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v-maximal extensions, henselian fields and conservative fields

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

In this paper we continue the investigation of v-maximal extensions [2](or maximal spectral extensions [9]) in connexion with some Henselian fields and conservative fields (introduced here for the first time). We also apply our results to generalize and put in a new light some classical theorems on valued fields.

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Introduction

Let (K, v_K) be a perfect rank 1 valued field and let v be a fixed prolongation of v_K to a fixed algebraic closure $\overline{K}$ of K. In [2] the notion of a v_K-maximal extension was introduced. An intermediate subfield L, $K \subset L \subset \overline{K}$ is called a v_K-maximal extension if v_K has a unique extension v_L (the restriction of v to L in our case) to L and L is maximal with this property. In section 1 of this note we start with another more profitable definition of this notion and we present a general characterization of v_K-maximal extensions (see Theorem 1). In this new frame we recall the basic results of [2], [4] and [9] (see Theorems 2, 3 and 4).

In section 2 we introduce some notions and results connected with v_K-maximal extensions. In Theorem 5 we prove that any minimal polynomial of an element α algebraic over K, contained in the henselization $K(v)$ of (K, v_K) in $(\overline{K}, v)$, continues to remain irreducible over any v_K-maximal extension L. In Theorem 6 we give a topological characterization of the v_K-maximal extensions. In Definition 2 we introduce the notion of a $K(v)$-conservative field and in Theorems 7, 8, 9 and 10 we supply the most important connections between the $K(v)$-conservative fields, Henselian fields and the v_K-maximal extensions. In Theorem 11 we present a one to one and onto correspondence between the set of all intermediate subfields K_1, $K \subset K_1 \subset K(v)$ (the henselization of (K, v_K) in $(\overline{K}, v)$) and the set of all $K(v)$-conservative intermediate subfields

L_1, $L \subset L_1 \subset \overline{K}$, where L is a fixed v_K-maximal extension of (K, v_K). In Theorem 13 we prove that the only finite v_K-maximal extension of (K, v_K) is K itself.

In section 3 we give a new light on some classical results of Endler, Kaplansky, Schilling, Ribenboim and Warner (see Lemma 1, Corollary 1, Theorem 14 and Theorem 15).

1 Definitions and commentaries on some previous results

Let (K, v_K) be a perfect (any algebraic extension of K is separable) rank 1 valued field ($v_K : K \to \mathbb{R} \cup \{\infty\}$ is a Krull valuation) and let $\overline{K}$ be a fixed algebraic closure of K. Let v be a fixed prolongation of v_K to $\overline{K}$ and let $G_K = Gal(\overline{K}/K)$ be the absolute Galois group of K, i.e. the group of all the field automorphisms of $\overline{K}$ which fix the elements of K. The following mapping

$$x \to v^*(x) = \min\{v(\sigma(x)) : \sigma \in G_K\},$$

$x \in \overline{K}$, is a pseudovaluation on $\overline{K}$ which extends the valuation v_K. This means that v^* has the following properties:

i) $v^*(x) = \infty$ if and only if $x = 0$,

ii) $v^*(xy) \geq v^*(x) + v^*(y)$,

iii) $v^*(x + y) \geq \min\{v^*(x), v^*(y)\}$ and

iv) $v^*(a) = v_K(a)$ for any $a \in K$.

Since any valuation w on $\overline{K}$ is of the form $w = v \circ \mu$ for a $\mu \in G_K$ (see [8] or [7] for instance), we easily see that v^* depends only on (K, v_K) and not on the fixed prolongation v of v_K. This is why we call it the *spectral value of* K (see also [3] for an alternative definition). Since $v^*(x+y) \geq \min\{v^*(x), v^*(y)\}$ and $v^*(xy) \geq v^*(x)+v^*(y)$ for any $x, y \in \overline{K}$, one can immediately prove that v^* induces a topology on $\overline{K}$, in this way this last one becomes a topological field. If L is a subfield of $\overline{K}$ which contains K, one denotes v_L^* the restriction of v^* to L. If (L, v_L^*) is maximal with the property that v_L^* is a Krull valuation ($v_L^*(xy) = v_L^*(x) + v_L^*(y)$), we say that L is a *v_K-maximal extension of* K (see also [2]) or a *maximal spectral extension* (see also [9]) *of* K.

