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Filippov lemma for a certain differential inclusion of fourth-order

by
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Abstract

We consider a boundary value problem for a fourth-order nonconvex differential inclusion and we establish some Filippov type existence results for this problem.

Key Words: Boundary value problem, differential inclusion, contractive set-valued map.
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1 Introduction

This paper is concerned with the following boundary value problem for fourth-order differential inclusions

$$L_4x(t) + a(t)x(t) \in F(t, x(t)) \quad a.e. \ ([0, T]), \tag{1.1}$$

$$L_ix(0) = L_ix(T), \quad i = 0, 1, 2, 3, \tag{1.2}$$

where $L_0x(t) = a_0(t)x(t)$, $L_ix(t) = a_i(t)(L_{i-1}x(t))'$, $i = 1, 2, 3$,
$L_4x(t) = (a_3(t)(a_2(t)(a_1(t)(a_0(t)x(t))')')')'$, $a(.), a_i(.) : [0, T] \to \mathbb{R}$ are continuous mappings, $a_0(t) \equiv 1$, $a(t) \geq 0$, $a_i(t) > 0$, $i = 1, 2$, $t \in [0, T]$, $a_1(t) \equiv a_3(t)$ and $F(., .) : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map.

The present paper is motivated by relatively recent papers of Švec ([15]) and Arara, Benchohra, Ntouyas and Ouahab ([3]), where several existence results for problem (1.1)-(1.2) are provided. In [15], by means of the Ky Fan fixed point theorem it is obtained an existence result for problem (1.1)-(1.2), when $F(., .)$ is upper semicontinuous and has convex compact values. In [3] the situation when $F(., .)$ has nonconvex values is investigated and two existence results are obtained using the Covitz and Nadler multivalued contraction principle and the Bressan and Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values.

The aim of our paper is to present two additional existence results for problem (1.1)-(1.2) which are Filippov type existence results for this problem. The first one is obtained by the application of the set-valued contraction principle in the space of derivatives of solutions instead of the space of solutions as in [3]. In addition, as usual at a Filippov existence type theorem, our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. The idea of applying the set-valued contraction principle in the space of derivatives of the solutions belongs to Tallos ([11,16]) and it was already used for other classes of differential inclusions.

In our second approach we show that Filippov's ideas ([10]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([10]) consists in proving the existence of a solution starting from a given "quasi" solution.

For the motivation of study of problem (1.1)-(1.2) we refer to [3,15] and references therein. Several qualitative properties of the solutions of forth-order differential equations and inclusions may be found in [1,2,5,8,9,12,14] etc..

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space and consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},\ d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

The set-valued contraction principle ([7]) states that if X is complete, and $T : X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then $T(.)$ has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1 ([13]) *Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z \in X} d(T_1(z), T_2(z)).$$

Let $I = [0, T]$, we denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $||x(.)||_C = \sup_{t \in I} |x(t)|$ and $L^1(I, \mathbb{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbb{R}$ endowed with the norm $||u(.)||_1 = \int_0^T |u(t)|dt$. If $J \subset \mathbb{R}$ is an interval, by $AC^i(J, \mathbb{R})$, $i = 0, 1, 2, 3$ we denote the space of i-times differentiable functions $x(.) : J \to \mathbb{R}$ whose i^{th} derivative $x^{(i)}(.)$ is absolutely continuous.

A function $x(.) \in AC^3(I,\mathbb{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $f(.) \in L^1(I,\mathbb{R})$ with $f(t) \in F(t,x(t))$ *a.e.* (I) such that $L_i x(0) = L_i x(T)$, $i = 0,1,2,3$.

Lemma 2.2 ([15]) *The boundary value problem*

$$L_4 x(t) + a(t)x(t) = 0, \tag{2.1}$$

$$L_i x(0) = L_i x(T), \quad i = 0,1,2,3, \tag{2.2}$$

has only the trivial solution $x(t) \equiv 0$ *on* I.

Therefore, if $f(.) : [0,T] \to \mathbb{R}$ *is an integrable function, there exists the Green function* $G(.,.)$ *for problem*

$$L_4 x(t) + a(t)x(t) = f(t), \tag{2.3}$$

$$L_i x(0) = L_i x(T), \quad i = 0,1,2,3, \tag{2.4}$$

and the solution of problem (2.3)-(2.4) *is given by*

$$x(t) = \int_0^T G(t,s)f(s)ds. \tag{2.5}$$

According to [15] the Green function $G(.,.)$ is bounded. Let $G_0 := \sup_{t,s\in I} |G(t,s)|$.

3 The main results

We study first problem (1.1)-(1.2) with fixed point techniques. In order to do this we introduce the following hypothesis.

