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Functions preserving spheres in $\mathbb{Q}_p$

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

We study functions $f : A \to \mathbb{Q}_p$, which preserve spheres with center 0, where A is an open subset of $\mathbb{Q}_p$ which contains 0.

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1 Introduction

Let A be an open subset of $\mathbb{C}_p$. A continuous one-to-one function $f : A \to \mathbb{C}_p$ preserving the distance between the points is an *isometry* (see [2], [3], [6] or [7] for terminology and classic results). Bishop characterized in [1] the isometries $f : \mathbb{Q}_p \to \mathbb{Q}_p$. Thus he showed that, if $f(0) = 0$, then f is an isometry if and only if, for each positive real number R, f permutes the balls with center 0 and radius R. Brussel studied the fixed points of certain families of isometries $[\ ]_q$ defined on the unit ball of $\mathbb{C}_p$ (see [4]). The restriction of $[\ ]_q$ (called q-bracket) to $\mathbb{Z}_p$ is an interpolation of the arithmetic function on the set of nonnegative integers given by

$$[n]_q = 1 + q + ... + q^{n-1}.$$

Here q is an element of the ball with center 1 and radius $p^{-\frac{1}{p-1}}$.

In this paper we study the functions $f : A \to \mathbb{Q}_p$, where A is an open subset of $\mathbb{Q}_p$ and $0 \in A$, which preserve spheres with center 0. Every isometry f such that $f(0) = 0$ belongs to the set of these functions. There are functions preserving all the spheres with center 0 which are neither continuous nor one-to-one (see Section 2).

Theorem 1 from Section 2 characterizes the functions preserving spheres which are isometries. Hence we obtain the Bishop's result quoted above. Then in Theorem 2 we give a representation of continuous functions preserving a ball by means of Mahler series. Section 3 deals with analytic functions preserving spheres (see Theorem 3). If $f : \mathbb{Q}_p \to \mathbb{Q}_p$ is an entire function, then f preserves all the spheres included in a fixed ball if and only if it preserves a finite number of spheres (see Remark 2). A construction of an entire function preserving all the spheres is given in Theorem 4. All counterexamples of Liouville's Theorem are constructed by means of entire functions having this property (see Remark 3).

2 Continuous functions preserving spheres

Let p be a fixed prime and consider $|\ |$ the normalized p-adic absolute value defined on $\mathbb{Q}$ (i.e. $|p| = \frac{1}{p}$). If $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\ |$, then every nonzero element $x \in \mathbb{Q}_p$ has the representation

$$x = \sum_{i=m}^{\infty} a_m p^m,\ m \in \mathbb{Z},\ a_i \in \{0, 1, ..., p-1\}, a_m \neq 0. \tag{2.1}$$

We denote $a_m = x^{(0)}$ and $m = v_p(x)$, that is the p-adic valuation of x.

Let A be an open subset of $\mathbb{Q}_p$ which contains 0 and let $f : A \to \mathbb{Q}_p$ be a function such that

$$|f(x)| = |x|, \tag{2.2}$$

for every $x \in A$. If R is a positive real number, we denote by $B(R) = \{x \in \mathbb{Q}_p : |x| \leq R\}$ and $S(R) = \{x \in \mathbb{Q}_p : |x| = R\}$ the ball with circumference and the sphere, with center 0 and radius R, respectively. Then by (2.2) it follows that $f(0) = 0$ and f preserves every sphere, with center 0, included in A. Moreover, for every $x \in A$, x different from 0, there exists a unique p-adic unit u_x, such that

$$f(x) = u_x x. \tag{2.3}$$

We put $\mathcal{U}_f = \{u_x\}_{x \in A^*}$, where $A^* = A \backslash \{0\}$. Thus f is uniquely defined by the set of p-adic units $\mathcal{U}_f$.

We call $\mathcal{U}_f$ *admissible*, if, for every $x, y \in A^*$ such that $|x| = |y|$, it follows that

$$|x^{(0)} p^{v_p(x)} (u_x - u_y) + u_y^{(0)} (x - y)| = |x - y|. \tag{2.4}$$

Remark 1. Consider $f : \mathbb{Q}_p \to \mathbb{Q}_p$ an isometry (i.e. $|f(x) - f(y)| = |x - y|$, for every $x, y \in \mathbb{Q}_p$) with $f(0) = 0$. Then f is a continuous function which verifies (2.2), where $A = \mathbb{Q}_p$.

There exist functions verifying (2.2) which are not continuous functions. Thus, for example, we take p an odd prime, $A = \mathbb{Q}_p$, and for $x = \sum\limits_{j=m}^{\infty} a_j(x) p^j$, with $m \in \mathbb{Z}$, $a_j(x) \in \{0, 1, ..., p-1\}$, $a_m(x) \neq 0$, we define

$$f(x) = \sum_{j=m}^{\infty} b_j(x) p^j,$$

where $b_j(x) = a_j(x)$, for all $j > m$, and

$$b_m(x) = \begin{cases} a_m(x),\ \text{if either } a_j(x) = 0,\ \forall\ j > m,\ \text{or } a_j(x) \neq 0,\ \forall\ j > m, \\ a_{j_0}(x),\ \text{otherwise, with } j_0 = \min\limits_{a_j(x) \neq 0} \{j : j > m\}. \end{cases}$$

It follows easily that f verifies (2.2) and if we take $x_n = 1 + \sum_{j=1}^{n} 2p^j$, $n \geq 1$, then $f(x_n) = 2 + \sum_{j=1}^{n} 2p^j$. Since

$$x^* = \lim_{n\to\infty} x_n = 1 + \sum_{j=1}^{\infty} 2p^j$$

and

$$f(x^*) = x^* \neq \lim_{n\to\infty} f(x_n) = 2 + \sum_{j=1}^{\infty} 2p^j,$$

it follows that f is not a continuous function. It is easy to see that f is not one-to-one.