Theorem 1. *With the above notation, the following statements are equivalent:*

a) L is a v_K-maximal extension of (K, v_K).

b) v_K has a unique prolongation (as a Krull valuation!) to L and L is maximal with this property.

c) Any K-embedding $\theta : L \to \overline{K}$ is continuous w.r.t. any valuation w of L, which is an extension of v_K to L and L is maximal with this property.

Proof: To prove that a) is equivalent to b), it is sufficient (the others are trivial!) to prove that if v_L^* is a Krull valuation, then v_K has a unique prolongation to L. Let us assume that v_L^* is a Krull valuation on L, i.e. $v_L^*(xy) = v_L^*(x) + v_L^*(y)$ for any $x, y \in L$. We suppose that there exists $\alpha \in L$ and $n \geq 2$ distinct prolongations $v_1, ..., v_n$ of v_K to $K[\alpha]$. Let $a_1, a_2, ..., a_n$, $b_1, b_2, ..., b_n$ be $2n$ elements in $K[\alpha]$ with the following properties:

$$v_1(a_1) < 0 < v_2(a_2) \leq v_3(a_3) \leq ... \leq v_n(a_n),$$

$$v_2(b_2) < 0 < v_1(b_1) \leq v_3(b_3) \leq ... \leq v_n(b_n)$$

and

$$v_1(a_1b_1) \geq 0,\ v_2(a_2b_2) \geq 0.$$

Let M be a real number with $M > \max\{v_n(a_n), v_n(b_n)\}$. Since $v_1, ..., v_n$ are independent as valuations, we can apply the Approximation Theorem (see [6] or [7] for instance) to find $x, y \in K[\alpha] \subset L$, with

$$v_i(x - a_i) > M, v_i(y - b_i) > M$$

for any $i = 1, 2, ..., n$. We see that $v_i(x) = v_i(a_i)$ and $v_i(y) = v_i(b_i)$ for any $i = 1, 2, ..., n$. Since $v_i(a_ib_i) \geq 0$ for any $i = 1, 2, ..., n$, one has that $v^*(xy) \geq 0$. Since

$$v^*(x) = \min\{v_i(x) : i = 1, 2, ..., n\} = \min\{v_i(a_i) : i = 1, 2, ..., n\} = v_1(a_1)$$

and $v^*(y) = v_2(b_2)$, one has that

$$v^*(xy) \geq 0 > v^*(x) + v^*(y),$$

i.e. v^* is not a valuation on L, which is a contradiction to our assumption.

The continuity of θ w.r.t. w is equivalent to the fact that $w \circ \theta$ and w are equivalent as valuations (see [7], or [8]). Since both of them coincide on K, this last equivalence means equality (see [7], or [8]) i.e. all the valuations w which extends v_K to L are one and the same. The maximality appears both in b) and c). Thus, it is not difficult to prove that b) and c) are equivalent. □

In the following we preserve the above definitions and notation.

Theorem 2. *(see also [2] and [9]) Let $(\widetilde{K}, \widetilde{v}_K)$ be a fixed completion of (K, v_K) and let $\overline{\widetilde{K}}$ be a fixed algebraic closure of $\widetilde{K}$, which also contains $\overline{K}$. Let $L \subset \overline{K}$ be a v_K-maximal extension of (K, v_K). Then $\widetilde{K}L = \overline{\widetilde{K}}$ and $L \cap \widetilde{K} = K$.*

Example 1. *If (K, v_K) is a Henselian field, then $\overline{K}$ is the unique v_K-maximal extension of (K, v_K).*

Example 2. *Let v_p be the p-adic valuation on $\mathbb{Q}$ and $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial over $\mathbb{Q}_p$. Then $\mathbb{Q}[\alpha]$ is always contained in a v_p-maximal extension of $(\mathbb{Q}, v_p)$ (apply Zorn's Lemma).*

In [4] the following result is proved. In fact only the statement iii) is new. The others, i) and ii) are particular cases of Theorem 2.

