg(Q_i) - \deg(Q_0)}{i} = \deg(P_0) + p_Y(F_1),\\ p_Y(G_2) &= \deg(P_0) + \max_{1\le i\le d_2} \frac{\deg(R_i) - \deg(R_0)}{i} = \deg(P_0) + p_Y(F_2).\end{aligned}$$

Let us put

$$\begin{aligned}m &= (d-1)\deg(P_0) + \deg(P_d),\\ m_1 &= \deg(R_0) + (d_1-1)\deg(P_0) + \deg(Q_{d_1}),\\ m_2 &= \deg(Q_0) + (d_2-1)\deg(P_0) + \deg(R_{d_2}).\end{aligned}$$

We obviously have $m = m_1 + m_2$ and $d = d_1 + d_2$, while from Theorem 1 in [8] we have $p_Y(G) = \max\{p_Y(G_1), p_Y(G_2)\}$. Therefore

$$\frac{m_1}{d_1} \le \frac{m}{d} = \frac{m_1+m_2}{d_1+d_2},$$

from which we deduce that $m_1d_2 \le m_2d_1$. Similarly, we have

$$\frac{m_2}{d_2} \le \frac{m}{d} = \frac{m_1+m_2}{d_1+d_2},$$

from which we deduce now that $m_1d_2 \geq m_2d_1$. This yields

$$\frac{m_1}{d_1} = \frac{m_2}{d_2} = \frac{m_1 + m_2}{d_1 + d_2} = \frac{m}{d},$$

which shows that $p_Y(G) = p_Y(G_1) = p_Y(G_2)$. All that remains now is to subtract $\deg(P_0)$ in these equalities, which leads us to $p_Y(F) = p_Y(F_1) = p_Y(F_2)$, and completes the proof. ∎

Remark: For the particular case when P_0 is a non-zero constant, our Proposition 2 gives Proposition 2 of Panaitopol-Ştefănescu in [8].

Corollary 3. *Let* $F(X,Y) = \sum_{i=0}^{d} P_i(X)Y^{d-i} \in k[X,Y]$, *with* $P_0P_d \neq 0$, *and* $d \geq 1$. *If* $p_Y(F) = \frac{\deg(P_d)-\deg(P_0)}{d}$ *and* $\gcd(\deg(P_d) - \deg(P_0), d) = 1$, *then* F *is irreducible in* $k[X,Y]$.

Proof: We may obviously assume that F is not divisible by a polynomial $f \in k[X]$. Let us assume now that $F = F_1F_2$ is a non-trivial factorization of F in $k[X,Y]$, $d_1 = \deg_Y(F1)$, $d_2 = \deg_Y(F2)$, and that F_1 and F_2 are represented as in the proof of Proposition 2. We have

$$p_Y(F_1) = p_Y(F_2) = p_Y(F) = \frac{\deg(P_d) - \deg(P_0)}{d}.$$

If follows that there exists an index $i \in \{1, \ldots, d_1\}$ such that

$$p_Y(F_1) = \frac{\deg(Q_i) - \deg(Q_0)}{i} = \frac{\deg(P_d) - \deg(P_0)}{d} = p_Y(F).$$

We put now $a = \deg(Q_i) - \deg(Q_0)$ and $m = \deg(P_d) - \deg(P_0)$, and we have

$$\frac{a}{i} = \frac{m}{d} \quad \text{with} \quad \gcd(m,d) = 1.$$

Therefore $ad = mi$, and since m and d are coprime integers, there exists a positive integer u such that $i = ud$, so in particular we must have $i \geq d$. On the other hand we obviously have $i \leq d_1 < d$, a contradiction. This shows that F must be irreducible in $k[X,Y]$, and completes the proof. ∎

Corollary 4 (Panaitopol–Ştefănescu, 1990)**.** *Let*

$$F(X,Y) = cY^d + \sum_{i=1}^{d} P_i(X)Y^{d-i},$$

where $c \in k \setminus \{0\}$, $d \in \mathbb{N}^*$, *and* $P_i(X) \in k[X]$. *If* $P_Y(F) = \deg(P_d)/d$ *and* $\gcd(\deg(P_d), d) = 1$, *the polynomial* F *is irreducible in* $k[X,Y]$.

Remark: Another extension of Corollary 4 was obtained by Bhatia–Khanduja [4].

We consider now bivariate polynomials F in $k[X,Y]$ for which the degree index is not equal to $(\deg(P_d)-\deg(P_0))/d$.

Theorem 5. *Let*

$$F(X,Y) = \sum_{i=0}^{d} P_i(X)Y^{n-i} \in k[X,Y],\ P_0P_d \neq 0$$

and assume that there exists an index $s \in \{1,2,...,d\}$ such that the following conditions are fulfilled:

(a) $\dfrac{\deg(P_i)-\deg(P_0)}{i} \leq \dfrac{deg(P_s)-\deg(P_0)}{s}$ *for all* $i \in \{1,2,...,d\}$*;*

(b) $\gcd(\deg(P_s)-\deg(P_0),s)=1$*;*

(c) $\dfrac{\deg(P_s)-\deg(P_0)}{s} - \dfrac{\deg(P_d)-\deg(P_0)}{d} = \dfrac{1}{sd}$.