The following result shows which are the functions satisfying (2.2) which are isometries.

Theorem 1. *Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be a function such that $f(0) = 0$. Then f is an isometry if and only if f verifies (2.2) and $\mathcal{U}_f$ is admissible.*

Proof: Suppose that f is an isometry. Then by (2.3) we obtain

$$|u_x x - u_y y| = |x - y|.$$

Since $|u_x x - u_y y| = |x(u_x - u_y) + u_y(x - y)|$ and $|u_y| = 1$ it follows that $|x(u_x - u_y)| \leq |x - y|$. Hence we get (2.4) and $\mathcal{U}_f$ is admissible.

Conversely, we suppose that f verifies (2.2) and $\mathcal{U}_f$ is admissible. If $|x| \neq |y|$, then by (2.2)

$$|f(x) - f(y)| = \max\{|f(x)|, |f(y)|\} = \max\{|x|, |y|\} = |x - y|.$$

Consider $|x| = |y|$. By (2.3), it follows that

$$|f(x) - f(y)| = |x(u_x - u_y) + u_y(x - y)|.$$

Since $\mathcal{U}_f$ is admissible, it follows that $|x^{(0)} p^{v_p(x)}(u_x - u_y)| \leq |x - y|$. Hence

$$|f(x) - f(y)| = |x^{(0)} p^{v_p(x)}(u_x - u_y) + u_y^{(0)}(x - y)| = |x - y|$$

and f is an isometry. □

Corollary 1. *([1], Theorem 1.2) Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be a function such that $f(0) = 0$. Then f is an isometry if and only if there exist permutations*

$$\sigma_i : \{1, 2, ..., p-1\} \to \{1, 2, ..., p-1\},\ i \in \mathbb{Z}, \tag{2.5}$$

and

$$\tau_\alpha : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\},\ \alpha \in \bigcup_{i\in\mathbb{Z}} \mathbb{Q}_p/p^i, \tag{2.6}$$

such that f is given by

$$f(x) = \sigma_m(a_m)p^m + \sum_{i=m+1}^{\infty} \tau_{a_m p^m + \ldots + a_{i-1}p^{i-1}}(a_i)p^i, \qquad (2.7)$$

where x is given by (2.1).

Proof: Suppose f is an isometry. Then by Theorem 1 f satisfies (2.2) and $\mathcal{U}_f$ is admissible. Consider $v_p(x) = v_p(y) = m$ and the canonical representations of the form (2.1)

$$y = \sum_{i=m}^{\infty} b_i p^i,\ u_x = \sum_{i=0}^{\infty} c_i p^i,\ u_y = \sum_{i=0}^{\infty} d_i p^i,\ f(x) = \sum_{i=m}^{\infty} \alpha_i p^i,\ f(y) = \sum_{i=m}^{\infty} \beta_i p^i. \qquad (2.8)$$

By (2.3) and (2.8), for every $t \geq m$, we obtain

$$\sum_{i=m}^{t} \left(\sum_{j=0}^{i-m} c_j a_{i-j} \right) p^i \equiv \sum_{i=m}^{t} \alpha_i p^i \pmod{p^{t+1}}, \qquad (2.9)$$

and

$$\sum_{i=m}^{t} \left(\sum_{j=0}^{i-m} d_j b_{i-j} \right) p^i \equiv \sum_{i=m}^{t} \beta_i p^i \pmod{p^{t+1}}. \qquad (2.10)$$

Suppose $v_p(x-y) = s \geq m$. Then, for every $i = m, ..., s-1$, $a_i = b_i$, $a_s \neq b_s$ and, by (2.4), this is equivalent to

$$c_i = d_i.\ \text{for}\ i = 0, 1, ..., s-m-1,\ a_m(c_{s-m} - d_{s-m}) + d_0(a_s - b_s) \not\equiv 0 (\text{mod}\ p). \qquad (2.11)$$

Thus by (2.9) and (2.10) this is equivalent to $\alpha_i = \beta_i$, for $i = m, m+1, ..., s-1$ and $\alpha_s \neq \beta_s$. We define $\sigma_m(a_m) := \alpha_m$, $\tau_{a_m p^m + \ldots + a_{i-1}p^{i-1}}(a_i) := \alpha_i$, $i = m, m+1, ..., s-1$. Now, by induction on s, it follows (2.7).

