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On the norm of the trace functions and applications

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

Given a prime number p and the Galois orbit $O(x)$ of a transcendental element x of $\mathbb{C}_p$, the topological completion of the algebraic closure of the field of p-adic numbers, we give an estimation for the norm of the trace functions defined on the complement of $O(x)$ with values in $\mathbb{C}_p$. Then we give some applications to transcendental functions.

Key Words: p-adic analytic functions, local fields, Galois orbits, distributions.
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1 Introduction

Let p be a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers and let $|\cdot|$ be the usual p-adic absolute value. This absolute value can be uniquely extended to an absolute value (denoted also by $|\cdot|$) on $\overline{\mathbb{Q}}_p$, a fixed algebraic closure of $\mathbb{Q}_p$. Further, consider the Tate field $\mathbb{C}_p$, which is the completion of $(\overline{\mathbb{Q}}_p, |\cdot|)$, and use the same notation $|\cdot|$ for the unique p-adic absolute value that extends the p-adic absolute value $|\cdot|$ on $\overline{\mathbb{Q}}_p$. Denote $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, endowed with the Krull topology. Then G acts continuously on $\overline{\mathbb{Q}}_p$, and it is easy to see that G is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. Let $O(x)$ be the orbit of a transcendental element x of $\mathbb{C}_p$ with respect to the Galois group G. We are interested to give an estimation for the norm of the trace functions defined on $\mathbb{P}^1(\mathbb{C}_p) \setminus O(x)$ with values in $\mathbb{C}_p$, which are equivariant with respect to the absolute Galois group, with applications to transcendental functions.

The paper consists of four sections. The first section is usually an introduction in the framework of the paper. The second section contains some background material. In the third section we consider the class of the trace functions defined on $\mathbb{P}^1(\mathbb{C}_p) \setminus O(x)$ with values in $\mathbb{C}_p$, which are in fact Cauchy transforms or derivatives of Cauchy transforms on $O(x)$ with respect to Galois equivariant strongly Lipschitz distributions. We give an estimation for the norm of the functions in the above class. The estimation is made on the complement of a neighborhood of the orbit of x and it gives some informations about the behavior of this type of p-adic analytic functions at boundary, see Theorem 1. The last section contains some important applications of the main result in the study of transcendental functions. We have that a function in the above class is transcendental over $\mathbb{Q}_p(Z)$, see Proposition 1. Moreover, any finite set of functions in the above class of different orders is linearly independent over $\mathbb{Q}_p(Z)$, see Propositon 2. Finally, for any Krasner analytic function, which is defined on the complement of the orbit of a transcendental element of $\mathbb{C}_p$ such that it can be represented as an infinite series with terms in the above class with different orders and with coefficients in $\mathbb{Q}_p(Z)$, we give an estimation of its norm on the complement of a neighborhood of the orbit of x and then we obtain that the above representation is unique, see Proposition 3.

2 Background material

Let p be a prime number and $\mathbb{Q}_p$ the field of p-adic numbers, endowed with the p-adic absolute value $|\cdot|$, normalized such that $|p| = 1/p$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and denote by the same symbol $|\cdot|$ the unique extension of $|\cdot|$ to $\overline{\mathbb{Q}}_p$. Further, denote by $(\mathbb{C}_p, |\cdot|)$ the completion of $(\overline{\mathbb{Q}}_p, |\cdot|)$ (see [4], [5]). Let $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the absolute Galois group endowed with the Krull topology. The group G is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. In what follows we shall identify these two groups. For any $x \in \mathbb{C}_p$ denote $O(x) = \{\sigma(x) : \sigma \in G\}$, the orbit of x, and let $\widetilde{\mathbb{Q}_p[x]}$ be the topological closure of the ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$.

For any closed subgroup H of G denote $Fix(H) = \{x \in \mathbb{C}_p : \sigma(x) = x \text{ for all } \sigma \in H\}$. Then $Fix(H)$ is a closed subfield of $\mathbb{C}_p$. Denote $H(x) = \{\sigma \in G : \sigma(x) = x\}$. Then $H(x)$ is a subgroup of G, and $Fix(H(x)) = \widetilde{\mathbb{Q}_p[x]}$.