Theorem 3. *(see [4]) Let $(\mathbb{Q}, v_p)$ be the valued field which appeared above (see example 2), $G = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $G_p = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $L^{(p)}$ be a v_p-maximal extension of $(\mathbb{Q}, v_p)$. Then:*

i) $L^{(p)}$ is dense in $\mathbb{C}_p$, the usual complex p-adic field which contains $\overline{\mathbb{Q}}_p$.

ii) $L^{(p)}\mathbb{Q}_p = \overline{\mathbb{Q}}_p$, $L^{(p)} \cap \mathbb{Q}_p = \mathbb{Q}$.

iii) Any $\mu \in G$ can be uniquely written as $\mu = \sigma\tau$, where $\sigma \in G_p$ and $\tau \in G_{L^{(p)}} = Gal(\overline{\mathbb{Q}}/L^{(p)})$.

This last result was generalized in [2] as follows.

Let (K, v_K) be a perfect rank 1 valued field, $(\widetilde{K}, \widetilde{v}_K)$ be a fixed completion of (K, v_K), $(\overline{\widetilde{K}}, \overline{\widetilde{v}}_K)$ be a fixed algebraic closure of $(\widetilde{K}, \widetilde{v}_K)$, where $\overline{\widetilde{v}}_K$ is the unique extension of $\widetilde{v}_K$ to $\overline{\widetilde{K}}$ and let $(\overline{K}, v)$ be the algebraic closure of K in $\overline{\widetilde{K}}$, where v is the restriction of $\overline{\widetilde{v}}_K$ to $\overline{K}$. We call this v *the standard extension of* v_K *to* $\overline{K}$. Let L be an intermediate field between K and $\overline{K}$ and let v_L be the restriction of v to L.

Definition 1. *Let (K, v_K) be as above. We say that L/K is a Henselian extension w.r.t. v_K if v_L is the unique extension of v_K to L. A subfield M of $\overline{K}$ is said to be a Henselian field if the extension $\overline{K}/M$ is a Henselian extension w.r.t. v_M, i. e. v, the standard extension defined above, is the unique extension of v_M to $\overline{K}$. Here v_M is the restriction of v to M.*

Let $G(v) = \{\sigma \in G \overset{def}{=} Gal(\overline{K}/K) : v \circ \sigma = v\}$ be the decomposition group of v and let $K(v)$ be the decomposition field of v, i.e. the fixed field of $G(v)$. It is in fact the henselization of (K, v_K), i.e. the least Henselian field in $(\overline{K}, v)$ which contains (K, v_K). It is equal to $\widetilde{K} \cap \overline{K}$, i.e. the algebraic closure of K in $\widetilde{K}$ and it is also equal to the v-topological closure of K in $\overline{K}$.

Before we state the main result of [2], we give two remarks on the structure of the v_p-maximal extensions of $(\mathbb{Q}, v_p)$.

Proposition 1. *(see [2]) i) There exist infinite many non-isomorphic v_p-maximal extensions of $(\mathbb{Q}, v_p)$ in $\overline{\mathbb{Q}}$.*

ii) Each v_p-maximal extension L of $(\mathbb{Q}, v_p)$ in $\overline{\mathbb{Q}}$ contains at most one quadratic subfield $\mathbb{Q}[\sqrt{d}]$.

Theorem 4. *(see [2]) With the above notation, let L be a v_K-maximal extension of (K, v_K). Then,*

i) $LK(v) = \overline{K}$, $L \cap K(v) = K$, where v is the standard extension of v_K to $\overline{K}$.

ii) L is dense in $\overline{K}$ relative to the topology induced by v.

iii) Any $\sigma \in Gal(\overline{K}/K)$ can be uniquely written as $\sigma = \tau_L \circ h_L$, where $\tau_L \in G(v)$ $(= Gal(\overline{K}/K(v))$ and $h_L \in G_L$, the absolute Galois group $Gal(\overline{K}/L)$ of L.

