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The category of linear modular lattices

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

In this paper we introduce and investigate a new category whose objects are the bounded modular lattices and whose morphisms are the so called linear morphisms.

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Introduction

The aim of this paper is to introduce and investigate a new category whose objects are the bounded modular lattices and whose morphisms are the so called linear morphisms.

In Section 0 we present some notation and terminology on lattices. Section 1 is devoted to the introduction of a special class of maps between lattices that evoke the morphisms of modules, and for this reason we call them linear morphisms.

In Section 2 we show that the class of all bounded modular lattices together with morphisms between two such lattices as linear morphisms has a natural structure of a category we denote $\mathcal{LM}$. We also describe the monomorphisms and epimorphisms of this category and the subobjects of any of its objects.

In Section 3 we present a latticial version of the concept of an injective module with respect to a class of modules, namely that of a linear $\mathcal{C}$-injective lattice, where $\mathcal{C}$ is a nonempty class of lattices, by using the linear morphims defined in Section 1. In particular one obtains the concept of a linear injective lattice; when restricting to bounded modular lattices, the linear injective lattices are precisely the injective objects of the category $\mathcal{LM}$ investigated in Section 2. Then, we define the concept of a linear $\mathcal{C}$-injective hull of a lattice and show that it is unique up to a lattice isomorphism.

Section 4 provides some examples and counter-examples of linear $\mathcal{C}$-injective lattices and investigates the connection between linear $\mathcal{C}$-injectivity and $\mathcal{C}$-injectivity. Several open questions on these concepts are also presented.

0 Preliminaries

All lattices considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1, in other words they are bounded. For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{ \ x \in L \mid a \leqslant x \leqslant b \ \}.$$

An *initial interval* (resp. a *quotient interval*) of b/a is any interval c/a (resp. b/c) for some $c \in b/a$. In particular, $c/0$ (resp. $1/c$) is an initial (resp. quotient) interval of $L = 1/0$ for any $c \in L$.

We denote by $\mathcal{L}$ (resp. $\mathcal{M}$) the class of all lattices (resp. modular lattices). Throughout this paper, a lattice will always mean a member of $\mathcal{L}$, and $(L, \leqslant, \wedge, \vee, 0, 1)$, or more simply, just L, will always denote such a lattice.

The class $\mathcal{L}$ is in fact a category, whose morphisms are the usual lattice morphisms. If the lattices L and L' are isomorphic (in the category $\mathcal{L}$) we denote this by $L \simeq L'$. Note that a surjective lattice morphism, in particular a lattice isomorphism, preserves both the least and last elements. This is not the case in general as the following simple example shows: the map $f : [0, 1] \longrightarrow [0, 3]$ between the intervals $[0, 1]$ and $[0, 3]$ of the set $\mathbb{R}$ of real numbers defined by $f(x) = x + 1$, $x \in [0, 1]$, is a morphism of complete lattices, that preserves neither the least nor the last elements.

For all undefined notation and terminology on lattices, the reader is referred to Birkhoff [3], Crawley and Dilworth [4], and/or Grätzer [5].

Throughout this paper R will denote an associative ring with nonzero identity element, and Mod-R the category of all unital right R-modules. The notation M_R will be used to designate a unital right R-module M, and $N \leqslant M$ will mean that N is a submodule of M. The lattice of all submodules of a module M_R will be denoted by $\mathcal{L}(M_R)$.

1 Linear lattice morphisms

In this section we define the concept of a linear morphism between lattices, that evokes the property of a linear map $\varphi : M \longrightarrow N$ between modules M_R and N_R to have a kernel $\mathrm{Ker}\,\varphi$ and to verify the Fundamental Theorem of Isomorphism: $M/\mathrm{Ker}\,\varphi \simeq \mathrm{Im}\,\varphi$.

Definitions 1.1. *Let $f : L \longrightarrow L'$ be a map between the lattice L with least element 0 and last element 1 and the lattice L' with least element $0'$ and last element $1'$.*

(1) *f is called a* linear morphism *if there exist $k \in L$, called a* kernel *of f, and $a' \in L'$ such that the following two conditions are satisfied.*

i) $f(x) = f(x \vee k), \forall\, x \in L$.

ii) f *induces an isomorphism of lattices* $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x), \forall\, x \in 1/k$.

(2) f *is called a* linear monomorphism (*resp.* essential linear monomorphism) *if there exists an* $a' \in L'$ (*resp. an essential element* $a' \in L'$) *such that* f *induces an isomorphism of lattices* $\bar{f} : L \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x), \forall\, x \in L$.

By definition, a linear monomorphism of lattices is an injective morphism of lattices, but the converse is false. For example, the canonical inclusion map of the set $\mathbb{Q}$ of all rational numbers into the set $\mathbb{R}$ of all real numbers is an injective morphism of lattices, but it is not a linear monomorphism of lattices.

Note that any lattice isomorphism is a (bijective) linear morphism and for any linear morphism $f : L \longrightarrow L'$ one has $f(L) = a'/0'$, where $a' = f(1)$.