The map $x \rightsquigarrow \sigma(x)$ from G to $O(x)$ is continuous, and it defines a homeomorphism from $G/H(x)$ (endowed with the quotient topology) to $O(x)$ (endowed with the induced topology from $\mathbb{C}_p$) (see for instance [2]). In such a way $O(x)$ is a closed compact and totally disconnected subspace of $\mathbb{C}_p$, and the group G acts continuously on $O(x)$: if $\sigma \in G$ and $\tau(x) \in O(x)$ then $\sigma \star \tau(x) := (\sigma\tau)(x)$.

For any real number $\varepsilon > 0$ denote $B(x, \varepsilon) = \{y \in \mathbb{C}_p : |y - x| < \varepsilon\}$ and $B[x, \varepsilon] = \{y \in \mathbb{C}_p : |y - x| \leq \varepsilon\}$. Also denote $E(x, \varepsilon) = \{y \in \mathbb{C}_p \cup \{\infty\} : |y - t| \geq \varepsilon, \text{ for all } t \in O(x)\}$. The complement of $E(x, \varepsilon)$ in $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ is denoted by $V(x, \varepsilon)$. Both sets $E(x, \varepsilon)$ and $V(x, \varepsilon)$ are open and closed, and one has: $\cap_\varepsilon V(x, \varepsilon) = O(x)$. Denote $E(x) = \cup_\varepsilon E(x, \varepsilon) = \mathbb{P}^1(\mathbb{C}_p) \setminus O(x)$. For $\varepsilon > 0$ denote $H(x, \varepsilon) = \{\sigma \in G : |\sigma(x) - x| < \varepsilon\}$, and let S_ε be a complete system of representatives for the right cosets of G with respect to $H(x, \varepsilon)$. One knows that for any $0 < \varepsilon' < \varepsilon$, $|S_\varepsilon|$ divides $|S_{\varepsilon'}|$, see [3]. Then $V(x, \varepsilon) = \cup_{\sigma \in S_\varepsilon} B(\sigma(x), \varepsilon)$.

If X is a compact subset of $\mathbb{C}_p$, then by an open ball in X we mean a subset of the form $B(x, \varepsilon) \cap X$ where $x \in \mathbb{C}_p$ and $\varepsilon > 0$. Let us denote by $\Omega(X)$ the set of subsets of X which are open and compact. It is easy to see that any $D \in \Omega(X)$ can be written as a finite union of open balls in X, any two disjoint.

Definition 1. *By a distribution on X with values in $\mathbb{C}_p$ we mean a map $\mu : \Omega(X) \to \mathbb{C}_p$ which is finitely additive, that is, if $D = \cup_{i=1}^n D_i$ with $D_i \in \Omega(X)$ for $1 \leq i \leq n$ and $D_i \cap D_j = \emptyset$ for $1 \leq i \neq j \leq n$, then $\mu(D) = \sum_{i=1}^n \mu(D_i)$. (See also [8].)*

The norm of μ is defined by $\|\mu\| := \sup\{|\mu(D)| : D \in \Omega(X)\}$. If $\|\mu\| < \infty$ we say that μ is a measure on X.

Definition 2. *We say that a distribution μ on X is Lipschitz if*

$$\lim_{\varepsilon \to 0} \varepsilon \max |\mu(B(a, \varepsilon))| = 0,$$

where the "max" is taken over all the balls $B(a, \varepsilon)$ from $\Omega(X)$.

Remark 1. *Any measure on X is a Lipschitz distribution and any Lipschitz function on X is Riemann integrable with respect to any Lipschitz distribution.*

Definition 3. *An element $x \in \mathbb{C}_p$ is called Lipschitz if and only if $\lim_{n\to\infty} \frac{\varepsilon}{|N(x,\varepsilon)|} = 0$, where $N(x, \varepsilon)$ is the number of balls of radius ε that cover the orbit of x.*

Remark 2. *When x is a Lipschitz element of $\mathbb{C}_p$ then the Haar distribution π_x, which is defined on the open and compact subsets of the orbit of x, is also Lipschitz. Here π_x is defined by $\pi_x(B) = \frac{1}{N(x,\varepsilon)}$ for any open and compact ball B of $\Omega(O(x))$ of radius $\varepsilon > 0$ and then it is extended by additivity to the whole $\Omega(O(x))$.*

Definition 4. *An element $x \in \mathbb{C}_p$ is called p-bounded if there exists an $s \in \mathbb{N}$ such that p^s does not divide the number $N(x, \varepsilon)$, for any $\varepsilon > 0$. In this case π_x is a measure.*