Conversely, suppose that f is given by (2.7). Then it follows that f satisfies (2.2). It is enough to prove that $\mathcal{U}_f$ is admissible. Consider $x, y \in \mathbb{Q}_p$ such that $v_p(x) = v_p(y) = m$ and $v_p(x-y) = s \geq m$. Then (2.8) holds, where $\alpha_m = \sigma_m(a_m)$, $\beta_m = \sigma_m(b_m)$, $\alpha_i = \tau_{a_m p^m + \ldots + a_{i-1}p^{i-1}}(a_i)$, $\beta_i = \tau_{b_m p^m + \ldots + b_{i-1}p^{i-1}}(b_i)$, for $i \geq m+1$. Thus, for $t = m, m+1, ..., s$, by (2.9) and (2.10) we find

$$c_0 a_m \equiv \sigma_m(a_m) (\text{mod}\ p),\ \sum_{j=0}^{i-m} c_j a_{i-j} \equiv \tau_{a_m p^m + \ldots + a_{i-1}p^{i-1}}(a_i) \pmod{p} \qquad (2.12)$$

and

$$d_0 b_m \equiv \sigma_m(b_m) (\text{mod}\ p),\ \sum_{j=0}^{i-m} d_j b_{i-j} \equiv \tau_{b_m p^m + \ldots + b_{i-1}p^{i-1}}(b_i) \pmod{p}. \qquad (2.13)$$

Hence, for $s \geq m$, because σ_m and $\tau_{a_m p^m + \ldots + a_{i-1} p^{i-1}}$ are one-to-one, we get

$$d_0(a_s - b_s) + a_m(c_{s-m} - d_{s-m}) \not\equiv 0 \pmod{p}. \tag{2.14}$$

This implies (2.4) and, by Theorem 1, f is an isometry. ∎

Now we study functions preserving spheres which are continuous functions.

Theorem 2. *Let $f : B(p^m) \to B(p^m)$ be a function such that $f(0) = 0$, where m is an integer. Then f is a continuous function which verifies (2.2), for every $x \in B(p^m)$, if and only if f has the representation*

$$f(x) = p^{-m} \sum_{k=1}^{\infty} a_k \begin{pmatrix} p^m x \\ k \end{pmatrix}, \; a_k \in \mathbb{Q}_p, \tag{2.15}$$

where

$$\lim_{k\to\infty} |a_k| = 0, \tag{2.16}$$

$$\begin{pmatrix} x \\ k \end{pmatrix} = \frac{x(x-1)...(x-k+1)}{k!}, \tag{2.17}$$

the series in (2.15) converges uniformly on $\mathbb{Z}_p$, and, for every nonnegative integer n,

$$\left| \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a_k \right| = |n|. \tag{2.18}$$

Proof: We denote $\mathcal{C}(B(p^m))$ the set of continuous functions $f : B(p^m) \to B(p^m)$. If $m = 0$, $B(1) = \mathbb{Z}_p$, and by Mahler Theorem (see, for example, [5] or [6], p. 173), for every $f \in \mathcal{C}(B(p^m))$ it follows that, (2.15), (2.16) hold and the series in (2.15) converges uniformly on $\mathbb{Z}_p$.

Define $\Phi : \mathcal{C}(B(p^m)) \to \mathcal{C}(\mathbb{Z}_p)$ by

$$\Phi(f)(x) := p^m f\left(\frac{x}{p^m}\right), \; f \in \mathcal{C}(B(p^m)), \; x \in \mathbb{Z}_p,$$

and $\Psi : \mathcal{C}(\mathbb{Z}_p) \to \mathcal{C}(B(p^m))$ by

$$\Psi(g)(x) := p^{-m} g\left(p^m x\right), \; g \in \mathcal{C}(\mathbb{Z}_p), \; x \in B(p^m).$$

It follows easily that Ψ is the inverse function of Φ. Hence Φ is a bijective function and by (2.15), (2.16) written for $m = 0$ we obtain (2.15), (2.16), for every m.

By (2.15) we obtain

$$f(np^{-m}) = p^{-m} \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a_k \tag{2.19}$$

and by (2.2) we obtain (2.18).

Conversely, we suppose that (2.15), (2.16) and (2.18) are fulfilled. Because Ψ is a bijective function, then by Mahler Theorem f is a continuous function and by (2.18) we obtain that (2.2) holds for all nonnegative integers. Since the set of numbers np^{-m}, where m, n are nonnegative integers is a dense subset in $B(p^m)$ it follows the theorem. ∎

By Theorem 2 and (2.19) one has the following result.

Corollary 2. *Let $f : B(p^m) \to B(p^m)$ be a function such that $f(0) = 0$. Then f is an isometry if and only if (2.15), (2.16) hold and, for every positive integers k, l,*

$$\left| \sum_{s=1}^{k} \begin{pmatrix} k \\ s \end{pmatrix} a_s - \sum_{t=1}^{l} \begin{pmatrix} l \\ t \end{pmatrix} a_t \right| = |k - l|, \ k > l. \tag{2.20}$$