Examples 1.2. (1) Let $\varphi : M_R \longrightarrow M'_R$ be a morphism of modules, and consider the map $f : \mathcal{L}(M_R) \longrightarrow \mathcal{L}(M'_R)$ defined by $f(N) = \varphi(N)$ for every $N \leqslant M$. Then f is a linear morphism with kernel $\operatorname{Ker}\varphi$. □

(2) For any lattice L and any $a \leqslant b$ in L, the map $p : b/0 \longrightarrow b/a$, $p(x) := x \vee a$, is a surjective linear morphism with kernel a, as it can be easily seen. This linear morphism is the latticial counterpart of the canonical surjective map from any module M_R to the factor module M/N, where N is any submodule of M. □

Examples 1.2 (1) show that a linear morphism is not necessarily a morphism of lattices. However, it is a morphism of join-semilattices, as this is shown in the next result.

Proposition 1.3. *The following assertions hold for a linear morphism* $f : L \longrightarrow L'$ *with a kernel* k.

(1) *For* $x, y \in L$, $f(x) = f(y) \iff x \vee k = y \vee k$.

(2) $f(k) = 0'$ *and* k *is the last element of* L *having this property; so, the kernel of a linear morphism is unique.*

(3) *If* $a \in L$ *is such that* $f(a) = 0'$, *then* f *induces a linear morphism*

$$h : 1/a \longrightarrow L',\ h(x) = f(x), \forall\, x \in 1/a.$$

(4) $f(x \vee y) = f(x) \vee f(y), \forall\, x, y \in L$.

Proof: By definition, there exists $a' \in L'$ such that f induces an isomorphism of lattices $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x), \forall\, x \in 1/k$.

(1) "$\Longrightarrow$" Suppose that $f(x) = f(y)$ for $x, y \in L$. Then $f(x \vee k) = f(y \vee k)$. Since $x \vee k$ and $y \vee k$ are both in $1/k$, it follows that $\bar{f}(x \vee k) = \bar{f}(y \vee k)$. But $\bar{f}$ is an isomorphism, so $x \vee k = y \vee k$.

"$\Longleftarrow$" We have $f(x) = f(x \vee k) = f(y \vee k) = f(y)$.

(2) Since $\bar{f} : 1/k \overset{\sim}{\longrightarrow} a'/0'$ is an isomorphism of lattices for some $a' \in L'$, we have $f(k) = \bar{f}(k) = 0'$. If $l \in L$ is such that $f(l) = 0'$, then $f(k) = f(l)$. Using now (1) we obtain $k = k \vee k = l \vee k$, and so $l \leqslant k$.

(3) Since $f(a) = 0'$, we have $a \leqslant k$ by (2); so, we can take $h := i \circ \bar{f} \circ p$, where p is the canonical linear morphism $p : 1/a \longrightarrow 1/k$, $p(x) = x \vee k, \forall\, x \in 1/a$, described in Examples 1.2 (2) and $i : a'/0' \hookrightarrow L'$ is the canonical inclusion map. Indeed,

$$h(x) = (i \circ \bar{f} \circ p)(x) = \bar{f}(p(x)) = \bar{f}(x \vee k) = f(x \vee k) = f(x), \forall\, x \in 1/a,$$

so h is a linear morphism with kernel k.

(4) Set $x_1 = x \vee k$ and $y_1 = y \vee k$. Notice that $x_1 \in 1/k$ and $y_1 \in 1/k$, so $x_1 \vee y_1 \in 1/k$. We have

$$f(x \vee y) = f(x \vee y \vee k) = f(x_1 \vee y_1) = \bar{f}(x_1 \vee y_1).$$

Since $\bar{f}$ is an isomorphism of lattices, it follows that

$$\bar{f}(x_1 \vee y_1) = \bar{f}(x_1) \vee \bar{f}(y_1).$$

Thus

$$f(x \vee y) = \bar{f}(x_1) \vee \bar{f}(y_1) = f(x \vee k) \vee f(y \vee k) = f(x) \vee f(y),$$

as desired. □

Corollary 1.4. *A linear morphism $f : L \longrightarrow L'$ between two lattices is an increasing map.*

Proof: Let $x \leqslant y$ in L. Then, $f(y) = f(x \vee y) = f(x) \vee f(y)$ by Proposition 1.3 (4), so $f(x) \leqslant f(y)$. □

Corollary 1.5. *For any linear morphism $f : L \longrightarrow L'$ one has $f(0) = 0'$.*

Proof: Let k be the kernel of f. Then $0' \leqslant f(0) \leqslant f(k) = 0'$ by Corollary 1.4 and Proposition 1.3 (2), so $f(0) = 0'$. □

Corollary 1.6. *The following assertions are equivalent for a map $f : L \longrightarrow L'$ between two lattices L and L'.*

(1) *f is a linear monomorphism.*

(2) *f is an injective linear morphism.*

(3) *f is a linear morphism with kernel zero.*

Proof: (1)$\Longrightarrow$(2): By definition.

(2)$\Longrightarrow$(3): Let k be the kernel of f. Then $f(0) = f(0 \vee k) = f(k)$, so $k = 0$ because f is injective.

(3)$\Longrightarrow$(1): If k is the kernel of f, then $k = 0$ by hypothesis, so $1/k = 1/0 = L \simeq a'/0'$ for some $a' \in L'$, and then, f is a linear monomorphism by definition. □

2 The category $\mathcal{LM}$ of linear modular lattices

In this section we show that the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, and called the category of linear modular lattices, with morphisms between two such lattices as linear morphisms. Then, we describe the monomorphisms and epimorphisms of this category and the subobjects of any of its objects. Note that the concept of a linear lattice appears sometimes in the literature with a completely other meaning than ours (see, e.g., Grätzer [5, p. 467]).