Remark 1. *Let now w be an arbitrary extension of a rank 1 valuation v_K, defined on a perfect field K, to a fixed algebraic closure $\overline{K}$ of K. Let $\left(\widetilde{\overline{K}}, \widetilde{w}\right)$ be a fixed completion of $(\overline{K}, w)$. Since $\widetilde{\overline{K}}$ is algebraically closed and complete (see [7] or [8] for instance), it contains an algebraic closure $\overline{\widetilde{K}}$ of $\widetilde{K}$, where $\widetilde{K}$ is the topological closure (with respect to w) of K in $\left(\widetilde{\overline{K}}, \widetilde{w}\right)$. Let $\widetilde{w}_{\widetilde{K}}$ be the restriction of $\widetilde{w}$ to $\widetilde{K}$. Since $\left(\widetilde{K}, \widetilde{w}_{\widetilde{K}}\right)$ is a completion of (K, v_K), we see that w can be viewed as a standard extension of v_K to $\overline{K}$. This is why one can replace the standard extension v of v_K to $\overline{K}$, which appears in Theorem 4, with an arbitrary extension w of v_K to $\overline{K}$.*

2 Some results on v-maximal extensions, Henselian fields and conservative fields

Let (K, v_K) be a rank 1 perfect valued field, let $\overline{K}$ be a fixed algebraic closure of K and let v be a fixed arbitrary extension of v_K to $\overline{K}$. Let $G(v)$, $K(v)$ be as above, the decomposition group of v and the decomposition field of v (the henselization of (K, v_K) in $\overline{K}$) respectively. We preserve these hypotheses and notation along this section.

Theorem 5. *Let $\alpha \in K(v)$, where $K(v)$ is the henselization of (K, v_K) in $\overline{K}$ and let L, $K \subset L \subset \overline{K}$ be a v_K-maximal extension of (K, v_K). Then, i) $\deg_K \alpha = \deg_L \alpha$, and ii) $Irr_K\alpha = Irr_L\alpha$, i.e. the monic minimal polynomials of α over K and over L respectively are identical.*

Proof: Since $i)$ implies $ii)$, it is enough to prove $i)$. Let

$$g_\alpha(X) = a_0 + a_1X + ... + a_{t-1}X^{t-1} + X^t \in L[X]$$

be the irreducible monoc polynomial of α over L and let $K[\beta] = K[a_0, a_1, ..., a_{t-1}]$, $\beta \in L$. Let us look at the following inclusions of fields:

$$K \subset K[\alpha] \subset K[\alpha][\beta],$$

$$K \subset K[\beta] \subset K[\beta][\alpha].$$

Since $\beta \in L$ one has that $K \subset K[\beta]$ is a Henselian extension (because L is a v_K-maximal extension). Thus, the minimal polynomial of β over K remains irreducible over $\widetilde{K}$, the completion of K w.r.t. v, in particular it remains irreducible over $K(v)$ (which is included in $\widetilde{K}$). So one has the following equalities:

$$\deg_K \beta = \deg_{K(v)} \beta = \deg_{K[\alpha]} \beta.$$

Hence,

$$\deg_K \alpha = \deg_{K[\beta]} \alpha = \deg_L \alpha.$$

□

Theorem 6. *With the above notation, let the following tower of valued fields, $(K, v_K) \subset (L, v_L) \subset (\overline{K}, v)$ be such that v_L is the unique extension of v_K to L, i.e. the extension of valued fields $K \subset L$ is Henselian. Then L is a v_K-maximal extension if and only if (L, v_L) is dense in $(\overline{K}, v)$.*

Proof: $\Rightarrow$) Since $LK(v) = \overline{K}$ and since K is dense in $K(v)$ (see Theorem 4, Remark 1 and the inclusions $K \subset K(v) \subset \widetilde{K}$), one has that $L = LK$ is dense in $\overline{K}$.

$\Leftarrow$) If L is dense in $\overline{K}$, the henselization of L in $\overline{K}$ is $\overline{K}$ itself. Let $\gamma \in \overline{K}$ be such that $L \subset L[\gamma]$ is a Henselian extension and let g_γ be the minimal polynomial of γ over L. Since $\overline{K}$ is the henselization of L, g_γ is still irreducible over $\overline{K}$ (see [7], or [8]) i.e. it is of degree one, so $\gamma \in L$. Thus L is a v_K-maximal extension. □