Then $F(X,Y)$ is either irreducible in $k[X,Y]$, or has a factor whose degree with respect to Y is a multiple of s.

Proof: Let us suppose that there exists a non-trivial factorization $F = F_1F_2$ of the polynomial F in $k[X,Y]$. Let us put now $m = \deg(P_d)-\deg(P_0)$, $a = \deg(P_s)-\deg(P_0)$ and observe that by hypothesis (a) we have $p_Y(F) = a/s$, while by hypothesis (c) we have

$$\frac{a}{s} - \frac{m}{d} = \frac{1}{sd}.$$

Therefore $ad - sm = 1$. We suppose that

$$\begin{aligned} F_1(X,Y) &= Q_0(X)Y^{d_1} + \sum_{i=1}^{d_1} Q_i(X)Y^{d_1-i}, \\ F_2(X,Y) &= R_0(X)Y^{d_2} + \sum_{i=1}^{d_2} R_i(X)Y^{d_2-i}. \end{aligned}$$

and we also put $m_1 = \deg(Q_{d_1}) - \deg(Q_0)$ and $m_2 = \deg(R_{d_2}) - \deg(R_0)$.

By Proposition 1 we have

$$p_Y(F) = \max\{p_Y(F_1), p_Y(F_2)\},$$

hence we must have the inequalities

$$\frac{m_1}{d_1} \leq \frac{a}{s}, \quad \frac{m_2}{d_2} \leq \frac{a}{s}.$$

We deduce that

$$\frac{m_2}{d_2} = \frac{(\deg(P_d)-\deg(P_0)) - (\deg(Q_{d_1})-\deg(Q_0))}{d-d_1} = \frac{m-m_1}{d-d_1} \leq \frac{a}{s},$$

so $sm - sm_1 \leq ad - ad_1$, and it follows that

$$0 \leq ad_1 - sm_1 \leq ad - sm = 1.$$

Therefore $ad_1 - sm_1 \in \{0, 1\}$.

Let us first suppose that $ad_1 - sm_1 = 0$. In this case, since a and s are coprime integers, s must divide $d_1 = \deg_Y(F_1)$.

In the remaining case we have $ad_1 - sm_1 = 1$, which may be also written as $a(d - d_2) - s(m - m_2) = 1$, or equivalently that $(ad - sm) + (sm_2 - ad_2) = 1$. Now, since $ad - sm = 1$, we deduce that $sm_2 = ad_2$, and since a and s are coprime integers, s must divide $d_2 = \deg_Y(F_2)$.

We therefore conclude that either F is irreducible, or has a factor whose degree with respect to Y must be a multiple of s. This completes the proof of the theorem. ∎

Examples

1. Let $F(X,Y) = p(X)Y^d + q(x)Y + r(X) \in k[X,Y]$ where $p \neq 0$, $d > 1$. We suppose that $\deg(p) = 1$ and $\deg(q) = \deg(r) = 2$. We have

$$\frac{\deg(q) - \deg(p)}{d-1} = \frac{1}{d-1},$$

$$\frac{\deg(r) - \deg(p)}{d} = \frac{1}{d},$$

$$\frac{\deg(q) - \deg(p)}{d-1} - \frac{\deg(r) - \deg(p)}{d} = \frac{1}{(d-1)d}.$$

Since $\gcd(1, d-1) = 1$ we can apply Theorem 5 for $s = d-1$. Therefore either F is irreducible in $k[X,Y]$, or has a divisor whose degree with respect to Y is a multiple of $d-1$. Such a divisor must obviously have degree equal to $d-1$, so if F is reducible, then it must have a linear factor with respect to Y.

Therefore either F is irreducible, or has a divisor of the form $s(X)Y - t(X)$ with $s, t \in k[X]$, $s \neq 0$.

2. Let $F(X,Y) = P(X)Y^d + Q(X)Y^3 + R(X)Y + S(X) \in k[X,Y]$, where $P(X) \neq 0$, $d > 3$. If

$$\deg(P) = m \geq 1,\ \deg(Q) = n \leq m \quad \text{and} \quad \deg(R) = \deg(S) = m+1$$

the polynomial F is either irreducible in $k[X,Y]$, or has a divisor of degree 1 with respect to Y.

Proof: Here we will apply Theorem 5 too. We first observe that

$$\frac{\deg(Q) - \deg(P)}{d-3} = \frac{n-m}{d-3} \leq 0 < \frac{1}{d-1} = \frac{m+1-m}{d-1} = \frac{\deg(R) - \deg(P)}{d-1}.$$

On the other hand

$$\frac{\deg(R) - \deg(P)}{d-1} - \frac{\deg(S) - \deg(P)}{d} = \frac{1}{d-1} - \frac{1}{d} = \frac{1}{(d-1)d}.$$

Therefore $p_Y(F) = \dfrac{1}{d-1}$, and the hypotheses of Theorem 5 are satisfied. ∎

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