Lemma 2.1. *Let L, L', L'' be lattices, and let $f : L \longrightarrow L'$, $g : L' \longrightarrow L''$ be linear morphisms (resp. essential linear monomorphisms). If the lattice L' is modular, then $g \circ f$ is also a linear morphism (resp. essential linear monomorphism).*

Proof: Let k be the kernel of f and k' the kernel of g. Then, $\exists\, a' \in L'$, $a'' \in L''$ such that $\forall\, x \in L$, $x' \in L'$ we have

$$f(x) = f(x \vee k) \quad \text{and} \quad 1/k \simeq a'/0',$$

$$g(x') = g(x' \vee k') \quad \text{and} \quad 1'/k' \simeq a''/0''.$$

Denote by $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$ the lattice isomorphism induced by f and by $\bar{g} : 1'/k' \xrightarrow{\sim} a''/0''$ the lattice isomorphism induced by g. Set $b' := k' \wedge a'$, $c'' := g(a')$, and $l := \bar{f}^{-1}(b')$. We claim that

$$(g \circ f)(x \vee l) = (g \circ f)(x), \forall\, x \in L,$$

and $g \circ f$ induces a lattice isomorphism $1/l \simeq c''/0''$.

Indeed, if we set $x' := f(x)$, by Proposition 1.3 (4) and Corollary 1.4, we have

$$(g \circ f)(x \vee l) = g(f(x) \vee f(l)) = g(x' \vee b') = g(x' \vee (k' \wedge a')) \leqslant g(x' \vee k') = g(x') = (g \circ f)(x).$$

On the other hand, again by Corollary 1.4, we have

$$(g \circ f)(x) = g(f(x)) \leqslant g(f(x) \vee f(l)) = g(x' \vee b') = (g \circ f)(x \vee l).$$

This shows that

$$(g \circ f)(x \vee l) = (g \circ f)(x), \forall\, x \in L,$$

as desired.

We are now going to prove that $g \circ f$ induces a lattice isomorphism

$$1/l \simeq c''/0'',$$

where $c'' := g(a') = g(a' \vee k')$. Since $\bar{g} : 1'/k' \xrightarrow{\sim} a''/0''$ is the lattice isomorphism induced by the increasing map g, it produces by restriction an isomorphism $(a' \vee k')/k' \simeq c''/0''$. Using the modularity of L', we deduce the following lattice isomorphisms induced by the linear morphisms g and f:

$$c''/0'' \simeq (a' \vee k')/k' \simeq a'/(a' \wedge k') = a'/b' \simeq 1/l,$$

which shows that $g \circ f$ is a linear morphism.

Finally, we show that if f and g are both essential linear monomorphisms, then so is also $g \circ f$. First, observe that, by Corollary 1.6, the linear morphism $g \circ f$ is a linear monomorphism. Clearly, we have $k = 0$ and $k' = 0'$. Then $b' = 0'$, and so $l = k = 0$. To finish the proof, we have to show that and $c'' = g(a')$ is an essential element of L''. Indeed, since a' is an essential element of $L' = 1'/0'$, using the lattice isomorphism $\bar{g} : 1'/0' \xrightarrow{\sim} a''/0''$ induced by g, we deduce that $c'' = \bar{g}(a') = g(a')$ is an essential element of $a''/0''$. On the other hand, a'' is an essential element $L'' = 1''/0''$ because g is an essential linear monomorphism by hypothesis. Since the relation of "being essential" is transitive, we conclude that c'' is an essential element of $L'' = 1''/0''$, as desired. □

Proposition 2.2. *The following statements hold.*

(1) *The class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, if for any $L, L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'.*

(2) *The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices.*

(3) *The monomorphisms in the category $\mathcal{LM}$ are exactly the injective linear morphisms, i.e., the linear monomorphisms defined in 1.1.*

(4) *The epimorphisms in the category $\mathcal{LM}$ are exactly the surjective linear morphisms.*

(5) *The subobjects of $L \in \mathcal{LM}$ can be taken as the intervals $a/0$ for any $a \in L$.*

Proof: (1) First, notice that the identity map 1_L of any $L \in \mathcal{M}$ is clearly a linear morphism. Then, use Lemma 2.1 to deduce that $\mathcal{LM}$ is indeed a category.

(2) is obvious.

(3) Let $L, L' \in \mathcal{LM}$, and let $f : L \longrightarrow L'$ be an injective linear morphism, i.e., a linear monomorphism by Corollary 1.6. Then clearly f is a monomorphism in the category $\mathcal{LM}$.

Assume now that $f : L \longrightarrow L'$ is a monomorphism in the category $\mathcal{LM}$ and show that it is an injective map. By Corollary 1.6, it is sufficient to show that the kernel k of f is zero. Assume that $k \neq 0$, and denote $K := k/0$. Consider the diagram

$$K \underset{o}{\overset{\iota}{\rightrightarrows}} L \xrightarrow{f} L',$$

where ι is the inclusion map and o is the zero map $o(x) = 0, \forall\, x \in K$. Note that ι and o are linear morphisms and $f \circ \iota = f \circ o$ since $0' \leqslant (f \circ \iota)(x) = f(x) \leqslant f(k) = 0'$ and $(f \circ o)(x) = f(0) = 0', \forall\, x \in K$, by Corollary 1.5. But $\iota \neq o$ because we assumed that $k \neq 0$, which contradicts the fact that f is a monomorphism in $\mathcal{LM}$. This proves that f is an injective map, as desired.