Remark 2. *Let $\mathbb{C}_p$ be the p-adic complex number field, i.e. the completion of $\overline{\mathbb{Q}}_p$, an algebraic closure of $\mathbb{Q}_p$ (the p-adic number field), w.r.t. the unique prolongation of the usual p-adic valuation on $\mathbb{Q}_p$. Let v_p be the standard p-adic valuation on $\mathbb{C}_p$. In [1] it is proved that there exists a transcendental (over $\mathbb{Q}_p$) element $t \in \mathbb{C}_p$ such that the completion (w.r.t. v_p) of $\mathbb{Q}_p(t)$ is exactly the entire $\mathbb{C}_p$. Let us denote by K the field $\mathbb{Q}_p(t)$. Since $\mathbb{C}_p$ is algebraically closed (see [7], or [8] for instance), the algebraic closure $\overline{K}$ of K in $\mathbb{C}_p$ is an algebraic closure of K. Since K is dense in $\overline{K}$, the henselization of K in $\overline{K}$ is $\overline{K}$ itself. Now, theorem 6 says that K is a v_p-maximal extension in $\overline{K}$. This means that the restriction of v_p to K splits in any subfield T of $\overline{K}$, which contains K.*

Definition 2. *A field $L_1 \subset \overline{K}$ is called a $K(v)$-conservative field if for any $\alpha \in L_1$, the minimal polynomial of α over $K(v)$ (the henzelization of (K, v_K) in $\overline{K}$) has coefficients in L_1, i.e. in $L_1 \cap K(v)$.*

Example 3. *If $L_1 \subset K(v)$, or if $K(v) \subset L_1$, then L_1 is a $K(v)$-conservative field.*

Example 4. *If L_1 is normal over K, or over $K(v)$, then L_1 is $K(v)$-conservative.*

Example 5. *There is an infinite number of primes p such that for $v = v_p$ defined on $\overline{\mathbb{Q}}_p$, $L_1 = \mathbb{Q}[\sqrt[3]{2}] \subset \mathbb{Q}(v_p) \subset \overline{\mathbb{Q}}$ is $\mathbb{Q}(v_p)$-conservative, but L_1 is not a normal extension of $\mathbb{Q}$.*

Theorem 7. *Let $K \subset L_1 \subset \overline{K}$ be a tower of fields. Then L_1 is a $K(v)$-conservative field if and only if $L_1 \cap K(v) \subset L_1$ is a Henselian extension with respect to the restrictions of the valuation v to $L_1 \cap K(v)$ and to L_1 respectively.*

Proof: $\Rightarrow$) Since the henselization of $L_1 \cap K(v)$ is also $K(v)$, it is enough to see that the minimal polynomial g_α of an element $\alpha \in L_1$ over $L_1 \cap K(v)$ is also irreducible over $K(v)$. Since L_1 is a $K(v)$-conservative field, the minimal polynomial of α over $K(v)$ has coefficients in $L_1 \cap K(v)$, so it is exactly g_α which is irreducible. Thus the extension $L_1 \cap K(v) \subset L_1$ is a Henselian extension w.r.t. v.

$\Leftarrow$) Let $\beta \in L_1$ and let f_β be its minimal polynomial over $K(v)$. In general, this last polynomial is a divisor in $K(v)[X]$ of the minimal polynomial h_β of β over $L_1 \cap K(v)$. If $L_1 \cap K(v) \subset L_1$ is a Henselian extension, then h_β is also irreducible over $K(v)$, the henselization of $L_1 \cap K(v)$. Thus, $h_\beta = f_\beta$ and so the coefficients of f_β are also in L_1. Hence L_1 is a $K(v)$-conservative field. □

Theorem 8. *Let $K \subset K_1 \subset K(v)$ and let $K \subset L \subset \overline{K}$ be a v_K-maximal extension. Then the extension $K_1 \subset LK_1$ is a v_{K_1}-maximal extension.*

Proof: Since K_1 is dense in $K(v)$, one has that LK_1 is dense in $LK(v) = \overline{K}$ (Theorem 4). Hence, it will be enough to prove that $K_1 \subset LK_1$ is a Henselian extension (see Theorem 6). Let $\gamma \in L$ and let f_γ be the minimal polynomial of γ over K. It is also irreducible over $K(v)$ (L is a v_K-maximal extension), in particular it is also irreducible over K_1. Thus $K_1 \subset K_1[\gamma]$ is a Henselian extension for any $\gamma \in L$. Hence $K_1 \subset LK_1$ is a Henselian extension. □