Remark 3. *In [3] it is introduced the concept of p-bounded algebraic extension of $\mathbb{Q}_p$. Precisely, an algebraic extension $\mathbb{Q}_p \subseteq L$ is called p-bounded if there exists a natural number s such that p^s does not divide $[K : \mathbb{Q}_p]$ for any finite extension K of $\mathbb{Q}_p$ with $K \subseteq L$. By taking $x \in \mathbb{C}_p$ such that $x \in \widetilde{L}$, where L is a p-bounded algebraic extension of $\mathbb{Q}_p$, one has $\widetilde{\mathbb{Q}_p[x]} \subseteq \widetilde{L}$ and, for any $\varepsilon > 0$, $FixH(x, \varepsilon) \subset L$ and $[FixH(x, \varepsilon) : \mathbb{Q}_p] < \infty$. It is clear that x is p-bounded.*

Definition 5. *Let X be a compact subset of $\mathbb{C}_p$ and, for any $\varepsilon > 0$ let X_ε denote the ε-neighborhood of X in $\mathbb{C}_p$. A function $f : \mathbb{P}^1(\mathbb{C}_p) \setminus X \to \mathbb{C}_p$ is said to be Krasner analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus X$ provided that for any $\varepsilon > 0$ there is a sequence of rational functions with all their poles in X_ε that converges uniformly to f on $\mathbb{P}^1(\mathbb{C}_p) \setminus X_\varepsilon$, see [4], [6] and [9]. We denote by $\mathcal{A}(\mathbb{P}^1(\mathbb{C}_p) \setminus X, \mathbb{C}_p)$ the set of all Krasner analytic functions defined on $\mathbb{P}^1(\mathbb{C}_p) \setminus X$ with values in $\mathbb{C}_p$.*

The set $X \subset \mathbb{C}_p$ is said to be G-equivariant, or equivariant with respect to the absolute Galois group, provided that $\sigma(x) \in X$ for any $x \in X$ and any $\sigma \in G$. ($X = O(x)$ is such an example.)

Definition 6. *Let X be a G-equivariant compact subset of $\mathbb{C}_p$ and μ a distribution on X with values in $\mathbb{C}_p$. We say that μ is G-equivariant, or equivariant with respect to the absolute Galois group, if $\mu(\sigma(B)) = \sigma(\mu(B))$, for any ball B in X and any $\sigma \in G$.*

Remark 4. *On a Galois orbit $O(x)$ there exists a unique G-equivariant probability distribution with values in $\mathbb{Q}_p$, namely the Haar distribution π_x.*

Because $O(x)$ is a compact set, the image of the distance function $d_x : O(x) \to \mathbb{R}_+$, $d_x(y) := |y-x|$, $y \in O(x)$, is a set of the form $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots, 0\}$ with $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_n > \dots > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. The sequence $\{\varepsilon_n\}_{n\geq 1}$ is called the fundamental sequence associated with the orbit $O(x)$.

Definition 7. *A G-equivariant Lipschitz distribution μ on $O(x)$ with values in $\mathbb{C}_p$ is a strongly Lipschitz distribution if the following condition holds: there exists $N(\mu) \in \mathbb{N}$ such that the sequence $\{\varepsilon_n|\mu(B(x,\varepsilon_n))|\}_{n\geq N(\mu)}$ is strictly decreasing to zero, where the sequence $\{\varepsilon_n\}_{n\geq 1}$ is the fundamental sequence associated with the orbit $O(x)$.*

Remark 5. *Any G-equivariant measure μ on $O(x)$ with values in $\mathbb{Q}_p$ is a strongly Lipschitz distribution. A more interesting example is the Haar distribution π_x with $x \in \widetilde{L}$, where L is a p-bounded algebraic extension of $\mathbb{Q}_p$ as in Remark 3. In this case the sequence of positive real numbers $|\pi_x(B(x,\varepsilon_n))|$ is constant for n large enough and by this π_x is a strongly Lipschitz distribution.*

As in [3], a Krasner analytic function defined on a subset X of $\mathbb{C}_p$ is called *equivariant* if for any $z \in X$ one has $O(z) \subset D$ and $f(\sigma(z)) = \sigma(f(z))$ for all $\sigma \in G$.