(4) Clearly, any surjective linear morphism is an epimorphism in the category $\mathcal{LM}$. Conversely, let $f : L \longrightarrow L'$ be an epimorphism in the category $\mathcal{LM}$ and prove that it is an surjective map. Then, $f(L) = a'/0'$ for some $a' \in L'$ because f is a linear morphism, so it is sufficient to show that $a' = 1'$. Assume that $a' < 1'$ and set $C := 1'/a'$. Consider the diagram

$$L \xrightarrow{f} L' \underset{\alpha}{\overset{p}{\rightrightarrows}} C,$$

where $p : L' \longrightarrow 1'/a', p(x') = x' \vee a',$ is the canonical surjective linear morphism defined in Examples 1.2 (2), and α is the linear morphism $\alpha(x') = a', \forall\, x' \in L'$. We have $p \circ f = \alpha \circ f$ because $a' \leqslant (p \circ f)(x) = p(f(x)) \leqslant p(a') = a' = (\alpha \circ f)(x), \forall\, x \in L$, but $p \neq \alpha$ because $1' = p(1') \neq \alpha(1') = a'$; this contradicts the fact that f is an epimorphism in $\mathcal{LM}$, and so, f is necessarily a surjective map.

(5) Let (S, α) be a subobject of $L \in \mathcal{LM}$. This means that $S \in \mathcal{M}$ and $\alpha : S \rightarrowtail L$ is a "representing" monomorphism in the category $\mathcal{LM}$. By definition, α induces a lattice isomorphism $S \simeq a/0$, and so, in the equivalence class of (S, α) we may choose the representing monomorphism $(a/0, i)$, where i is the inclusion map $a/0 \hookrightarrow L$. □

Remark 2.3. We have seen in Section 1 that for any linear morphism $f : L \longrightarrow L'$ the kernel k of f and the element $f(1) = a' \in L'$ such that f induces a lattice isomorphism $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$ are uniquely determined by f. For this reason, in analogy with the usual notation from Module Theory, we may denote

$$\mathrm{Ker}\, f := k, \;\; \mathrm{Im}\, f := a', \;\; \mathrm{Coim}\; f := 1/k, \;\; \mathrm{Coker}\; f := 1'/a'.$$

So, though the category $\mathcal{LM}$ is far to be pre-additive, it satisfies the axioms AB 1) and AB 2) of Grothendieck.

3 Linear injective lattices

Recall that a module Q_R is said to be M-*injective*, where M_R is another module, if for every submodule $N \leqslant M$, every morphism $N \longrightarrow Q$ can be extended to a morphism $M \longrightarrow Q$. If $\mathcal{A}$ is a nonempty class of right R-modules, then Q is called $\mathcal{A}$-*injective* if it is M-injective for every $M \in \mathcal{A}$.

When thinking to obtain latticial counterparts of these module-theoretical concepts, there are at least two options, depending on what kind of morphisms are we taking into account: usual lattice morphisms or linear morphisms of lattices. The closest latticial counterpart to the module case is however the second option because the main feature of module morphisms are modeled, in our opinion, by linear morphisms of lattices.

The purpose of this section is to define and investigate the concept of a linear $\mathcal{C}$-injective lattice, where $\mathcal{C}$ is a nonempty class of lattices.

Definitions 3.1. *Let* $Q, L \in \mathcal{L}$*. The lattice* Q *is said to be* linear L-injective *if for every element* $a \in L$*, every linear morphism* $a/0 \longrightarrow Q$ *can be extended to a linear morphism* $L \longrightarrow Q$*.*

The lattice Q *is said to be* L-injective *if for every sublattice* S *of* L*, every lattice morphism* $S \longrightarrow Q$ *can be extended to a lattice morphism* $L \longrightarrow Q$*.*

If $\mathcal{C}$ *is a nonempty class of lattices, then* Q *is said to be* linear $\mathcal{C}$-injective (*resp.* $\mathcal{C}$-injective) *if it is* linear L-injective (*resp.* L-injective) *for every* $L \in \mathcal{C}$*.*

The lattice Q *is called* linear injective (*resp.* injective) *if it is linear* $\mathcal{L}$*-injective* (*resp.* $\mathcal{L}$-injective)*.* □

Note that the injective lattices are exactly the injective objects of the category $\mathcal{L}$. If we restrict now our considerations from the class $\mathcal{L}$ of all lattices to the class $\mathcal{M}$ of all modular lattices, then, in view of Proposition 2.2, the linear injective modular lattices are precisely the injective objects of the category $\mathcal{LM}$.

Examples and counter-examples of linear $\mathcal{C}$-injective lattices and $\mathcal{C}$-injective lattices will be provided in Section 4. As expected, we will see there that there is no connection between these two sorts of injectivity.

The next result presents a characterization, possibly known, of $\mathcal{A}$-injective modules in terms of essential submodules, where $\mathcal{A}$ is a nonempty class of right R-modules.

Proposition 3.2. *The following assertions are equivalent for a module* Q_R *and a nonempty class* $\mathcal{A}$ *of right* R*-modules which is closed under factor modules.*

(1) Q_R *is* $\mathcal{A}$*-injective.*

(2) *For every module* $M_R \in \mathcal{A}$ *and for every essential submodule* N *of* M*, every monomorphism* $N \longrightarrow Q$ *can be extended to a monomorphism* $M \longrightarrow Q$*.*

Proof: (1) $\Longrightarrow$ (2): Suppose that Q_R is $\mathcal{A}$-injective. Let $M_R \in \mathcal{A}$ be a module, let N be an essential submodule of M, and let $f : N \longrightarrow Q$ be a monomorphism. Since Q_R is $\mathcal{A}$-injective, f can be extended to a morphism $g : M \longrightarrow Q$. Now $(\operatorname{Ker} g) \cap N = \operatorname{Ker} f = 0$. Since N is an essential submodule of M, we deduce that $\operatorname{Ker} g = 0$, so g is a monomorphism.