Theorem 9. *Let L be a v_K-maximal extension and let $L \subset L_1 \subset \overline{K}$ be a tower of fields. Then, $L_1 \cap K(v) \subset L_1$ is a Henselian extension if and only if $[L_1 \cap K(v)]L = L_1$.*

Proof: $\Rightarrow$) From Theorem 8 the extension $L_1 \cap K(v) \subset [L_1 \cap K(v)]L$ is a $v_{L_1 \cap K(v)}$-maximal extension. If $L_1 \cap K(v) \subset L_1$ is a Henselian extension and since $L_1 \cap K(v) \subset [L_1 \cap K(v)]L \subset L_1$, one has that $[L_1 \cap K(v)]L = L_1$.

$\Leftarrow$) Assume now that $[L_1 \cap K(v)]L = L_1$. From Theorem 8, $L_1 \cap K(v) \subset [L_1 \cap K(v)]L$ is a Henselian extension. Thus, $L_1 \cap K(v) \subset L_1$ is also a Henselian extension. □

Theorem 10. *Let $K \subset L \subset L_1 \subset \overline{K}$ be a tower of fields, where L is a v_K-maximal extension. Then the following assertions are equivalent:*

i) L_1 is a $K(v)$-conservative field.

ii) $L_1 \cap K(v) \subset L_1$ is a Henselian extension with respect to the restrictions of the valuation v to $L_1 \cap K(v)$ and L_1 respectively.

iii) $[L_1 \cap K(v)]L = L_1$.

Proof: Here is nothing else to prove but a direct application of Theorems 7, 8 and 9. □

Theorem 11. *Let L be a fixed v_K-maximal extension in $\overline{K}$. The mappings $K_1 \to LK_1$ and $L_1 \to L_1 \cap K(v)$, where $K \subset K_1 \subset K(v)$ and $L \subset L_1 \subset \overline{K}$ are two towers of fields, supply a one-to-one and onto correspondence between the K-subextensions K_1 of $K(v)$ and the conservative superfields L_1 of L which are contained in $\overline{K}$.*

Proof: Theorem 8 says that the extension $K_1 \subset LK_1$ is a v_{K_1}-maximal extension. Let us take an element $\alpha \in L_1$ $(= K_1 L)$. Since the henselization of K_1 is also $K(v)$, its minimal polynomial f_α over $K(v)$ is also the minimal polinomial of α over K_1, thus the coefficients of f_α are in K_1 which is included in L_1 $(= K_1 L)$. Hence $K_1 L$ is a $K(v)$-conservative field. Let us prove now that $LK_1 \cap K(v) = K_1$. Since $K_1 \subset LK_1 \cap K(v)$, it remains to prove that $K_1 \supset LK_1 \cap K(v)$. Let us take $\gamma \in LK_1 \cap K(v)$. Since $K_1 \subset LK_1$ is a Henselian extension (see theorem 8), the minimal polynomial f_γ of γ over K_1 is also irreducible over $K(v)$, i.e. it is also the minimal polynomial of γ over $K(v)$. Since $\gamma \in K(v)$, $\deg f_\gamma = 1$, thus $\gamma \in K_1$. Therefore, $K_1 \supset LK_1 \cap K(v)$. The equality $[L_1 \cap K(v)]L = L_1$ is clear because of Theorem 10 iii) (L_1 is $K(v)$-conservative). □

Theorem 12. *Let L be a fixed v_K-maximal extension in $\overline{K}$ and let $K \subset K_1 \subset K(v)$ be a tower of fields and let $L_1 = K_1 L \subset \overline{K}$. Let $G_{K_1} = Gal(\overline{K}/K_1)$, $G_{L_1} = Gal(\overline{K}/L_1)$. Then*

i) $L_1 K(v) = \overline{K}$ and $L_1 \cap K(v) = K_1$.

ii) Any $\mu \in G_{K_1}$ can be uniquely written as $\mu = \sigma\tau$, where $\sigma \in G(v)$ and $\tau \in G_{L_1}$.