Hypothesis 3.1 (i) $F(.,.) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty closed values and for every* $x \in \mathbb{R}$, $F(.,x)$ *is measurable.*

(ii) *There exists* $L(.) \in L^1(I,\mathbb{R}_+)$ *such that for almost all* $t \in I$, $F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \leq L(t)|x-y| \quad \forall\ x,y \in \mathbb{R}.$$

(iii) $d(0, F(t,0)) \leq L(t)$ *a.e.* (I)

Denote $L_0 := \int_0^1 L(s)ds$.

Theorem 3.2 *Assume that Hypothesis* 3.1 *is satisfied and* $G_0 L_0 < 1$. *Let* $y(.) \in AC^3(I,\mathbb{R})$ *be such that* $L_i y(0) = L_i y(T)$, $i = 0,1,2,3$ *and there exists* $q(.) \in L^1(I,\mathbb{R}_+)$ *with* $d(L_4 y(t) + a(t)y(t), F(t,y(t))) \leq q(t)$, *a.e.* (I).

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem* (1.1)-(1.2) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{G_0}{1 - G_0 L_0} \int_0^T q(t)dt + \varepsilon.$$

Proof. For $u(.) \in L^1(I,\mathbb{R})$ define the following set-valued maps

$$M_u(t) = F(t, \int_0^T G(t,s)u(s)ds), \quad t \in I,$$

$$T(u) = \{\phi(.) \in L^1(I,\mathbb{R}); \quad \phi(t) \in M_u(t) \quad a.e.\ (I)\}.$$

It follows from Lemma 2.2 that $x(.)$ is a solution of problem (1.1) if and only if $L_4x(.) + a(.)x(.)$ is a fixed point of $T(.)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I,\mathbb{R})$. The fact that the set valued map $M_u(.)$ is measurable is well known. For example the map $t \to \int_0^T G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [6]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [6]) we infer that $M_u(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, \int_0^T G(t,s)u(s)ds)) \leq$$

$$\leq L(t)(1 + G_0 \int_0^T |u(s)|ds),$$

which shows that $\phi \in L^1(I,\mathbb{R})$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(.)$ is a contraction on $L^1(I,\mathbb{R})$.

Let $u, v \in L^1(I,\mathbb{R})$ be given and $\phi \in T(u)$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \{x \in \mathbb{R}; \quad |\phi(t) - x| \leq L(t)|\int_0^T G(t,s)(u(s) - v(s))ds|\}.$$

From Proposition III.4 in [6], $H(.)$ is measurable and from Hypothesis 3.1 ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_0^T |\phi(t) - \psi(t)|dt \leq \int_0^T L(t)(\int_0^T |G(t,s)|.|u(s) - v(s)|ds)dt$$

$$= \int_0^T (\int_0^T L(t)|G(t,s)|dt)|u(s) - v(s)|ds \leq G_0L_0||u - v||_1.$$

We deduce that

$$d(\phi, T(v)) \leq G_0L_0||u - v||_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq G_0L_0||u - v||_1,$$

thus $T(.)$ is a contraction on $L^1(I,\mathbb{R})$.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + q(t)[-1,1], \quad (t,x) \in I \times \mathbb{R},$$

$$M_u^1(t) = F_1(t, \int_0^T G(t,s)u(s)ds),$$

$$T_1(u) = \{\psi(.) \in L^1(I, \mathbb{R}); \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\}, \quad u(.) \in L^1(I, \mathbb{R}).$$

Obviously, $F_1(.,.)$ satisfies Hypothesis 3.1.

Repeating the previous step of the proof we obtain that T_1 is also a G_0L_0-contraction on $L^1(I, \mathbb{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq \int_0^T q(t)dt. \tag{3.1}$$

Let $\phi \in T(u)$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbb{R}; \quad |\phi(t) - z| \leq q(t)\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_1(u)$ and one has

$$||\phi - \psi||_1 = \int_0^T |\phi(t) - \psi(t)|dt \leq \int_0^T q(t).$$

As above we obtain (3.1).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1 - G_0L_0} \int_0^T q(t)dt.$$

Since $v(.) = L_4y(.) + a(.)y(.) \in Fix(T_1)$ it follows that for any $\varepsilon > 0$ there exists $u(.) \in Fix(T)$ such that

$$||v - u||_1 \leq \frac{1}{1 - G_0L_0} \int_0^T q(t)dt + \frac{\varepsilon}{G_0}.$$

We define $x(t) = \int_0^T G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq \int_0^T |G(t,s)|.|u(s) - v(s)|ds \leq \frac{G_0}{1 - G_0L_0} \int_0^T q(t)dt + \varepsilon$$

which completes the proof.

The assumption in Theorem 3.2 is satisfied, in particular, for $y(.) = 0$ and therefore, via Hypothesis 3.1 (iii), with $q(.) = L(.)$. We obtain the following consequence of Theorem 3.2.