(2) $\Longrightarrow$ (1): Suppose that (2) is satisfied. First we consider a module $M \in \mathcal{A}$, a submodule $N \leqslant M$, and a monomorphism $f : N \longrightarrow Q$. We will show that f can be extended to a morphism $g : M \longrightarrow Q$. To do this, pick T a complement of N in M, i.e., T is maximal in the set of all submodules P of M with $P \cap N = 0$. There is a canonical isomorphism $\varphi : N \longrightarrow (N+T)/T$. If we set $\bar{f} = f \circ \varphi^{-1}$, then clearly $\bar{f}$ is a monomorphism. Note that $(N+T)/T$ is an essential submodule of M/T (see, e.g., Wisbauer [8, 17.6]), and $M/T \in \mathcal{A}$ because the class $\mathcal{A}$ is closed under factor modules. By hypothesis, it follows that $\bar{f}$ can be extended to $\bar{g} : M/T \longrightarrow Q$. If $q : M \longrightarrow M/T$ denotes the canonical epimorphism, then $g := \bar{g} \circ q$ extends f to M.

Next, we drop the assumption that the morphism $f : N \longrightarrow Q$ is monomorphism. We are going to show that f can be extended to a morphism $g : M \longrightarrow Q$, which will finish the proof. To see this, let $K := \operatorname{Ker} f$, $p : N \longrightarrow N/K$ and $s : M \longrightarrow M/K$ be the canonical epimorphisms. There is a canonical monomorphism $h : N/K \longrightarrow Q$ such that $f = h \circ p$.

Since $M/K \in \mathcal{A}$, by using the property we just proved, we deduce that h can be extended to a morphism $\bar{h} : M/K \longrightarrow Q$. Now, $g := \bar{h} \circ s$ extends f to M, as desired. □

The concept of a K-injective module for some module K_R can be formulated in a way that allow to apply Proposition 3.2 for the class $\mathcal{A}_K$ of right R-module defined by

$$\mathcal{A}_K := \{ M_R \mid M \simeq K/N, N \leqslant K \}.$$

Note that $\mathcal{A}_K$ is the least subclass of Mod-R that contains K and is closed under factor modules.

Indeed, as it is well-known, if a module Q_R is K-injective, then it is also K''-injective for any exact sequence $K \longrightarrow K'' \longrightarrow 0$ in Mod-R (see, e.g., Anderson and Fuller [1, Proposition 16.13]), so Q is $\mathcal{A}_K$-injective too.

By an *abstract class of lattices* we mean any nonempty subclass class $\mathcal{C}$ of $\mathcal{L}$ which is closed under isomorphisms.

The next result is the latticial counterpart of the module case characterization in Proposition 3.2.

Proposition 3.3. *The following are equivalent for a lattice Q and an abstract class $\mathcal{C}$ of upper continuous modular lattices which is closed under factor intervals.*

(1) *Q is a linear $\mathcal{C}$-injective lattice.*

(2) *For any $P \in \mathcal{C}$ and any essential element $e \in P$, any linear monomorphism $e/0 \longrightarrow Q$ can be extended to a linear monomorphism $P \longrightarrow Q$.*

Proof: (1) $\Longrightarrow$ (2): Let $P \in \mathcal{C}$, e an essential element of P, and let $f : e/0 \longrightarrow Q$ be a linear monomorphism. By assumption, there exists a linear morphism $g : P \longrightarrow Q$ which extends f. Consider the kernel k of g. By Proposition 1.3 and Corollary 1.4 we have $g(k \wedge e) = g(k) = 0$, so $f(k \wedge e) = 0 = f(0)$. Then $k \wedge e = 0$ because f is injective. Since e is an essential element of P, we deduce that $k = 0$. By Corollary 1.6, g is a linear monomorphism, as desired.

(2) $\Longrightarrow$ (1): First we show that for every $P \in \mathcal{C}$ and $p \in P$, any linear monomorphism $f : p/0 \longrightarrow Q$ can be extended to a linear morphism $P \longrightarrow Q$. To see this, consider a pseudo-complement t of p in P. By modularity, the map $\varphi : p/0 \longrightarrow (p \vee t)/t$ defined by $\varphi(x) := x \vee t$ is an isomorphism of lattices. If we set $\bar{f} := f \circ \varphi^{-1}$, then clearly $\bar{f}$ is a linear morphism. Note that $1/t \in \mathcal{C}$ because $\mathcal{C}$ is closed under factor intervals. Moreover, since t is a pseudo-complement of p in P, it follows that $p \vee t$ is an essential element of $1/t$. By hypothesis, there exists a linear monomorphism $\bar{g} : 1/t \longrightarrow Q$ extending $\bar{f}$. Define $g : P \longrightarrow Q$ by $g(x) := \bar{g}(x \vee t)$, $x \in P$. Then, g is a linear morphism (having the kernel t) which extends f.