3 Analytic functions preserving spheres

For a fixed integer t, let

$$f = \sum_{i=0}^{\infty} c_i X^i, \ c_i \in \mathbb{Q}_p, \tag{3.1}$$

be a convergent series on $B(p^{-t})$. Since every coefficient $c_i \neq 0$ has the form

$$c_i = u_i p^{i\theta(i)}, \ u_i \in \mathbb{Q}_p, \ i\theta(i) \in \mathbb{Z}, \tag{3.2}$$

where $|u_i| = 1$, and the series converges for $x = p^t$ we obtain

$$\lim_{i \to \infty} (\theta(i) + t)i = \infty. \tag{3.3}$$

Because (3.1) can be written as

$$f = \sum_{k \geq 0} u_{i_k} \left(p^{\theta(i_k)} X \right)^{i_k}, \ |u_{i_k}| = 1, \tag{3.4}$$

we denote

$$I_f = \{i_k\}_{k \geq 0}, \ \Theta_f = \{\theta(i_k)\}_{k \geq 0}.$$

Define $h_t : I_f \times \Theta_f \to \mathbb{N}$ such that

$$h_t(i, \theta(i)) = (\theta(i) + t)i. \tag{3.5}$$

Since $i\theta(i) \in \mathbb{Z}$, by (3.3), there exists

$$m_t = \min_{k \geq 0} \{h_t(i_k, \theta(i_k))\}. \tag{3.6}$$

From (3.3), (3.5) and (3.6) it follows that there exists $N_f \in \mathbb{Z} \cup \{\infty\}$ such that for all $t \geq N_f$, $m_t > t$, and for all $t < N_f$, $m_t \leq t$.

If $m_t \leq t$, for $r = m_t, m_t + 1, ..., t$, denote

$$V_t^{(r)} = \{i_k \in I_f : h_t(i_k, \theta(i_k)) = r\}, \ V_t = \bigcup_{r=m_t}^{t} V_t^{(r)}. \tag{3.7}$$

By (3.3), it follows that $V_t^{(r)}$ is a finite set, for every $r = m_t, m_t + 1, ..., t$.

Let K be a field and

$$P = \sum_{k=0}^{d} a_k X^k \tag{3.8}$$

a polynomial with coefficients in K of degree equal to d. For $j \geq 0$, we denote

$$T^j(P) = \sum_{k=j}^{d} a_k \binom{k}{j} X^{k-j}. \tag{3.9}$$

Lemma 1. *(Taylor's formula) Let K be a field, $\alpha \in K$ a fixed element and P a polynomial given by (3.8). Then, for every $x \in K$,*

$$P(x) = \sum_{j=0}^{d} T^j(P)(\alpha)(x-\alpha)^j \tag{3.10}$$

Proof: For every nonnegative integer k

$$X^k = (X - \alpha + \alpha)^k = \sum_{j=0}^{k} \binom{k}{j} \alpha^{k-j}(X-\alpha)^j.$$

Then, by (3.8), we get

$$P(x) = \sum_{k=0}^{d}\sum_{j=0}^{k} a_k \binom{k}{j} \alpha^{k-j}(x-\alpha)^j$$

$$= \sum_{j=0}^{d} T^j(P)(\alpha)(x-\alpha)^j.$$

□

Now we prove the following result:

Lemma 2. *Let $f : B(p^{-t}) \to B(p^{-t})$ be the function defined by the convergent series (3.4). Then f satisfies (2.2), for every $x \in S(p^{-t})$, if and only if, $m_t \leq t$, and for all $a_i \in \{0, 1, 2, ..., p-1\}$, $a_0 \neq 0$,*

$$\sum_{s=0}^{t-m_t} \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \sum_{\gamma=0}^{t-m_t-s} \sum_{J_q \in M(i_k,\gamma)} p^{s+\gamma} \binom{i_k}{j_0 \, ... \, j_q} a_0^{j_0} ... a_q^{j_q} \in S(p^{-t+m_t}), \tag{3.11}$$

where

$$M(i,\gamma) = \{J_q = (j_0, ..., j_q) \in \mathbb{N}^{q+1} : j_0 + j_1 + ... + j_q = i,\ j_1 + 2j_2 + ... + qj_q = \gamma,$$

$$q = 0, 1, ..., \gamma\}, \tag{3.12}$$

and

$$\left(\begin{array}{c} i \\ j_0 \ ... \ j_q \end{array}\right) = \frac{i!}{j_0! j_1! ... j_q!}$$

are multinomial coefficients. Moreover (2.2) holds if for all $\alpha \in \{0, 1, .., t - m_t\}$*, all* $\tau \in \{0, 1, .., \alpha\}$*,* $\tau \neq t - m_t$*, and all* $a_0 \in \{1, 2, ..., p - 1\}$*,*

$$\sum_{s=0}^{\alpha-\tau} p^{\tau+s} T^{\tau}(P_{s,t})(a_0) \in B(p^{-\alpha-1}), \tag{3.13}$$

and

$$T^{t-m_t}(P_{0,t})(a_0) \in S(1), \tag{3.14}$$

where

$$P_{s,t} = \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} x^{i_k}.$$

Proof: Because every $x \in S(p^{-t})$ can be represented as

$$x = \sum_{j=0}^{\infty} a_j p^{t+j}, \ a_j \in \{0, 1, ..., p - 1\}, \ a_0 \neq 0, \tag{3.15}$$

by (3.4), it follows that (3.8) holds if and only if

$$\left| \sum_{r=m_t}^{t} \sum_{i_k \in V_t^{(r)}} u_{i_k} \left(\sum_{j=0}^{\infty} a_j p^{\theta(i_k)+t+j} \right)^{i_k} \right| = \frac{1}{p^t}, \tag{3.16}$$