Corollary 3.3 *Assume that Hypothesis* 3.1 *is satisfied and* $G_0L_0 < 1$. *Then for every* $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem* (1.1)-(1.2) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{G_0L_0}{1 - G_0L_0} + \varepsilon. \tag{3.2}$$

Remark 3.4 Corollary 3.3 is an improvement of Theorem 3.3 in [3]. According to Theorem 3.3 in [3] if Hypothesis 3.1 is satisfied, $G_0L_0 < 1$ and $F(.,.)$ has compact values then problem (1.1)-(1.2) has at least a solution. Obviously, in Corollary 3.3 we do not assume that the values of $F(.,)$ are compact. Moreover, in (3.2) we obtained a priori bounds for the solution.

We present next the main result of this paper. Its proof uses Filippov's construction of successive approximations ([10]).

Theorem 3.5 *Assume that Hypothesis* 3.1 (i), (ii) *is satisfied and* $G_0L_0 < 1$*. Let* $y(.) \in AC^3(I,\mathbb{R})$ *be such that* $L_iy(0) = L_iy(T)$, $i = 0,1,2,3$ *and there exists* $q(.) \in L^1(I,\mathbb{R}_+)$ *with* $d(L_4y(t) + a(t)y(t), F(t,y(t))) \leq q(t)$, *a.e.* (I).

Then there exists $x(.)$ *a solution of problem* (1.1)-(1.2) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{G_0}{1 - G_0L_0}\int_0^T q(t)dt. \tag{3.3}$$

Proof. The set-valued map $t \to F(t,y(t))$ is measurable with closed values and

$$F(t,y(t)) \cap \{L_4y(t) + a(t)y(t) + q(t)[-1,1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows (e.g., Theorem 1.14.1 in [4]) that there exists a measurable selection $f_1(t) \in F(t,y(t))$ *a.e.* (I) such that

$$|f_1(t) - L_4y(t) - a(t)y(t)| \leq q(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t) = \int_0^T G(t,s)f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq G_0\int_0^T q(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I,\mathbb{R})$, $f_n(.) \in L^1(I,\mathbb{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^T G(t,s)f_n(s)ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t,x_{n-1}(t)) \quad a.e.\,(I),\ n \geq 1, \tag{3.6}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad a.e.\,(I), n \geq 1. \tag{3.7}$$

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^T |G(t,t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \leq$$

$$G_0\int_0^T L(t_1)|x_n(t_1) - x_{n-1}(t_1)|dt_1 \leq G_0\int_0^T L(t_1)\int_0^T |G(t_1,t_2)|.$$

$$|f_n(t_2) - f_{n-1}(t_2)|dt_2 \leq G_0^2 \int_0^T L(t_1) \int_0^T L(t_2)|x_{n-1}(t_2) - x_{n-2}(t_2)|dt_2 dt_1$$

$$\leq (G_0)^n \int_0^T L(t_1) \int_0^T L(t_2) ... \int_0^T L(t_n)|x_1(t_n) - y(t_n)|dt_n ... dt_1 \leq$$

$$\leq (G_0 L_0)^n G_0 \int_0^T q(t)dt.$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I, \mathbb{R})$, hence converging uniformly to some $x(.) \in C(I, \mathbb{R})$. Therefore, by (3.7), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{aligned} &|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq \\ &G_0 \int_0^T q(t)dt + \textstyle\sum_{i=1}^{n-1} (G_0 \int_0^T q(t)dt)(G_0 L_0)^i = \frac{G_0 \int_0^T q(t)dt}{1 - G_0 L_0}. \end{aligned} \tag{3.8}$$

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all $t \in I$

$$\begin{aligned} &|f_n(t) - L_4 y(t) - a(t)y(t)| \leq \textstyle\sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - L_4 y(t) \\ &-a(t)y(t)| \leq L(t) \frac{G_0 \int_0^T q(t)dt}{1 - G_0 L_0} + q(t). \end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I, \mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that $x(.)$ is a solution of (1.1)-(1.2). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.5)-(3.7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I, \mathbb{R})$ and $f_n(.) \in L^1(I, \mathbb{R})$, $n = 1, 2, ...N$ satisfying (3.5), (3.7) for $n = 1, 2, ...N$ and (3.6) for $n = 1, 2, ...N - 1$. The set-valued map $t \to F(t, x_N(t))$ is measurable. Moreover, the map $t \to L(t)|x_N(t) - x_{N-1}(t)|$ is measurable. By the lipschitzianity of $F(t, .)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1, 1]\} \neq \emptyset.$$

Theorem 1.14.1 in [4] yields that there exist a measurable selection $f_{N+1}(.)$ of $F(., x_N(.))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad a.e.\ (I).$$

We define $x_{N+1}(.)$ as in (3.5) with $n = N + 1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete.

Remark 3.6 Obviously, Theorem 3.5 extends Theorem 3.2. We do not suppose that $d(0, F(t, 0)) \leq L(t)$ *a.e.* (I) and the estimate in (3.3) is better than the one in Theorem 3.2.

Even if Theorem 3.5 improves Theorem 3.2, we chosen to present both results; on one hand because the methods used in their proofs are different and on the other hand to show that there exists situations when the fixed point approaches are less powerful.

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