Next we show that for every $P \in \mathcal{C}$ and $p \in P$, any linear morphism $f : p/0 \longrightarrow Q$ can be extended to a linear morphism $P \longrightarrow Q$. To see this, note that, by definition, f induces an isomorphism $f' : p/k \xrightarrow{\sim} q/0$, for k the kernel of f and some $q \in Q$. Thus $h : p/k \longrightarrow Q$ defined by $h(x) := f'(x)$, $x \in p/k$, is a linear monomorphism. Moreover, $1/k \in \mathcal{C}$ because $\mathcal{C}$ is closed under factor intervals. Now, by using the property we just proved, we obtain a linear morphism $\bar{h} : 1/k \longrightarrow Q$ which extends h. Let k' be the kernel of $\bar{h}$, and let $a \in P$ be such that $\bar{h}$ induces an isomorphism $1/k' \simeq a/0$. Define $g : P \longrightarrow Q$ by $g(x) := \bar{h}(x \vee k)$, $\forall\, x \in P$. For every $x \in P$ we have

$$g(x) = \bar{h}(x \vee k) = \bar{h}((x \vee k) \vee k') = \bar{h}((x \vee k') \vee k) = g(x \vee k')\,.$$

Moreover, g induces the same isomorphism $1/k' \simeq a/0$ as $\bar{h}$. Hence g is a linear morphism with kernel k'. Then, for every $x \in p/0$, we have

$$f(x) = f(x \vee k) = h(x \vee k) = \bar{h}(x \vee k) = g(x).$$

So g extends f, and we are done. $\square$

The next result is the latticial counterpart of the well-known result in Module Theory saying that any M-injective module is both M'-injective and M''-injective for any exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in Mod-R (see, e.g., Anderson and Fuller [1, Proposition 16.13]).

Proposition 3.4. *Let $Q, L \in \mathcal{L}$ and assume that Q is linear L-injective. Then Q is linear I-injective for any initial or quotient interval I of L, and so, it is linear S-injective for any subfactor $S = d/c$ of L, $c \leqslant d$ in L.*

Proof: We have to prove that for every $a \in L$, Q is both linear $a/0$-injective and linear $1/a$-injective. The linear $a/0$-injectivity follows immediately from definitions.

We are now going to show that Q is linear $1/a$-injective. This means that for every $b \in L$ with $a \leqslant b$, every linear morphism $f : b/a \longrightarrow Q$ can be extended to a linear morphism $g : 1/a \longrightarrow Q$. Consider the commutative diagram

$$\begin{array}{ccc} b/0 & \xrightarrow{\ i\ } & 1/0 = L \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow \\ b/a & \xrightarrow{\ j\ } & 1/a \end{array}$$

where i, j are the canonical inclusion maps, and p, q are the canonical surjection maps described in Examples 1.2 (2).

It is easily seen that $f \circ p : b/0 \longrightarrow Q$ is a linear morphism with $\mathrm{Ker}\,(f \circ p) = \mathrm{Ker}\, f$, so there exists a linear morphism $h : L \longrightarrow Q$ such that $f \circ p = h \circ i$. But

$$h(a) = (f \circ p)(a) = f(p(a)) = f(a) = 0$$

because $a = \mathrm{Ker}\, p$ and $f(a) = 0$. By Proposition 1.3 (3), h induces a linear morphism $g : 1/a \longrightarrow Q$. Then $g \circ q = h$ because

$$(g \circ q)(x) = g(q(x)) = g(x \vee a) = h(x \vee a) = h(x) \vee h(a) = h(x) \vee 0 = h(x), \forall\, x \in L.$$

It follows that

$$f \circ p = h \circ i = g \circ q \circ i = g \circ j \circ p,$$

so $g \circ j = f$ because p is a surjective map, and we are done. $\square$

Corollary 3.5. *Let $L \in \mathcal{L}$, and set $\mathcal{C}_L := \{ K \in \mathcal{L} \mid K \simeq 1/a, \forall a \in L \}$. Then $Q \in \mathcal{L}$ is L-injective if and only if Q is $\mathcal{C}_L$-injective.*

Proof: Apply Proposition 3.4. □

We introduce now the latticial counterpart of the concept of injective hull from Module Theory.

Definition 3.6. *Let L be a lattice, and let $\mathcal{C}$ be an abstract class of lattices. We say that a lattice Q is a* linear $\mathcal{C}$-injective hull *of L if Q satisfies the following two conditions.*

(i) *There exists an essential linear monomorphism $L \longrightarrow Q$.*

(ii) *Q is $\mathcal{C}$-injective.* □

The next result shows that, as in the module case, the linear $\mathcal{C}$-injective hull is unique up to a lattice isomorphism.

Proposition 3.7. *Let L be a lattice, and let $\mathcal{C}$ be an abstract class of lattices. Suppose that Q and Q' are two linear $\mathcal{C}$-injective hulls of L. If $Q, Q' \in \mathcal{C}$, then Q and Q' are isomorphic lattices.*