for all $a_j \in \{0, 1, ..., p - 1\}$, $a_0 \neq 0$. Since

$$\left| \left(\sum_{j=t-m_t+1}^{\infty} a_j p^{\theta(i_k)+t+j} \right)^{i_k} \right| = \left| \left(\sum_{j=t-m_t+1}^{\infty} a_j p^{\frac{h_t(i_k,\theta(i_k))}{i_k}+j} \right)^{i_k} \right| < \frac{1}{p^t},$$

by (3.7) and (3.16) it follows that (3.8) holds if and only if

$$\left| \sum_{s=0}^{t-m_t} \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \left(\sum_{j=0}^{t-m_t} a_j p^{\frac{m_t+s}{i_k}+j} \right)^{i_k} \right| = \frac{1}{p^t}. \tag{3.17}$$

Then (3.17) is equivalent to

$$\left| \sum_{s=0}^{t-m_t} \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \sum_{j_0+...+j_{t-m_t}=i_k} \left(\begin{array}{c} i_k \\ j_0 ... j_{t-m_t} \end{array}\right) a_0^{j_0} ... a_{t-m_t}^{j_{t-m_t}} p^{m_t+s+\gamma} \right| = \frac{1}{p^t}, \tag{3.18}$$

where $\gamma = j_1 + 2j_2 + ... + (t - m_t)j_{t-m_t}$ depends on $j_1, j_2, ..., j_{t-m_t}$. Hence we get

$$\left| \sum_{s=0}^{t-m_t} \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \sum_{\substack{j_0 + ... + j_{t-m_t} = i_k \\ \gamma \le t - m_t - s}} \binom{i_k}{j_0 ... j_{t-m_t}} a_0^{j_0} ... a_{t-m_t}^{j_{t-m_t}} p^{s+\gamma} \right| = \frac{1}{p^{t-m_t}},$$

for all $a_i \in \{0, 1, ..., p-1\}$, $a_0 \neq 0$. Since $s + \gamma \in \{0, 1, ..., t - m_t\}$, this implies that (2.2) is equivalent to (3.11).

Now we suppose that (3.13) and (3.14) hold. Then, because

$$\binom{i_k}{j_0\ j_1\ ...\ j_q} = \frac{\tau!}{j_1! ... j_q!} \binom{i_k}{\tau}, \ \tau = j_1 + ... + j_q,$$

we get

$$\sum_{s=0}^{t-m_t} \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \sum_{\gamma=0}^{t-m_t-s} \sum_{J_q \in M(i_k,\gamma)} p^{s+\gamma} \binom{i_k}{j_0\ ...\ j_q} a_0^{j_0} ... a_q^{j_q}$$

$$= \sum_{\tau=0}^{t-m_t-1} \sum_{\gamma_1=0}^{t-m_t} \sum_{s=0}^{t-m_t-\tau-\gamma_1} \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \binom{i_k}{\tau} p^{\tau+s} a_0^{i_k-\tau} p^{\gamma_1} \cdot$$

$$\left(\sum_{\substack{j_1 + ... + j_q = \tau \\ j_2 + ... + (q-1)j_q = \gamma_1}} \frac{\tau!}{j_1! ... j_q!} a_1^{j_1} ... a_q^{j_q} \right)$$

$$+ \sum_{i_k \in V_t^{(m_t)}} u_{i_k} \binom{i_k}{t - m_t} p^{t-m_t} a_0^{i_k - t + m_t}.$$

Since the expressions in the big parentheses are integers, by (3.13) with $\alpha = t - m_t - \gamma_1$, and by by (3.14) it follows (3.11). Thus (2.2) holds, for every $x \in S(p^{-t})$. □

Remark 2. By Lemma 2 it follows that an analytic function $f : B(p^{-n}) \to B(p^{-n})$ given by (3.4) verifies (2.2) on $S(p^{-t})$, where t is a fixed integer greater than or equal to n if and only if a finite number of coefficients u_i verify (3.11). In fact if we denote

$$P_{m_t+s,t} = \sum_{i_k \in V_t^{(m_t+s)}} u_{i_k} \left(p^{\theta(i_k)} X\right)^{i_k}, \ s = 0, 1, ..., t - m_t, |u_{i_k}| = 1,$$

$$P_t = \sum_{s=0}^{t-m_t} P_{m_t+s,t},$$

then the coefficients u_{i_k} of P_t verify (3.11) and P_t satisfies (2.2) on $S(p^{-t})$. By (3.5) it follows that, for every t,

$$V_t \subset V_{t-1}. \tag{3.19}$$

Suppose that f satisfies (2.2) on $S(p^{-t})$, for every $t \geq n$. Because

$$h_{t+1}(i, \theta(i)) = h_t(i, \theta(i)) + i \tag{3.20}$$

it follows that either $t + 1 - m_{t+1} < t - m_t$ or $V_{t+1} = V_t = \{1\}$. Thus by (3.19) it follows that it is enough to verify (2.2) only for a finite number of $t \geq n$. This number is less than $n - m_n$.