For a G-equivariant subset X of $\mathbb{C}_p$, let $\mathcal{A}^G(\mathbb{P}^1(\mathbb{C}_p)\setminus X, \mathbb{C}_p)$ be the set of equivariant Krasner analytic functions on $\mathbb{P}^1(\mathbb{C}_p)\setminus X$ with values in $\mathbb{C}_p$, and $\mathcal{A}_0^G(\mathbb{P}^1(\mathbb{C}_p)\setminus X, \mathbb{C}_p)$ its subset consisting of those functions that vanish at ∞.

3 Main result

Let x be a transcendental element of $\mathbb{C}_p$ and let μ be a strongly Lipschitz distribution defined on $O(x)$. For any positive integer s let us define the following function

$$F_{s,\mu}(z) = \int_{O(x)} \frac{1}{(z-t)^s} \mathrm{d}\mu(t). \tag{1}$$

The above function is well defined and, moreover, $F_{s,\mu} \in \mathcal{A}_0^G(\mathbb{P}^1(\mathbb{C}_p) \setminus O(x), \mathbb{C}_p)$, see [1] and [10]. In fact, $F_{1,\mu}$ is the Cauchy transform on $O(x)$ with respect to μ and it is the trace function of x associated with μ, see [3] and [11].

For any $F \in \mathcal{A}_0^G(\mathbb{P}^1(\mathbb{C}_p) \setminus O(x), \mathbb{C}_p)$ and any $\varepsilon > 0$ let us denote by $\|F\|_{E(x,\varepsilon)}$ the sup norm of F on $E(x,\varepsilon)$. Let $\{\varepsilon_n\}_{n\geq 1}$ be the fundamental sequence associated with the orbit $O(x)$ and denote, for the sake of simplicity, $H_n = H(x,\varepsilon_n)$ and $S_n = S_{\varepsilon_n}$. Here S_n is a complete system of representatives for the right cosets of G with respect to H_n.

Our goal of this section is to calculate $\|F_{s,\mu}\|_{E(x,\varepsilon_n)}$, for any $n \geq N(\mu)$, where $N(\mu)$ is a natural number which depends only on μ as in Definition 7, in terms of the fundamental sequence and the distribution considered above. By considering the Riemman sum

$$\Gamma_n(z) = \sum_{\sigma\in S_n} \frac{1}{(z-\sigma(x))^s} \mu(B(\sigma(x),\varepsilon_n)) \tag{2}$$

on $E(x,\varepsilon_n)$, one has from Mittag-Leffler's theorem that

$$\|\Gamma_n\|_{E(x,\varepsilon_n)} = \frac{|\mu(B(x,\varepsilon_n))|}{\varepsilon_n^s}, \tag{3}$$

for any $n \geq 1$. We estimate $\|\Gamma_{n+1} - \Gamma_n\|_{E(x,\varepsilon_n)}$ for any $n \geq N(\mu)$. We have that $\Gamma_{n+1}(z) - \Gamma_n(z)$ is a sum of terms which are conjugate with $T_{n+1}(z)$, where

$$T_{n+1}(z) = \sum_{\sigma \in H_n/H_{n+1}} \left[\frac{1}{(z-x)^s} - \frac{1}{(z-\sigma(x))^s} \right] \mu(B(\sigma(x), \varepsilon_{n+1})) \tag{4}$$

and $z \in E(x, \varepsilon_n)$. Because $\{\varepsilon_n\}_{n \geq 1}$ is the fundamental sequence and σ is in a complete system of representatives for the right cosets of H_n with respect to H_{n+1} one has that $|\sigma(x) - x| = \varepsilon_{n+1}$. From this, it is easy to see that

$$\left| \frac{1}{(z-x)^s} - \frac{1}{(z-\sigma(x))^s} \right| \leq \frac{\varepsilon_{n+1}}{\varepsilon_n^{s+1}}, \tag{5}$$

for any $z \in E(x, \varepsilon_n)$ and $n \geq N(\mu)$. By (4) and (5) one obtains

$$\|\Gamma_{n+1} - \Gamma_n\|_{E(x,\varepsilon_n)} \leq \frac{\varepsilon_{n+1}}{\varepsilon_n^{s+1}} \cdot |\mu(B(x, \varepsilon_{n+1})| < \frac{|\mu(B(x, \varepsilon_n))|}{\varepsilon_n^s}. \tag{6}$$

The strict inequaliy in (6) holds because μ is strongly Lipschitz distribution. By (3) and (6) we obtain that

$$\|\Gamma_{n+1}\|_{E(x,\varepsilon_n)} = \frac{|\mu(B(x, \varepsilon_n))|}{\varepsilon_n^s}. \tag{7}$$