Proof: Consider the essential linear monomorphisms $i : L \longrightarrow Q$ and $i' : L \longrightarrow Q'$ guaranteed by definition. Then $e := i(1)$ is an essential element of Q and $e' := i'(1)$ is an essential element of Q'. Moreover, $e/0 \simeq L \simeq e'/0'$, where $0'$ is the least element of Q'. Denote by $\varphi : e/0 \xrightarrow{\sim} e'/0'$ this isomorphism. Since $Q \in \mathcal{C}$ and Q' is $\mathcal{C}$-injective, it follows that there exists a linear morphism $f : Q \longrightarrow Q'$ that extends φ. Because e is an essential element of Q, the linear morphism f is necessarily injective (see the proof of $(1) \Longrightarrow (2)$ in Proposition 3.3). Denote by q the last element of Q and consider $a' := f(q) \in Q'$. Since f is an increasing map by Corollary 1.4, we have $e' = \varphi(e) = f(e) \leqslant f(q) = a'$. Because e' is an essential element of Q', so is also a'. By definition, f induces an isomorphism $\psi : Q \xrightarrow{\sim} a'/0'$. Since $Q' \in \mathcal{C}$ and Q is $\mathcal{C}$-injective, it follows that there exists a linear morphism $g : Q' \longrightarrow Q$, necessarily injective, that extends ψ^{-1}. If we denote by q' the last element of Q', then we have $q = g(a') \leqslant g(q')$. Thus $g(a') = g(q')$ and, since g is injective, we deduce that $a' = q'$. Hence ψ is an isomorphism between Q and Q', as desired. □

4 Some examples and open questions

In this section we provide a large variety of examples and counter-examples of linear $\mathcal{C}$-injective lattices and $\mathcal{C}$-injective lattices, relate the concept of linear $\mathcal{C}$-injectivity with that of $\mathcal{C}$-injectivity, and also present several open questions on them.

4.1. Examples of linear $\mathcal{C}$-injective lattices. For any ordinal α we set

$$W_\alpha := \alpha/0 = [\,0, \alpha\,] = \{\,\beta \,|\, \beta \text{ ordinal}, \ 0 \leqslant \beta \leqslant \alpha\,\}.$$

Note that W_α is a well-ordered set with order type $\alpha + 1$, so a bounded upper continuous modular lattice having all elements essential. The reader is referred to Rosenstein [7] for terminology and facts on ordinals.

We claim that if α and γ are any two ordinals such that $\alpha \leqslant \gamma$, then W_γ is linear W_α-injective. Indeed, according to Proposition 3.3, it is sufficient to prove that for any ordinal $\beta < \alpha$, any linear monomorphism $f : \beta/0 \longrightarrow W_\gamma$ can be extended to a linear monomorphism $\bar{f} : \alpha/0 \longrightarrow W_\gamma$. Indeed, by definition of a linear morphism, the image of f is exactly the interval $\beta/0$ of W_γ. Using the properties of well-ordered sets we must then have $f(\delta) = \delta$, $\forall\, \delta \in [\,0, \beta\,]$. Clearly, we may extend f to (a unique) $\bar{f} : W_\alpha \longrightarrow W_\gamma$ by putting $\bar{f}(\delta) = \delta$, $\forall\, \delta \in [\,\beta + 1, \alpha\,]$, and clearly $\bar{f}$ is a linear monomorphism.

Observe that, with similar arguments, one can prove that if α and γ are two ordinals such that $\alpha > \gamma$, then W_γ is not linear W_α-injective. Consequently, W_γ is linear W_α-injective if and only if $\alpha \leqslant \gamma$.

For any ordinal β denote by $\mathcal{W}_\beta$ the class of all well-ordered sets having order type $\beta + 1$. The example above shows that for any two ordinals α and γ with $\alpha \leqslant \gamma$, every $W \in \mathcal{W}_\gamma$ is linear $\mathcal{W}_\alpha$-injective.

4.2. Examples of $\mathcal{C}$-injective lattices. We show that for any bounded well-ordered set W, any lattice L is W-injective. Let S be a sublattice of W; in case the last element 1 of W is not in S, we adjoin it to S, so without loss of generality we may assume that $1 \in S$.

We have to prove that any lattice morphism $f : S \longrightarrow L$ can be extended to a lattice morphism $\bar{f} : W \longrightarrow L$. First, observe that a map $h : C \longrightarrow L$ from any chain C to a lattice L is a lattice morphism if and only if it is increasing.

For every $x \in W$ set $S_x := \{\,s \in S \,|\, x \leqslant s\,\}$. Note that $S_x \neq \varnothing$ because $1 \in S$, and denote by $p(x)$ the least element of S_x. Thus, we obtain a map $p : W \longrightarrow S$. It is easily verified that $x \leqslant p(x)$, $\forall\, x \in W$, $p(s) = s$, $\forall\, s \in S$, and p is an increasing map, so a lattice morphism. Now, if we take $\bar{f} := f \circ p$, then $\bar{f}$ is a lattice morphism that extends f, as desired.

4.3. Examples of $\mathcal{C}$-injective lattices which are not linear $\mathcal{C}$-injective. By 4.2, any finite totally ordered set (i.e., chain) F having at least two elements is W_ω-injective, where ω is the first transfinite ordinal (that is, the order type of the set $\mathbb{N} = \{0,\, 1,\, 2,\, \ldots\}$ of natural numbers), but F is not linear W_ω-injective.

Indeed, assume that F is linear W_ω-injective. Without loss of generality, we may assume that $F = \{\,0,\, 1,\, \ldots,\, n\,\}$ for some $n \in \mathbb{N}$, $n \geqslant 1$. Then, the identity map 1_F on F can be extended to a linear morphism $f : W_\omega \longrightarrow F$.

Let $k = \operatorname{Ker} f$. If $n \leqslant k$, then $0 \leqslant f(n) \leqslant f(k) = 0$, so $1 \leqslant n = 1_F(n) = f(n) = 0$, which is a contradiction. It follows that we must have $k < n$, and then, $0 = f(k) = 1_F(k) = k$, so f is a linear monomorphism, in particular an injective map, which is a contradiction because W_ω is an infinite set. This proves that F is not linear W_ω-injective.