Theorem 3. *Let $f : B(p^{-n}) \to B(p^{-n})$ be the function defined by the convergent series (3.4). Then (2.2) holds for every $x \in B(p^{-n})$, if and only if, $f(0) = 0$, $N_f = \infty$, for all $t \geq n$, and all $a_i \in \{0, 1, 2, ..., p-1\}$, $a_0 \neq 0$, (3.11) holds. Moreover (2.2) holds for every $x \in B(p^{-n})$ if $N_f = \infty$, for all $t \geq n$, all $\alpha \in \{0, 1, .., t - m_t\}$, all $\tau \in \{0, 1, .., \alpha\}$, $\tau \neq t - m_t$, and all $a_0 \in \{1, 2, ..., p-1\}$, (3.13) and (3.14) are fulfilled.*

Proof: Since $B(p^{-n}) = \{0\} \cup \bigcup_{t \geq n} S(p^{-t})$, by using the definition of N_f, the theorem follows by Lemma 2. ∎

Because $\mathbb{Q}_p = \bigcup_{t \in \mathbb{Z}} S(p^t)$, from Theorem 3 we derive the following:

Corollary 3. *Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be an entire function defined by (3.4). Then (2.2) holds for every $x \in \mathbb{Q}_p$, if and only if, $f(0) = 0$, $N_f = \infty$, and for all $t \in \mathbb{Z}$, and all $a_i \in \{0, 1, 2, ..., p-1\}$, $a_0 \neq 0$, (3.11) holds. Moreover (2.2) holds for every $x \in \mathbb{Q}_p$ if $N_f = \infty$, for all $t \in \mathbb{Z}$, all $\alpha \in \{0, 1, .., t - m_t\}$, all $\tau \in \{0, 1, .., \alpha\}$, $\tau \neq t - m_t$, and all $a_0 \in \{1, 2, ..., p-1\}$, (3.13) and (3.14) are fulfilled.*

Corollary 4. *Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be a polynomial function. Then (2.2) holds for every x from $\mathbb{Q}_p$, if and only if, there exists an element u from $\mathbb{Q}_p$, such that, for every $x \in \mathbb{Q}_p$,*

$$f(x) = ux, \ |u| = 1. \tag{3.21}$$

Proof: If f has at least two terms, then for t small enough, $V_t^{(m_t)}$ contains only one term and (3.11) does not hold. Hence $f(x) = ax^k$ with $a \in \mathbb{Q}_p$. If $k > 1$ or $|a| \neq 1$, then $N_f < \infty$ and the corollary follows by Corollary 3. ∎

In order to construct entire functions $f : \mathbb{Q}_p \to \mathbb{Q}_p$ defined by (3.1), which are not polynomial functions, and satisfy (3.13), (3.14), we need the following lemma:

Lemma 3. *(Hermite's interpolation) Let F_p be the field having p elements, r a nonnegative integer and $\{\beta_{i,j}, i = 1, ..., p-1, j = 1, ..., r\}$ arbitrary elements from F_p. Then there exists a unique polynomial $P \in F_p[X]$, of degree less than or equal to $d = rp + p - r - 2$ given by (3.8) such that, for every $j = 0, 1, ..., r$,*

$$T^j(P)(\gamma) = \beta_{i,j},\ \gamma = 1, 2, ..., p-1. \tag{3.22}$$

Proof: Consider $P \in F_p[X]$ given by (3.8), a fixed $\gamma \in \{1, 2, ..., p-1\}$ and r a nonnegative integer. Suppose that, for all $j = 0, 1, ..., r$,

$$T^j(P)(\gamma) = \sum_{k=j}^{d} a_k \left(\begin{array}{c} k \\ j \end{array} \right) \gamma^{k-j} = 0. \tag{3.23}$$

Then by (3.10) we get

$$P = (X-\gamma)^{r+1} Q, \tag{3.24}$$

where Q is a polynomial. Hence it follows that all polynomials of degree less than or equal to $(p-1)(r+1) - 1 = pr + p - r - 2$ which satisfy (3.22) for all $\beta_{ij} = 0$ can be written as

$$P = A \prod_{\gamma=1}^{p-1} (X-\gamma)^{r+1},\ A \in F_p.$$

Thus every polynomial of the form (3.8) of degree less than or equal to d which satisfies (3.22) for all $\beta_{ij} = 0$ vanishes identically. Hence it follows that the determinant of the matrix of the coefficients of the system (3.22), with respect to the unknowns a_k, is different from zero. This completes the proof of the lemma. □

Consider $P = X + \sum_{i=2}^{d} a_k X^i \in \mathbb{Z}_p[X]$. Then P satisfies (2.2) for every $x \in B(p^{-n})$, where n is a nonnegative integer. Given a polynomial verifying (2.2) on $B(p^{-n})$, in the proof of the following lemma we construct a polynomial satisfying (2.2) on $B(p^{-n+1})$.