Now, let us suppose that

$$\|\Gamma_{n+i}\|_{E(x,\varepsilon_n)} = \frac{|\mu(B(x, \varepsilon_n))|}{\varepsilon_n^s}, \tag{8}$$

for any $1 \leq i \leq k$. We have that $\Gamma_{n+k+1}(z) - \Gamma_{n+k}(z)$ is a sum of terms which are conjugate with $T_{n+k+1}(z)$ as in (4) with $n+k$ instead of n, for any $n \geq N(\mu)$ and any $z \in E(x, \varepsilon_n)$. As in (5) one has

$$\left| \frac{1}{(z-x)^s} - \frac{1}{(z-\sigma(x))^s} \right| \leq \frac{\varepsilon_{n+k+1}}{\varepsilon_n^{s+1}}, \tag{9}$$

for any $z \in E(x, \varepsilon_n)$ and σ in a complete system of representatives for the right cosets of H_{n+k} with respect to H_{n+k+1}. By (8) and (9) and because μ is strongly Lipschitz distribution we derive that

$$\|\Gamma_{n+k+1} - \Gamma_{n+k}\|_{E(x,\varepsilon_n)} \leq \frac{\varepsilon_{n+k+1}}{\varepsilon_n^{s+1}} \cdot |\mu(B(x, \varepsilon_{n+k+1})| < \frac{|\mu(B(x, \varepsilon_n))|}{\varepsilon_n^s} = \|\Gamma_{n+k}\|_{E(x,\varepsilon_n)},$$

so

$$\|\Gamma_{n+k+1}\|_{E(x,\varepsilon_n)} = \|\Gamma_{n+k}\|_{E(x,\varepsilon_n)} = \frac{|\mu(B(x, \varepsilon_n))|}{\varepsilon_n^s}.$$

The Principle of Mathematical Induction implies that (8) is true for any $i \geq 1$. By letting $i \to \infty$ one has that $\lim_{i \to \infty} \Gamma_{n+i}(z) = F_{s,\mu}(z)$, for any $z \in E(x, \varepsilon_n)$ and any $n \geq N(\mu)$. To sum up we obtain the main result.

Theorem 1. *Let μ be a strongly Lipschitz distribution defined on the orbit of a transcendental element x of $\mathbb{C}_p$. Let $\{\varepsilon_n\}_{n\geq 1}$ be the fundamental sequence associated with the orbit $O(x)$ and $E(x,\varepsilon_n) = \{z \in \mathbb{P}^1(\mathbb{C}_p) : |z-t| \geq \varepsilon_n,\ \text{for all}\ t \in O(x)\}$. Then, there exists a positive integer $N(\mu)$, which depends only on μ, such that for any positive integer s and any $n \geq N(\mu)$,*

$$\left\| \int_{O(x)} \frac{1}{(z-t)^s} \mathrm{d}\mu(t) \right\|_{E(x,\varepsilon_n)} = \frac{|\mu(B(x,\varepsilon_n))|}{\varepsilon_n^s},$$

where $B(x,\varepsilon_n)$ is the open ball of radius ε_n centered at x.

4 Applications

In this paragraph we use the same notation and definitions as in the previous paragraphs. Let μ be a strongly Lipschitz distribution defined on the orbit of a transcendental element x of $\mathbb{C}_p$ and let $\{\varepsilon_n\}_{n\geq 1}$ be the fundamental sequence associated with the orbit $O(x)$. By Definition 7, one has $|\mu(B(x,\varepsilon_n))| > 0$ for any $n \geq N(\mu)$. Because

$$|\mu(B(x,\varepsilon_n))| = \left| \textstyle\sum_{\sigma\in H_n/H_{n+1}} \mu(B(\sigma(x),\varepsilon_{n+1}))\right| \leq |\mu(B(x,\varepsilon_{n+1}))|,$$

we derive that the sequence $\{|\mu(B(x,\varepsilon_n))|\}_{n\geq 1}$ is increasing but it is not necessarily an upper bounded sequence. Now, it is clear that under the hypothesis of Theorem 1

$$\left\| \int_{O(x)} \frac{1}{(z-t)^s} \mathrm{d}\mu(t) \right\|_{E(x,\varepsilon_n)} = \frac{|\mu(B(x,\varepsilon_n))|}{\varepsilon_n^s} \geq \frac{|\mu(B(x,\varepsilon_{N(\mu)}))|}{\varepsilon_n^s}, \tag{10}$$

for any $n \geq N(\mu)$. By letting $n \to \infty$ in (10) one obtains that $\lim_{n\to\infty} \|F_{s,\mu}\|_{E(x,\varepsilon_n)} = \infty$. Using a similar argument as in the proof of the fact that the trace function of a transcendental Lipschitz element with respect to the Haar distribution is transcendental, see [3], we have the following result.