4.4. Examples of lattices which are not $\mathcal{C}$-injective. According to Banaschewski and Bruns [2, p. 376], any lattice $L \in \mathcal{L}$ having at least two elements (that is, non-trivial) is not

M_3-injective, where, as usually, M_3 denotes the *diamond lattice* (i.e., the five-element lattice of all subgroups of the Klein four-group). Another result, due to Nelson [6], states that any non-trivial lattice L is not N_5-injective, where N_5 is the so called *pentagon lattice.*

In particular, using the well-known characterization of distributive lattices saying that a lattice L is distributive if and only if L contains neither a pentagon nor a diamond (see, e.g., Grätzer [5, p. 80]), we deduce that any non-trivial lattice $L \in \mathcal{L}$ is not $\mathcal{C}$-injective for any nonempty class $\mathcal{C}$ of lattices containing at least one non-distributive lattice.

4.5. There are no non-trivial injective lattices. This follows immediately from 4.4.

4.6. An example of a linear $\mathcal{C}$-injective lattices which is not $\mathcal{C}$-injective. Denote by $0, a, b, c, 1$ the elements of the diamond lattice M_3. By 4.4, the two-element lattice $W_2 = \{0, 1\}$ is not M_3-injective.

We claim that W_2 is however linear M_3-injective. It is sufficient to prove that for every $x \in M_3 \setminus \{0, 1\}$, every linear morphism $f : x/0 \longrightarrow W_2$ can be extended to a linear morphism $\bar{f} : M_3 \longrightarrow W_2$. Without loss of generality, we may assume that $x = a$. Then $x/0 = \{0, a\}$.

Clearly, $f(0) = 0$. If $f(a) = 0$ we can take as $\bar{f} : M_3 \longrightarrow W_2$ the constant linear map carrying all elements of M_3 onto 0. Otherwise, $f(a) = 1$, and then we define $\bar{f}$ as follows:

$$\bar{f}(0) = \bar{f}(c) = 0 \quad \text{and} \quad \bar{f}(a) = \bar{f}(b) = \bar{f}(1) = 1.$$

Clearly the map $\bar{f}$ extends f, and it is easily verified that $\bar{f}$ is a linear morphism with kernel c.

4.7. There are no non-trivial linear injective lattices. Assume that there exists a non-trivial linear injective lattice Q. Consider a bounded chain C having the cardinal Card (C) strictly greater than the cardinal Card (Q) of Q, and denote by L the ordered direct (or disjoint) sum $Q \oplus C$. This means that $Q \cap C = \varnothing$, $L = Q \cup C$, $q < c, \forall\, q \in Q, \forall\, c \in C$, and the order on L extends both the orders on Q and C (see, e.g., Rosenstein [7, Definition 1.29]). Then, L is a lattice, which is modular (resp. upper continuous) if so is Q.

Denote by i the canonical injection $Q \hookrightarrow L$. Note that Q is an initial interval of L. Since we assumed that Q is linear injective, the identity map 1_Q on Q can be extended to a linear morphism $f : L \longrightarrow Q$.

We repeat now the argument used in 4.3. Let $k = \operatorname{Ker} f$. If $k \in C$, then $q < k, \forall\, q \in Q$, so $0 \leqslant f(q) \leqslant f(k) = 0$, and then f is the constant zero map on Q, which is a contradiction. Therefore, we must have $k \in Q$. Then $0 = f(0) = f(k) = 1_Q(k) = k$, and so $k = 0$, i.e., f is a linear monomorphism, in particular it is an injective map, which contradicts the inequality Card $(Q) <$ Card (C), and we are done.

4.8. Linear injectivity and ordered direct sums. It is known that if a module Q_R is both M_1-injective and M_2-injective, then Q is also $M_1 \oplus M_2$-injective. The latticial version of this result does not hold: W_2 is W_1-linear injective by 4.1, but W_2 is not linear $W_1 \oplus W_1$-injective. Indeed, $W_1 \oplus W_1 \simeq W_3$, and W_2 is not W_3-injective, again by 4.1.

4.9. Some open questions. We present below a list of six open questions, mainly asking when basic properties of modules related to injectivity do hold for their latticial counterparts.

1. *Find/characterize modules M_R, Q_R such that Q_R is M_R-injective implies that $\mathcal{L}(Q_R)$ is a linear $\mathcal{L}(M_R)$-injective lattice.*

2. *Find/characterize modules M_R, Q_R such that $\mathcal{L}(Q_R)$ is a linear $\mathcal{L}(M_R)$-injective lattice implies that Q_R is an M_R-injective module.*

3. *If $Q_1, Q_2 \in \mathcal{L}$ are both linear L-injective lattices, then does it follow that their ordered direct sum $Q_1 \oplus Q_2$ is also a linear L-injective lattice?*

4. *If $Q_1, Q_2 \in \mathcal{L}$ are both linear L-injective lattices, then does it follow that their direct product $Q_1 \times Q_2$ is also a linear L-injective lattice?*

5. *Find necessary or sufficient conditions for $L \in \mathcal{L}$ and nonempty classes of lattices $\mathcal{C}$ such that L has a $\mathcal{C}$-injective hull.*

6. *Find the connections between maximal linear essential extensions and linear injective lattices.*

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