Lemma 4. *Let n be an integer and let $P_1 = \sum_{i \in I_1} u_i^{(1)} \left(p^{\theta(i)} X\right)^i$ be a polynomial, written in the form (3.4), which verifies (2.2) on $S(p^{-t})$, for all $t \geq n$. If M is a positive integer, then there exists a polynomial $P_2 = \sum_{i \in I_2} u_i^{(2)} \left(p^{\theta(i)} X\right)^i$, written in the form (3.4), where $I_1 \subset I_2$, such that:*

(i) for every $i \in I_1$, $u_i^{(1)} = u_i^{(2)}$;
(ii) for every $t \geq n-1$, P_2 verifies (3.13) and (3.14);
(iii) for every $t \geq n$, m_t and $V_t^{(r)}$, $r = 0, 1, ..., t - m_t$, are the same for P_1 and P_2;
(iv) for every $i \in I_2 \backslash I_1$, $i > M$ and, if $n < 0$, then $\theta(i) > -\frac{n-1}{2}$.

Proof: Denote by $V_n(P_1)$, the set V_n defined for P_1,

$$\tilde{V}_{n-1}(P_1) = \{ i \in I_1 : h_{n-1}(i, \theta(i)) \leq n-1 \}.$$

Then, by (3.20), it follows that $\tilde{V}_{n-1}(P_1) \neq \emptyset$, and we choose

$$m_{n-1} \leq \min_{i \in \tilde{V}_{n-1}(P_1)} \{h_{n-1}(i, \theta(i))\}$$

By the definition of $\tilde{V}_{n-1}(P_1)$ we get $m_{n-1} \leq n-1$. For all $k = 0, 1, ..., n-1-m_{n-1}$, we put $r_k = n-1-m_{n-1}-k$, $d_k = (r_k+1)(p-1)-1$, $D_k = d + \sum\limits_{i=0}^{k} d_i$, where d is the degree of P_1. We seek the polynomials

$$\tilde{P}_{m_{n-1}+k,n-1} = \sum_{i=D_{k-1}+\delta_k}^{D_k+\delta_k} u_i^{(2)} X^i \in \mathbb{Z}_p[X],$$

where $D_{-1} = d$, $|u_i^{(2)}| \in \{0, 1\}$ and $\delta_k > 1$ are integers satisfying

$$D_{k-1} + \delta_k \equiv 0 \ (\mathrm{mod}\ (p-1)), \tag{3.25}$$

$$\delta_k > \max\left\{\frac{2m_{n-1}}{n-1}, M, n-m_{n-1}\right\} \tag{3.26}$$

such that, for all $\gamma \in \{1, 2, ..., p-1\}$

$$\begin{gathered} T^j(\tilde{P}_{m_{n-1},n-1})(\gamma) \equiv -T^j\left(\sum_{i \in \tilde{V}_{n-1}^{(m_{n-1})}} u_i^{(1)} x^i\right)(\gamma) \ (\mathrm{mod}\ p), \\ j = 0, ..., n-1-m_{n-1}-1, \end{gathered} \tag{3.27}$$

$$T^{n-1-m_{n-1}}(\tilde{P}_{m_{n-1},n-1})(\gamma) \equiv \gamma - T^{n-1-m_{n-1}}\left(\sum_{i \in \tilde{V}_{n-1}^{(m_{n-1})}} u_i^{(1)} x^i\right)(\gamma) \ (\mathrm{mod}\ p), \tag{3.28}$$

and for all $k = 1, 2, ..., n-1-m_{n-1}$, $j = 0, 1, ..., n-1-m_{n-1}-k$,

$$\begin{gathered} p^k T^j(\tilde{P}_{m_{n-1}+k,n-1})(\gamma) \equiv -\sum_{s=0}^{k-1} p^s \Big(T^j(\tilde{P}_{m_{n-1}+s,n-1})(\gamma) \\ -T^j(\sum_{i \in \tilde{V}_{n-1}^{(m_{n-1}+s)}} u_i x^i)(\gamma)\Big) - p^k T^j\left(\sum_{i \in \tilde{V}_{n-1}^{(m_{n-1}+k)}} u_i^{(1)} x^i\right)(\gamma) \ (\mathrm{mod}\ p^{k+1}), \end{gathered} \tag{3.29}$$

where $\tilde{V}_{n-1}^{(m_{n-1}+k)} = \{i \in \tilde{V}_{n-1} : h_{n-1}(i, \theta(i)) = m_{n-1}+k\}$.

To find the polynomials $\tilde{P}_{m_{n-1}+k,n-1}$ let us consider the images $\overline{R_{m_{n-1}+k,n-1}}$ of the polynomials $R_{m_{n-1}+k,n-1} = X^{-D_{k-1}-\delta_k} \tilde{P}_{m_{n-1}+k,n-1}$, in the residue field F_p. Thus, because for every nonzero $\overline{\gamma} \in F_p$, $\overline{\gamma}^{p-1} = \overline{1}$ and, by (3.25), $p-1$ divides $-D_{k-1} - \delta_k$, (3.27), (3.28) are equivalent to

$$T^j(\overline{R_{m_{n-1},n-1}})(\overline{\gamma}) \equiv -\gamma^{-D_{-1}-\delta_0} T^j \overline{\left(\sum_{i \in \tilde{V}_{n-1}^{(m_{n-1})}} u_i^{(1)} x^i \right)} (\gamma) \pmod p, \tag{3.30}$$

$$j = 0, ..., n-1-m_{n-1}-1,$$

$$T^{n-1-m_{n-1}}(\overline{R_{m_{n-1},n-1}})(\gamma)$$

$$\equiv \gamma^{-D_{-1}-\delta_0} \left(\gamma - T^{n-1-m_{n-1}} \overline{\left(\sum_{i \in \tilde{V}_{n-1}^{(m_{n-1})}} u_i^{(1)} x^i \right)} (\gamma) \right) \pmod p, \tag{3.31}$$