Proof: Since $K(v)$ is also the henselization of K_1 in $\overline{K}$, we simply apply Theorem 4 iii) to extension $K_1 \subset K_1L$. ∎

Remark 3. *$G(v)$ is never a normal subgroup of no G_{K_1} because $K(v)$ is never a normal extension of K or of K_1, except for the trivial cases $K(v) = K$ or $\overline{K}$ (see corollary 1 bellow). But G_{L_1} may be a normal subgroup of G_{K_1} when, for instance, L_1 is the normal closure $\widehat{L}$ of L in $\overline{K}$ and $K_1 = \widehat{L} \cap K(v)$.*

Theorem 13. *Let (K, v_K) be a rank 1 perfect valued field which is not algebraically closed and let v be an extension of v_K to a fixed algebraic closure $\overline{K}$ of K. Let (L, v_L) be a v_K-maximal extension of (K, v_K) in $\overline{K}$, such that L is finite dimensional over K. Then, $L = K$ and $K(v) = \overline{K}$.*

Proof: Assume that $[L : K] = n$ and say $L = K[\gamma]$, where $\gamma \in L$, $\deg_K \gamma = n$. Since $LK(v) = \overline{K}$ (see Theorem 12), one has that $[\overline{K} : K(v)] = n = 2$ (see the Artin-Schreier theory), if $L \neq K$. In this case $K(v)$ is a Henselian and a real closed field, which is impossible (see Ribenboim [11]).Thus, $L = K$ and $K(v) = \overline{K}$. ∎

3 Commentaries to some classical results

We start with two well known auxiliary results. In order to preserve the unity of our presentation, we also prove them in our specific way.

Lemma 1. *(see also [5]) Let (K, v_K) be a valued field and let $(K, v_K) \subset (M, w_K)$ be an algebraic extension of valued fields, $M \subset \overline{K}$, where M is normal over K. Here w_M is considered to be the restriction of a prolongation w of v_K to $\overline{K}$. Let v be an extension of v_K to $\overline{K}$ and let v_M be the restriction of v to M. Assume that (M, v_M) is a Henselian field. Then (M, w_M) is also a Henselian field.*

Proof: If (K, v_K) was a Henselian field, then $w_M = v_M$ and we would have nothing to prove. Thus, we can assume that (K, v_K) is not a Henselian field. Let us also suppose that (M, w_M) is not a Henselian field. Let $\sigma \in Gal(\overline{K}/K)$ such that $w = v \circ \sigma$ (see [8], or [7] for instance). If (M, w_M) is not a Henselian field, then there exist $\sigma_1, \sigma_2 \in Gal(\overline{K}/M)$ with $\sigma_1 \neq \sigma_2$ and $v \circ \sigma \circ \sigma_1 \neq v \circ \sigma \circ \sigma_2$. Let us consider the following valuations on $\overline{K}$:

$$v_1 = v \circ \sigma \circ \sigma_1 \circ \sigma^{-1}$$

and

$$v_2 = v \circ \sigma \circ \sigma_2 \circ \sigma^{-1}.$$

It is easy to see that $v_1 \neq v_2$ on $\overline{K}$ (take $y \in \overline{K}$ with $v(\sigma(\sigma_1(y))) \neq v(\sigma(\sigma_2(y)))$ and put $y = \sigma^{-1}(z)$). Let us prove that v_1 and v_2 are both extensions of v_M. Take $x \in M$. Then $\sigma^{-1}(x) \in M$ (M is normal over K). So,

$$v_1(x) = (v \circ \sigma \circ \sigma_1)\sigma^{-1}(x) = v(\sigma(\sigma^{-1}(x)) = v(x) = v_M(x).$$

Here we used the fact that $\sigma_1 \in Gal(\overline{K}/M)$. We also have $v_2(x) = v_M(x)$ for any $x \in M$. Since $v_1 \neq v_2$ and both extend v_M, we just obtained that (M, v_M) is not a Henselian field, a contradiction. Hence (M, w_M) must be also a Henselian field. ∎

Corollary 1. *(see also [5]) Let v be a valuation on $\overline{K}$ which extends v_K. Assume that (K, v_K) is not Henselian and that M/K, $M \subset \overline{K}$, is a normal extension. Let w be another valuation on $\overline{K}$ and w_M its restriction to M such that (M, w_M) is a Henselian field. Then $M = \overline{K}$.*

Proof: We simply apply Lemma 1 and the Uniqueness Theorem for Henselian fields (F. K. Schmidt, Kaplansky, Schilling) (see [6], [7], or [13]). ∎

Here is another application of our theory of v_K-maximal extensions.