Proposition 1. *Let μ be a strongly Lipschitz distribution defined on the orbit of a transcendental element x of $\mathbb{C}_p$. For any positive integer s, the function defined by*

$$F_{s,\mu}(z) = \int_{O(x)} \frac{1}{(z-t)^s} \mathrm{d}\mu(t) \tag{11}$$

is in $\mathcal{A}_0^G(\mathbb{P}^1(\mathbb{C}_p) \setminus O(x), \mathbb{C}_p)$ and it is transcendental over $\mathbb{Q}_p(Z)$.

Proposition 2. *Let k be a positive integer. For any positive integers $s_1 < s_2 < \cdots < s_k$, let $\mu_{s_1}, \mu_{s_2}, \ldots, \mu_{s_k}$ be strongly Lipschitz distributions defined on the orbit of a transcendental element x of $\mathbb{C}_p$. Then the functions defined by*

$$F_{s_i,\mu_{s_i}}(z) = \int_{O(x)} \frac{1}{(z-t)^{s_i}} \mathrm{d}\mu_{s_i}(t),\ 1 \leq i \leq k,$$

are linearly independent over $\mathbb{Q}_p(Z)$. Particularly, any function $F_{s,\mu}$ of type (11) cannot verify a differential equation in the form $\sum_{j=0}^m P_j F_{s,\mu}^{(j)} = 0$, where for any $0 \leq j \leq m$ we have $P_j \in \mathbb{Q}_p(Z)$ not all zero, and m is a positive integer.

Proof: For the sake of simplicity, we denote $F_i = F_{s_i,\mu_{s_i}}$, for any $1 \le i \le k$. Let us take a linear combination over $\mathbb{Q}_p(Z)$ in the form $G := \sum_{i=1}^{k} P_i F_i$ and suppose that $P_k \not\equiv 0$. Because $P_k \in \mathbb{Q}_p(Z)$ and x is transcendental, there exists a positive integer $n' \ge N(\mu)$ such that $\inf_{z \in V(x,\varepsilon_{n'})} |P_k(z)| > 0$. As in the proof of Theorem 1 and by the hypothesis of Proposition 2 it is easy to see that

$$\|G\|_{E(x,\varepsilon_n)\cap V(x,\varepsilon_{n'})} = \|P_k F_k\|_{E(x,\varepsilon_n)\cap V(x,\varepsilon_{n'})} > 0,$$

for n large enough. Now it is clear that the functions $\{F_i\}_{1\le i\le k}$ are linearly independent over $\mathbb{Q}_p(Z)$. The last part of the proposition results easily and the proof is done. □

By using a similar argument as in the proof of Proposition 2 we have the following result.

Proposition 3. *Let $\{s_i\}_{i\ge 1}$ be a strictly increasing sequence of positive integer. For any integer $i \ge 1$ one considers μ_{s_i} a strongly Lipschitz distribution defined on the orbit of a transcendental element x of $\mathbb{C}_p$. If a function $G : \mathbb{P}^1(\mathbb{C}_p) \setminus O(x) \to \mathbb{C}_p$ can be represented by an infinite series*

$$G(z) = \sum_{i=1}^{\infty} P_i(z) F_i(z),$$

which converges on any $E(x,\varepsilon_n)$, where $\{\varepsilon_n\}_{n\ge 1}$ is the fundamental sequence associated with the orbit $O(x)$, $P_i(z) \in \mathbb{Q}_p(z)$ and $F_i(z) = F_{s_i,\mu_{s_i}}(z) = \int_{O(x)} \frac{1}{(z-t)^{s_i}} \mathrm{d}\mu_{s_i}(t)$, $1 \le i \le k$, then for n large enough one has

$$\|G\|_{E(x,\varepsilon_n)} = \sup_{i\ge 1} \|P_i F_i\|_{E(x,\varepsilon_n)},$$

and the above representation of G is unique.

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