By Lemma 3, for $i \in [D_{-1} + \delta_0, D_0 + \delta_0]$, we find $\overline{u}_i^{(2)} \in F_p$ such that (3.30) and (3.31) hold. If $\overline{u}_i^{(2)} = \overline{\beta_i}$ with $\beta_i \in \{0, 1, ..., p-1\}$, we choose $u_i^{(2)} = \beta_i$. Hence we find $\tilde{P}_{m_{n-1},n-1}$ such that (3.27) and (3.28) are fulfilled.

Similarly, by recurrence, for $k = 1, ..., n-1-m_{n-1}$, because we can divide (3.29) by p^k, we find $\tilde{P}_{m_{n-1}+k,n-1}$ such that (3.29) holds.

We denote

$$V_{n-1}^{(m_{n-1}+k)} := \{i \in [D_{k-1} + \delta_k, D_k + \delta_k] : u_i \neq 0\} \cup \tilde{V}_{n-1}^{(m_{n-1}+k)}, \tag{3.32}$$

$V_{n-1} = \bigcup_{r=m_{n-1}}^{n-1} V_{n-1}^{(r)}$, $I_2 = I_1 \cup V_{n-1}$. For every $i \in V_{n-1}^{(m_{n-1}+k)} \backslash \tilde{V}_{n-1}^{(m_{n-1}+k)}$ we define

$$\theta(i) := \frac{m_{n-1} + k}{i} - n + 1. \tag{3.33}$$

Now we take

$$P_2 := \sum_{i \in I_2} u_i^{(2)} (p^{\theta(i)} X)^i, \tag{3.34}$$

where $u_i^{(2)} = u_i^{(1)}$, for $i \in I_1$. Hence (i) follows. By (3.27)-(3.29) and Lemma 2 it follows (ii). Since, by (3.26), for $i \in I_2 \backslash I_1$, $i > n - m_{n-1}$, from (3.33), we obtain (iii). (iv) follows easily by (3.26) because $\delta_k > \frac{2m_{n-1}}{n-1}$. Thus P_2 satisfies (i)-(iv). ∎

Theorem 4. *Let n be an integer and let $P = \sum_{i \in I_P} u_i \left(p^{\theta(i)} X\right)^i$, written in the form (3.4), such that $m_n(P) \leq n$ and, for every integer $t \in [n, 2n - m_n]$, P verifies (2.2) on $S(p^{-t})$. Then there exists an entire function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ such that:*

(i) f satisfies (2.2) for every $x \in \mathbb{Q}_p$;

(ii) if f is represented in the form (3.4), then for every $i \in I_P$ the corresponding terms of P and f coincide.

Proof: By Remark 2 it follows that P satisfies (2.2) on $S(p^{-t})$, for every $t \geq n$. By applying Lemma 4 to $P_1 = P$, $M > d_n$, where d_n is the degree of P, there exists a polynomial P_{n-1}

verifying the conditions (i)-(iv) of Lemma 4. Then by recurrence we find, for every $s \leq n-1$, a polynomial P_s such that it satisfies the conditions (i)-(iv) of Lemma 4.

We take

$$f := \sum_{i \in \bigcup_{s=-\infty}^{n} V_s} u_i (p^{\theta(i)} X)^i. \tag{3.35}$$

Then, by (ii) of Lemma 4 it follows that (3.13) and (3.14) hold for every $t \leq 0$. Moreover, by Lemma 4 (iv), we get, for every $i \in V_s \setminus V_{s+1}$,

$$\theta(i) = \frac{m_s + k}{i} - s \geq -s/2. \tag{3.36}$$

Hence it follows that $\lim_{i \to \infty} \theta(i) = \infty$ and f defines an entire function. Finally the theorem follows by (3.35) and Lemma 4. □

Remark 3. Suppose that $f : \mathbb{Q}_p \to \mathbb{Q}_p$, with $f(0) = 0$, is an entire function which is bounded, that is there exists a positive constant M_f and $|f(x)| \leq M_f$, for every $x \in \mathbb{Q}_p$. Then, for every $C = |y| > M_f$, $y \in \mathbb{Q}_p$, $g(x) = x\left(1 + \frac{f}{y}\right)$, is an entire function verifying (2.2) for every $x \in \mathbb{Q}_p$.

Conversely, if $g : \mathbb{Q}_p \to \mathbb{Q}_p$, with $g(0) = 0$ is an entire function verifying (2.2) for every $x \in \mathbb{Q}_p$, it follows easily that g is differentiable at 0 and $|g'(0)| = 1$. Then $f : \mathbb{Q}_p \to \mathbb{Q}_p$, such that, for $x \neq 0$, $f(x) = \frac{g(x)}{x}$, and $f(0) = g'(0)$ is a bounded entire function. Hence it follows that all counterexamples of Liouville's Theorem are constructed by means of entire functions verifying (2.2).

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