Theorem 14. *(see also Warner [14]) Let v be a prolongation of v_K to $\overline{K}$ such that (K, v_K) is not a Henselian field and $[K(v) : K] < \infty$. Then $K(v)$ is an algebraically closed field, $K(v) = K(\sqrt{-1})$ and K is a real closed field. In particular, if (M, v_M) is a finite Henselian extension of a rank 1 valued field (K, v_K), then either (K, v_K) is a Henselian field, or K is a real closed field and the least Henselian field which contains (K, v_K) is $(\overline{K}, v)$.*

Proof: Since any valuation v on $\overline{K}$ can be viewed as a standard extension of a v_K (its restriction to K), we can assume that v is a standard extension of v_K. If $[K(v) : K] = n$, then $[LK(v) = \overline{K} : L] = n$ (see Theorem 5 and Theorem 4), where L is any v_K-maximal extension of (K, v_K). Thus $[\overline{K} : L] = n$. From Artin-Schreier theory we get that $n = 2$, $\overline{K} = L[\sqrt{-1}]$ and L is a real closed field. Moreover, since $[K(v) : K] = 2$, one has that $K(v)/K$ is a normal extension. From Corollary 1 one has that $K(v) = \overline{K}$. Thus, $\sqrt{-1} \in K(v)$. But $\sqrt{-1}$ cannot be in K, otherwise $L = \overline{K}$ and $K(v) = K$, impossible! ((K, v_K) is not a Henselian field). Hence, $K(v) = K(\sqrt{-1})$ and so K is a real closed field. ∎

Here is the last application of our theory of v_K-maximal extensions.

This is a generalization of a Ribenboim's result [10].

Theorem 15. *Let K be a perfect field and let (K, v_K) be a rank 1 valued field such that $[\widetilde{K} : K] < \infty$, where $(\widetilde{K}, \widetilde{v}_K)$ is a completion of (K, v_K). Then, either $K = \widetilde{K}$, i.e. K is complete, or $K(v) = \overline{K} = \widetilde{K} = K[\sqrt{-1}]$, K is a real closed field and char $K = 0$. If (K, v_K) is discrete, then $[\widetilde{K} : K] < \infty$ implies $K = \widetilde{K}$ (see also Ribenboim [10]).*

Proof: Let v be the standard extension of v_K to $\overline{K}$, the algebraic closure of K in $\overline{\widetilde{K}}$, a fixed algebraic closure of $\widetilde{K}$. Then $K(v) \subset \widetilde{K}$ and, since $[\widetilde{K} : K] < \infty$, one has that $[K(v) : K] < \infty$ and we can apply Theorem 14. Thus, either $K = K(v) = \widetilde{K}$, or $K(v) = \overline{K} = \widetilde{K} = K[\sqrt{-1}]$ and so K is a real closed field and char $K = 0$. If (K, v_K) is discrete, then $K = \widetilde{K}$, otherwise $\overline{K} = \widetilde{K}$ would be discrete relative to v. But this last fact is impossible because the value group of any algebraically closed valued field is at least divisible. Thus $K = \widetilde{K}$ in the case when (K, v_K) is discrete. ∎

Remark 4. *If (K,v) is not discrete, one may have $\widetilde{K} \neq K$ and so $\widetilde{K} = K[\sqrt{-1}]$ is algebraic closed. Let for instance $\mathbb{C}_p$, the complex p-adic field. Since $\mathbb{C}_p$ is isomorphic with $\mathbb{C}$ as a field, let $\mathbb{R}^* \subset \mathbb{C}_p$ be a subfield of $\mathbb{C}_p$ which is isomorphic with $\mathbb{R}$, the usual real number field. Since $\mathbb{R}^*$ is not complete relative to v_p (otherwise it contains $\mathbb{Q}_p$ but this last one is not an ordered field), and since the completion of $(\mathbb{R}^*, v_p)$ is $\mathbb{C}_p$, we see that for $(K,v) = (\mathbb{R}^*, v_p)$, one has that $\widetilde{K} \neq K$ and so $\widetilde{K} = K[\sqrt{-1}]$ is algebraic closed.*

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